Miscellaneous Problems
End of Chapter 2

How to solve some 2^{nd} order DE's ?

- If $y(t)$ is a solution for a given DE then t is called independent variable (Indep. Var.) and y is called dependent variable (Dep. Var.)

Now if the 2^{nd} order DE misses the

[A] Dep. Var. y then let $\boxed{V = y'}$ and $\boxed{V' = y''}$

solve first for V then solve for y

[B] Indep. Var. t then let $\boxed{V = y'} = \frac{dy}{dt}$

and $V' = y'' = \frac{dV}{dt}$

$= \frac{dV}{dy}\frac{dy}{dt}$

$= \frac{dV}{dy} y'$

solve first for V
then solve for y

$$\boxed{y'' = \frac{dV}{dy} V}$$

Q36 / p.134

Exp Solve the DE :

$$t^2 y'' + 2t y' - 1 = 0 \quad , t > 0$$

Missing y $\Rightarrow$ Apply $\boxed{A}$ $\Rightarrow$ $y' = V$
$y'' = V'$

$$t^2 V' + 2t V - 1 = 0$$

$$V' + \underbrace{\frac{2}{t}}_{p(t)} V = \underbrace{\frac{1}{t^2}}_{g(t)} \quad \cdots B$$

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = t^2$$

$$V = \frac{1}{\mu}\left[\int \mu g \, dt + c_1\right]$$

$$= \frac{1}{t^2}\left[\int t^2 \frac{1}{t^2} \, dt + c_1\right]$$

$$= \frac{1}{t^2}\left[t + c_1\right]$$

$$y' = \frac{1}{t} + \frac{c_1}{t^2}$$

$$\boxed{y(t) = \ln t - \frac{c_1}{t} + c_2}$$

Exp Solve the IVP: Q49/p. 135

$$y'' - 3y^2 = 0 \quad , \; y(\underset{t_0}{0}) = \underset{y_0}{2} \; , \; y'_0 \, (\underset{t_0}{0}) = \underset{y'_0 = V_0}{4}$$

Missing t $\Rightarrow$ Apply $\boxed{B}$ $\Rightarrow$ $y' = V$

$$y'' = \frac{dV}{dy} V$$

$$\frac{dV}{dy} V - 3y^2 = 0 \qquad \text{"Seperable"}$$

$$\frac{dV}{dy} V = 3y^2$$

$$V \, dV = 3y^2 \, dy$$

$$\frac{V^2}{2} = y^3 + c_1$$

$$V^2 = 2y^3 + 2c_1$$

$$(y')^2 = 2y^3 + 2c_1 \qquad \text{To find } c_1 \Rightarrow$$

$$(4)^2 = 2(2)^3 + 2c_1$$

$$16 = 16 + 2c_1 \quad \Leftrightarrow \quad \boxed{c_1 = 0}$$

$$(y')^2 = 2y^3$$

$$y' = \pm\sqrt{2y^3}$$

$$4 = \boxed{+}\sqrt{2(2)^3}$$

$$y' = \sqrt{2y^3}$$

$$\frac{dy}{dt} = \sqrt{2}\, y^{\frac{3}{2}}$$

$$\int y^{-\frac{3}{2}} dy = \int \sqrt{2}\, dt$$

$$-2y^{-\frac{1}{2}} = \sqrt{2}\, t + c_2$$

$$\frac{-2}{\sqrt{y}} = \sqrt{2}\, t + c_2$$

To find $c_2 \Rightarrow$

$$\frac{-2}{\sqrt{2}} = \sqrt{2}(0) + c_2$$

$$-\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} = c_2$$

$$\boxed{c_2 = -\sqrt{2}}$$

$$\frac{-2}{\sqrt{y}} = \sqrt{2}\, t - \sqrt{2}$$

$$\frac{\sqrt{y}}{-2} = \frac{1}{\sqrt{2}\, t - \sqrt{2}}$$

$$\sqrt{y} = \frac{-2}{\sqrt{2}(t-1)}$$

$$\sqrt{y} = \frac{\sqrt{2}}{1-t}$$

$$\boxed{y(t) = \frac{2}{(1-t)^2}}$$

Exp (Q42) Solve the IVP

$$y y'' + (y')^2 = 0 \quad , y(0) = 1 \; , y'(0) = 1 \quad , y > 0$$

Missing $t \Rightarrow y' = v$ and $y'' = v \frac{dv}{dy}$

$$y\left(v \frac{dv}{dy}\right) + v^2 = 0$$

$$v\left[y \frac{dv}{dy} + v\right] = 0$$

either $v = 0 \Rightarrow y' = 0$ not possible since $y'_0 = 1$

or $y \frac{dv}{dy} + v = 0 \quad \Rightarrow \int \frac{dv}{v} = -\int \frac{dy}{y}$

$\ln|v| = -\ln|y| + c_1 \quad \Rightarrow$ To find c_1 we use IC's

$\ln|y'| = -\ln y + c_1$

$\ln 1 = -\ln 1 + c_1 \quad \Rightarrow 0 = 0 + c_1 \Rightarrow c_1 = 0$

$\ln|y'| = \ln y^{-1} \quad \Rightarrow |y'| = \frac{1}{y}$

either $y' = -\frac{1}{y}$ not possible since $y_0 = 1 = y'_0$

or $y' = \frac{1}{y} \quad \Rightarrow \frac{dy}{dt} = \frac{1}{y} \quad \Rightarrow \int y \, dy = \int dt$

$\frac{y^2}{2} = t + c_2 \quad \Rightarrow$ To find c_2 we use $y(0) = 1$

$\frac{1}{2} = 0 + c_2 \quad \Rightarrow c_2 = \frac{1}{2} \quad \Rightarrow \frac{y^2}{2} = t + \frac{1}{2}$

$y^2 = 2t + 1 \quad \Rightarrow y(t) = \pm\sqrt{2t+1} \quad \Rightarrow \boxed{y(t) = \sqrt{2t+1}}$ since $y_0 = 1$