Th 3.2.4 (Fundemental Set of Solution)

Assume y_1 and y_2 are solutions for the DE :

$y'' + p(t)\,y' + q(t)\,y = 0$ — ② on I. Then

the family of all solutions $c_1 y_1 + c_2 y_2$ satisfies ② iff

$\exists\ t_0 \in I$ s.t $W(y_1, y_2)(t_0) \neq 0$.

Proof: similar to proof of Th 3.2.3

Remark: If y_1 and y_2 satisfy Th 3.2.4, then

① y_1 and y_2 are solutions for ② and

② y_1 and y_2 are L. indep. since $W(y_1, y_2)(t_0) \neq 0$

So $\{y_1, y_2\}$ is called Fundemental set of solutions.

Exp Find the fundemental set of solutions for $y'' - y = 0$

ch. Eq $r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r_1 = 1 \Rightarrow y_1 = e^{t}$

$r_2 = -1 \Rightarrow y_2 = e^{-t}$

① e^{t}, e^{-t} are solutions

② $W(e^{t}, e^{-t})(t) = \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix} = -1 - 1 = -2 \neq 0$

$\Rightarrow e^{t}, e^{-t}$ are L. indep.

Hence, $\{e^{t}, e^{-t}\}$ is fundemental set of solutions

Exp Show that $y_1 = \sqrt{t}$ and $y_2 = \frac{1}{t}$ form fundemental set of solutions for the DE:

$$2t^2 y'' + 3t y' - y = 0 \quad , \quad t > 0$$

① we need to show y_1 and y_2 are solutions

$$y_1 = \sqrt{t} \Rightarrow y_1' = \frac{1}{2\sqrt{t}} = \frac{1}{2} t^{-\frac{1}{2}} \Rightarrow y_1'' = -\frac{1}{4} t^{-\frac{3}{2}}$$

$$2t^2 y_1'' + 3t y_1' - y_1 = 2t^2\left(-\frac{1}{4} t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2} t^{-\frac{1}{2}}\right) - t^{\frac{1}{2}}$$

$$= -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}}$$

$$= 0$$ so y_1 is solution

$$y_2 = \frac{1}{t} = t^{-1} \Rightarrow y_2' = -t^{-2} \Rightarrow y_2'' = 2t^{-3}$$

$$2t^2 y_2'' + 3t y_2' - y_2 = 2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1}$$

$$= 4t^{-1} - 3t^{-1} - t^{-1}$$

$$= 0$$ so y_2 is solution

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② $$W\left(\sqrt{t}, \frac{1}{t}\right)(t) = \begin{vmatrix} \sqrt{t} & \frac{1}{t} \\ \frac{1}{2\sqrt{t}} & -\frac{1}{t^2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2} t^{-\frac{3}{2}} = \frac{-3}{2\sqrt{t^3}} \neq 0$$

since $t > 0$

Hence, y_1 and y_2 are L. independent.

Thus they form fundemental set of solutions.

Exp Find the fundemental set of solutions for the DE:

$$3y'' + y' - 2y = 0$$

Ch. Eq.

$$3r^2 + r - 2 = 0$$

$$(3r-2)(r+1) = 0$$

$$r_1 = \frac{2}{3} \Rightarrow y_1 = e^{\frac{2}{3}t}$$

$$r_2 = -1 \Rightarrow y_2 = e^{-t}$$

$\Rightarrow y_1$ and y_2 are solutions

$$W\left(e^{\frac{2}{3}t}, e^{-t}\right)(t) = \begin{vmatrix} e^{\frac{2}{3}t} & e^{-t} \\ \frac{2}{3} e^{2t/3} & -e^{-t} \end{vmatrix}$$

$$= -e^{-t/3} - \frac{2}{3} e^{-t/3}$$

$$= \frac{-5}{3} e^{-t/3}$$

$$\neq 0 \quad \text{since } e^{-t/3} \neq 0$$

Hence, y_1 and y_2 are L. indep.

Thus, $\left\{ e^{\frac{2}{3}t}, e^{-t} \right\}$ form fundemental set of solutions

Th (Abel's Theorem)

Assume y_1 and y_2 are solutions for the DE:

$$y'' + p(t)y' + q(t)y = 0 \quad (2)$$

where $p(t)$ and $q(t)$ are cont. on interval I. Then the Wronskian of y_1 and y_2 is given by:

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)dt}$$

where c is constant that depends on the form of y_1 and y_2.

Furthermore, $W(y_1, y_2)(t) = 0 \quad \forall t \in I$

or $W(y_1, y_2)(t) \neq 0 \quad \forall t \in I$

Proof since y_1 and y_2 sol. for the DE (2) $\Rightarrow$

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad \dots A$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad \dots B$$

multiply A by $-y_2$
mutiply B by y_1
Thed add the results

$$\underbrace{\left(y_1 y_2'' - y_2 y_1''\right)}_{W'} + p(t)\underbrace{\left(y_1 y_2' - y_2 y_1'\right)}_{W} = 0$$

$$W' = y_1 y_2'' + \cancel{y_2' y_1'} - y_2 y_1'' - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_2 y_1''$$

$$W' + p(t)W = 0$$

$$\int \frac{W'}{W} = \int -p(t) \Rightarrow \ln|W| = -\int p(t) + d$$

$$|W| = e^{-\int p(t)dt} e^{d} \Rightarrow W = \pm e^{d} e^{-\int p(t)dt}$$

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)dt}$$

$W(y_1, y_2)(t) = 0$ iff $c = 0$ since $e^{-\int p(t)dt} \neq 0 \quad \forall t \in I$

Exp Find the wronskian for the solutions of the DE

1. $y'' - y' - 2y = 0$ $\quad p(t) = -1$

$W(y_1, y_2)(t) = c e^{-\int p(t)\,dt} = c e^{-\int -dt} = c e^{t}$ "we can find c" $\quad c = -3$

or ch.Eq. $r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0$

$r_1 = 2 \Rightarrow y_1(t) = e^{2t}$

$r_2 = -1 \Rightarrow y_2(t) = e^{-t}$

$W(e^{2t}, e^{-t})(t) = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{t} - 2e^{t} = -3e^{t}$

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2. $(t-1)y'' - ty' + y = 0 \quad , t > 1$

Compare this DE with (2) $\Rightarrow y'' - \left(\frac{t}{t-1}\right)y' + \left(\frac{1}{t-1}\right)y = 0$

$p(t) = -\frac{t}{t-1} \Rightarrow W(y_1, y_2)(t) = c e^{-\int p(t)\,dt}$

$= c e^{-\int \frac{-t}{t-1}\,dt}$

$W(y_1, y_2)(t) = c e^{\int \frac{t}{t-1}\,dt}$

$= c e^{\int \frac{t-1+1}{t-1}\,dt} = c e^{\int \left(1 + \frac{1}{t-1}\right)dt}$

$= c e^{t + \ln|t-1|} = c e^{t} e^{\ln(t-1)} \quad t > 1$

$= c(t-1)e^{t}$

To find $W(y_1, y_2)(t)$ $\begin{cases} W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} & \text{if we know } y_1 \text{ and } y_2 \\ W = c e^{-\int p(t)\,dt} & \text{if we don't know } y_1, y_2 \end{cases}$

<u>Exp</u> Assume y_1 and y_2 are solutions for the DE:

$$y'' - 2t y' + e^t y = 0 \ , \quad t > 0 \quad \text{with}$$

$W(y_1, y_2)(2) = 8$. Find $W(y_1, y_2)(3)$.

$p(t) = -2t$

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt} = c\, e^{-\int (-2t)\,dt}$$

$$= c\, e^{t^2}$$

$$W(y_1, y_2)(2) = c\, e^{2^2} = c\, e^4$$

$$8 = c\, e^4$$

$$c = \frac{8}{e^4} \Rightarrow W(y_1, y_2)(t) = \frac{8}{e^4}\, e^{t^2}$$

$$W(y_1, y_2)(3) = \frac{8}{e^4}\, e^{3^2}$$

$$= \frac{8}{e^4}\, e^9$$

$$= 8\, e^5$$