Ch3

3.1 2^{nd} order linear and homogenous DE with constant coefficients

- Any 2^{nd} order DE has the form

$$\boxed{y'' = f(t, y, y')} \quad (1)$$

- The DE (1) is linear if f is linear in y and y'. Otherwise, (1) is nonlinear.

- The solution of (1) is $y(t)$ where

  t is the indep. variable
  y is the depen. variable

- We will focus on the linear case of (1) which has the form

$$\boxed{y'' + p(t)\, y' + q(t)\, y = g(t)} \quad (2)$$

- If $g(t) = 0$, then (2) is called homogenous DE
- If $g(t) \neq 0$, then (2) is called nonhomogenous DE

- More precisely, we will learn how to solve (2) when $g(t) = 0$ and $p(t)$ and $q(t)$ are constants $\Rightarrow$

$$\boxed{ay'' + by' + cy = 0} \quad (3) \qquad y(t_0) = y_0, \; y'(t_0) = y'_0$$

- The DE ③ is 2^{nd} order linear homogenous with constant coefficients (2^{nd} OLHCC)

Question: How to solve the DE ③ ?

Answer: To find the solution of ③ $\Rightarrow$
we assume exponensial solution of the form:

$$\boxed{y(t) = e^{rt}}$$ where r is constant

To find r we substitute y, y', y'' in ③ $\Rightarrow$

$y'(t) = r e^{rt}$
$y''(t) = r^2 e^{rt}$

$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$

$$\boxed{a r^2 + b r + c = 0} \quad ④ \rightarrow$$ Characterstic Equation (Ch. Eq.)

Compare ④ with ③

- To solve the Ch. Eq. ④ for the roots r_1, r_2

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So we have three cases for the roots:

$\Downarrow$

[1] If $r_1 \neq r_2 \in \mathbb{R}$ "Real Different", then

the first solution is $y_1(t) = e^{r_1 t}$
and the second solution is $y_2(t) = e^{r_2 t}$

the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

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[2] If $r_1 = r_2 = r \in \mathbb{R}$ "Real Repeated", then

the first solution is $y_1(t) = e^{rt}$
and the second solution is $y_2(t) = t\, e^{rt}$

the general solution is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$y(t) = c_1 e^{rt} + c_2 t\, e^{rt}$$

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[3] If $r_{1,2} = \lambda \pm \mu i$ "Complex Roots", then

the first solution is $y_1(t) = e^{\lambda t} \cos(\mu t)$

and the second solution is $y_2(t) = e^{\lambda t} \sin(\mu t)$

the general solution is

$\mu > 0$

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

Exp Find the general solution of the following

① IVP: $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

2nd OLHCC

Ch. Eq $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -2$, $r_2 = -3$ "Real Different"

missing t

$y_1(t) = e^{-2t}$, $y_2(t) = e^{-3t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ To find $c_1, c_2 \Rightarrow$

$y(0) = \boxed{c_1 + c_2 = 2}$

$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$

$y'(0) = \boxed{-2c_1 - 3c_2 = 3}$

$c_1 = 9$
$c_2 = -7$

$y(t) = 9e^{-2t} - 7e^{-3t}$ unique solution

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② DE: $y'' - 4y' + 4y = 0$

2nd OLHCC

missing t

Ch. Eq. $r^2 - 4r + 4 = 0$

$(r-2)(r-2) = 0$

$r_1 = r_2 = 2$ "Real Repeated"

$y_1(t) = e^{2t}$, $y_2(t) = t e^{2t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$y(t) = c_1 e^{2t} + c_2 t e^{2t}$

3) DE: $y'' + 2y' + 2y = 0$

2^{nd} OLHCC

missing t

ch. Eq $r^2 + 2r + 2 = 0$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm \sqrt{4}\sqrt{-1}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$y_1(t) = e^{\lambda t}\cos(\mu t) = e^{-t}\cos t \qquad \lambda = -1, \mu = 1$$

$$y_2(t) = e^{\lambda t}\sin(\mu t) = e^{-t}\sin t$$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$= c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

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4) IVP: $2y'' + 3y' = 0$, $y(0) = 1$, $y'(0) = 3$

2^{nd} OLHCC

missing t
missing y

ch. Eq. $2r^2 + 3r = 0$

$$r(2r + 3) = 0$$

$r_1 = 0$, $r_2 = -\frac{3}{2}$ "Real Different"

$y_1(t) = 1$, $y_2(t) = e^{-\frac{3}{2}t}$

gen. sol. $y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$y(t) = c_1 + c_2 e^{-\frac{3}{2}t}$$

$$y'(t) = -\frac{3}{2} c_2 e^{-\frac{3}{2}t}$$

To find c_1 and c_2:

$y(0) = \boxed{c_1 + c_2 = 1}$

$y'(0) = -\frac{3}{2} c_2 = 3$

$\boxed{c_2 = -2}$

$\Rightarrow \boxed{c_1 = 3}$

$$y(t) = 3 - 2e^{-\frac{3}{2}t}$$

5 DE: $y'' + 9y = 0$ 2nd OLHCC missing t

Ch. Eq $r^2 + 9 = 0$

$r^2 = -9 \Rightarrow \sqrt{r^2} = \sqrt{-9}$

$|r| = \sqrt{9}\,\sqrt{-1}$

$|r| = 3i$

$r_{1,2} = \pm 3i$ $\lambda = 0$

$\mu = 3$

$y_1(x) = e^{\lambda x}\cos \mu x = \cos 3x$

$y_2(x) = e^{\lambda x}\sin \mu x = \sin 3x$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 \cos 3x + c_2 \sin 3x$

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6 IVP: $R'' + R = 0$, $R(0) = 3$, $R'(0) = 2$

Ch. Eq. $r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$ $\lambda = 0$

$\mu = 1$

2nd OLHCC

$R_1(x) = \cos x$

$R_2(x) = \sin x$

gen. sol. $R(x) = c_1 R_1(x) + c_2 R_2(x)$

$= c_1 \cos x + c_2 \sin x$

To find c_1 and $c_2 \Rightarrow$

$R'(x) = -c_1 \sin x + c_2 \cos x$

$R(0) = c_1 + 0 = 3 \Rightarrow c_1 = 3$

$R'(0) = 0 + c_2 = 2 \Rightarrow c_2 = 2$

$R(x) = 3\cos x + 2\sin x$