3.2 Solutions for linear DE's of order 2
(Wronskian, Fundemental Solutions, Abel's Theorem)

Th 3.2.1 (Existance and Uniquness)

Consider the IVP:

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad , \quad y(t_0) = y_0 \ , \ y'(t_0) = y'_0 \quad (1)$$

If $p(t), q(t), g(t)$ are cont. on an open interval I containing t_0, then $\exists$ a unique solution $y = \phi(t)$ satisfying the IVP (1) on I.

Exp Find the largest interval in which the solution of the following IVP's is valid (defined):

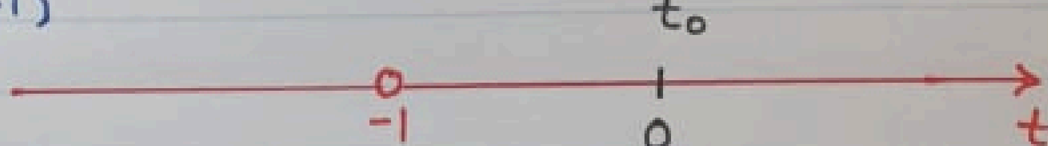
[A] $(t+1)\,y'' - (\cos t)\,y' = 1 - 3y \ , \ y(0) = 1 \ , \ y'(0) = 0$

Compare this IVP with (1) $\Rightarrow$

$$y'' - \left(\frac{\cos t}{t+1}\right) y' + \left(\frac{3}{t+1}\right) y = \frac{1}{t+1}$$

$$p(t) = -\frac{\cos t}{t+1} \quad , \quad q(t) = \frac{3}{t+1} \quad , \quad g(t) = \frac{1}{t+1}$$

All cont. on $\mathbb{R} \setminus \{-1\}$



$$I = (-1, \infty)$$

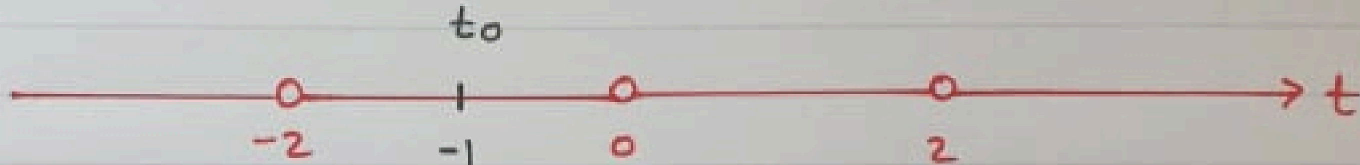
(B) $(t^2-4)y'' + (\sin t)y' + \ln|t|\, y = t \ ,\ y(-1)=0,\ y'(-1)=0$

Compare this IVP with (1) $\Rightarrow$

$$y'' + \underbrace{\left(\frac{\sin t}{t^2-4}\right)}_{p(t)} y' + \underbrace{\left(\frac{\ln|t|}{t^2-4}\right)}_{q(t)} y = \underbrace{\frac{t}{t^2-4}}_{g(t)}$$

All cont. on $\mathbb{R} \setminus \{-2, 0, 2\}$

$I = (-2, 0)$



(C) $\sqrt{2t+6}\, y'' + y' = \ln(2-t) \ ,\ y(1)=4 \ ,\ y'(1)=2$

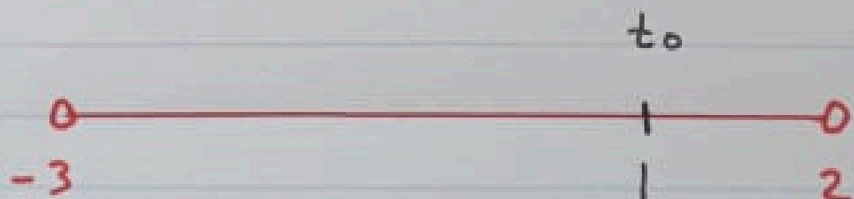
Compare this IVP with (1) $\Rightarrow$

$$y'' + \frac{1}{\sqrt{2t+6}}\, y' = \frac{\ln(2-t)}{\sqrt{2t+6}}$$

$2t+6>0$

$2t>-6$

$t>-3$



$2-t>0$

$2>t$

$I = (-3, 2)$

Exp Consider the IVP: $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = 0$, $y'(t_0) = 0$
where $p(t), q(t)$ are cont. on an open interval I contains t_0.
Find the solution of this IVP. Is it unique?

- $g(t) = 0$ which is cont. on $\mathbb{R}$ $\Rightarrow$ it's also cont. on I
- Conditions of Th3.2.1 hold $\Rightarrow \exists$ unique sol.
- The unique sol. must satisfy the DE and the IC's:

$y(t) = 0$ is the unique solution

Th 3.2.2 (Principle of Superposition)
Suppose y_1 and y_2 are solutions for the DE:

$y'' + p(t)y' + q(t)y = 0$. Then the linear combination

$c_1 y_1 + c_2 y_2$ is also solution for any constants c_1 and c_2.

Proof $(c_1 y_1 + c_2 y_2)'' + p(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2) =$

$c_1 y_1'' + c_2 y_2'' + p(t)(c_1 y_1' + c_2 y_2') + q(t)(c_1 y_1 + c_2 y_2) =$

$c_1 \underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{\text{zero since } y_1 \text{ sol.}} + c_2 \underbrace{(y_2'' + p(t)y_2' + q(t)y_2)}_{\text{zero since } y_2 \text{ sol.}} =$

$c_1(0) + c_2(0) = 0 \quad \Rightarrow c_1 y_1 + c_2 y_2$ is sol.

Remark The linear combination $c_1 y_1 + c_2 y_2$ is called the general solution and we write $y(t) = c_1 y_1(t) + c_2 y_2(t)$

• If the DE is supported by two IC's :

$y(t_0) = y_0$ and $y'(t_0) = y'_0$ then we can find the constants c_1 and c_2 in the general solution:

$$y(t) = c_1\, y_1(t) + c_2\, y_2(t)$$
$$y'(t) = c_1\, y'_1(t) + c_2\, y'_2(t)$$

$$y(t_0) = \boxed{c_1\, y_1(t_0) + c_2\, y_2(t_0) = y_0} \quad \text{---} ①$$
$$y'(t_0) = \boxed{c_1\, y'_1(t_0) + c_2\, y'_2(t_0) = y'_0} \quad \text{---} ②$$

We use Cramer's Rule to solve ① and ② for c_1 and c_2:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y_0\, y'_2(t_0) - y_2(t_0)\, y'_0}{y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0)} \quad \text{---} *^1$$

$$c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y_1(t_0)\, y'_0 - y_0\, y'_1(t_0)}{y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0)} \quad \text{---} *^2$$

For c_1 and c_2 to be well defined, we must have

$$W(y_1, y_2)(t_0) = y_1(t_0)\, y'_2(t_0) - y_2(t_0)\, y'_1(t_0) \neq 0$$

Def If y_1 and y_2 are solutions to the DE

$$y'' + p(t)y' + q(t)y = 0 \quad (2)$$

then the Wronskian of y_1 and y_2 is defined by

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)$$

$t \in I$.

Th 3.2.3 Assume y_1 and y_2 are solutions of (2).
Then $\exists$ c_1 and c_2 s.t $c_1 y_1 + c_2 y_2$ satisfies (2)
iff $W(y_1, y_2)(t_0) \neq 0'$, $t_0 \in I$.

Proof $\Rightarrow$ If $\exists$ c_1 and c_2 s.t $c_1 y_1 + c_2 y_2$ satisfies (2)
then c_1 and c_2 defined by $*^1$ and $*^2$
are well-defined
$\Rightarrow W(y_1, y_2)(t_0) \neq 0$

$\Leftarrow$ Assume y_1 and y_2 are solution of (2) $\Rightarrow$
by Th 3.2.2 $c_1 y_1 + c_2 y_2$ is also solution.

Since $W(y_1, y_2)(t_0) \neq 0$, $t_0 \in I$ $\Rightarrow$ we can
find c_1 and c_2 using Cramer's Rule $*^1$ and $*^2$

Def • $y_1, y_2, \cdots, y_n$ are Linearly Dependent if $\exists$

$c_1, c_2, \cdots, c_n$ not all zeros s.t

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

• $y_1, y_2, \cdots, y_n$ are Linearly Independent if

whenever $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$ implies that

$$c_1 = c_2 = \cdots = c_n = 0$$

Remark • If $W(y_1, y_2)(t) \neq 0$, then y_1 and y_2 are linearly independent

That is:

• If y_1 and y_2 are linearly dependent, then $W(y_1, y_2)(t) = 0$

Exp $y_1 = e^t$ and $y_2 = e^{-t}$ are Linearly Independent

since $W(e^t, e^{-t})(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$

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or $c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 e^t + c_2 e^{-t} = 0$

$$\left.\begin{aligned} t = 0 &\Rightarrow c_1 + c_2 = 0 \\ t = \ln 2 &\Rightarrow 2c_1 + \tfrac{1}{2} c_2 = 0 \end{aligned}\right\} \Rightarrow c_1 = c_2 = 0$$

Exp $y_1 = \sin 2x$ and $y_2 = \sin x \cos x$ are L. dependent

$c_1 y_1 + c_2 y_2 = 0 \Rightarrow c_1 \sin 2x + c_2 \sin x \cos x = 0 \Rightarrow \begin{matrix} c_1 = 1 \\ c_2 = -2 \end{matrix}$