Euler DE

Euler DE has the form $\boxed{x^2 y'' + \alpha x y' + \beta y = 0}$ — Ⓔ

where α and β are constant and $x \neq 0$ $(x > 0)$

Question: How to solve the Euler DE Ⓔ ?

Answer: Assume power solution of the form:

$$y = x^r \quad , \; r \in \mathbb{R} \text{ and consider } x > 0$$

To find r we substitute y, y', y'' in Ⓔ $\Rightarrow$

$$y' = r x^{r-1}$$
$$y'' = r(r-1) x^{r-2}$$

$t = \ln x$
$e^t = e^{\ln x}$
$e^t = x$
$e^{rt} = x^r$
$e^{rt} = y$

$$x^2 \, r(r-1) x^{r-2} + \alpha x \, r x^{r-1} + \beta x^r = 0$$

$$(r^2 - r) x^r + \alpha r x^r + \beta x^r = 0$$

$$x^r \left[r^2 - r + \alpha r + \beta \right] = 0 \qquad \text{note that } x^r \neq 0$$

$$\boxed{r^2 + (\alpha - 1) r + \beta = 0} \quad E^*$$

Now we solve the quadratic equation E^* for the roots r_1 and r_2

So we have three cases for these roots:

$$r_{1,2} = \frac{(1-\alpha) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

1E If $r_1 \neq r_2 \in \mathbb{R}$ "Real Different", then

$y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

2E If $r_1 = r_2 = r \in \mathbb{R}$ "Real Repeated", then

$y_1(x) = x^{r}$ and $y_2(x) = (\ln x) x^{r}$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{r} + c_2 (\ln x) x^{r}$$

3E If $r_{1,2} = \lambda \pm \mu i$ "Complex Roots", then

$y_1(x) = x^{\lambda} \cos(\mu \ln x)$ and $y_2(x) = x^{\lambda} \sin(\mu \ln x)$

The gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$y(x) = c_1 x^{\lambda} \cos(\mu \ln x) + c_2 x^{\lambda} \sin(\mu \ln x)$$

$\Downarrow$

$$y_1(x) = e^{\lambda t} \cos \mu t = e^{\lambda \ln x} \cos(\mu \ln x)$$

$$= e^{\ln x^{\lambda}} \cos(\mu \ln x)$$

$$= x^{\lambda} \cos(\mu \ln x)$$

same for $y_2(x)$

Exp Solve the DE

① $2x^2 y'' + 3xy' - y = 0 \quad , x > 0$ 2nd OLH → ⓔ

- This DE is Euler with $\alpha = \frac{3}{2}$ and $\beta = -\frac{1}{2}$
- Solve (E^*) $\Rightarrow r^2 + (\alpha - 1) r + \beta = 0$

$$r^2 + \left(\frac{3}{2} - 1\right) r - \frac{1}{2} = 0$$

$$r^2 + \frac{1}{2} r - \frac{1}{2} = 0$$

$$(r+1)\left(r - \frac{1}{2}\right) = 0$$

$r_1 = -1 \; , \; r_2 = \frac{1}{2}$ " Real Different "

$y_1(x) = x^{-1} = \frac{1}{x}$ and $y_2(x) = x^{\frac{1}{2}} = \sqrt{x}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$\boxed{y(x) = \frac{c_1}{x} + c_2 \sqrt{x}}$$

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② $x^2 y'' + 5xy' + 4y = 0 \; , \quad x > 0$ 2nd OLH → ⓔ

- This DE is Euler with $\alpha = 5$ and $\beta = 4$
- Solve (E^*) $\Rightarrow r^2 + (\alpha - 1) r + \beta = 0 \quad \Rightarrow r^2 + 4r + 4 = 0$

$\Rightarrow (r+2)(r+2) = 0$

$\Rightarrow r_1 = r_2 = r = -2$

$y_1(x) = x^{-2}$

$y_2(x) = (\ln x)\, x^{-2}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$\boxed{y(x) = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2}}$$

③ $x^2 y'' + x y' + y = 0$, $x > 0$ 2^{nd} OLH $\to$ (E)

• This DE is Euler with $\alpha = \beta = 1$

• solve (E^*) $\Rightarrow$ $r^2 + (\alpha - 1) r + \beta = 0$

$r^2 + 1 = 0$

$r_{1,2} = \pm i$ $\lambda = 0$, $\mu = 1$

$y_1(x) = x^{\lambda} \cos(\mu \ln x) = \cos(\ln x)$

$y_2(x) = x^{\lambda} \sin(\mu \ln x) = \sin(\ln x)$

gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)$

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④ $x y'' = \frac{1}{x} y$, $x > 0$

$x^2 y'' - y = 0$ $\Rightarrow$ This is Euler DE with $\alpha = 0$ and $\beta = -1$

$\Rightarrow$ solve (E^*): $r^2 + (\alpha - 1) r + \beta = 0$

$r^2 - r - 1 = 0$

$r_{1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ "Real Different"

$y_1(x) = x^{r_1}$ and $y_2 = x^{r_2}$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$y(x) = c_1 x^{\frac{1+\sqrt{5}}{2}} + c_2 x^{\frac{1-\sqrt{5}}{2}}$

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⑤ $x y'' = y'$, $x > 0$ $x^2 y'' - x y' = 0$ $\Rightarrow$ Euler

with $\alpha = -1$ and $\beta = 0$ $\Rightarrow$ $r^2 + (\alpha - 1) r + \beta = 0$ $\Rightarrow$ $r^2 - 2r = 0$

$r(r-2) = 0$ $\Rightarrow$ $r_1 = 0$, $r_2 = 2$ $\Rightarrow$ $y_1(x) = 1$ and $y_2(x) = x^2$

gen. sol. $y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 + c_2 x^2$
