3.3 Complex Roots

2^{nd} OLHCC

Exp Solve the DE: $y'' + 4y' + 5y = 0$

Ch. Eq. $r^2 + 4r + 5 = 0 \Rightarrow r_{1,2} = \dfrac{-4 \pm \sqrt{16-20}}{2}$

$$r_{1,2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

$\lambda = -2 \ , \ \mu = 1$

$$y_1(x) = e^{\lambda x} \cos \mu x = e^{-2x} \cos x$$
$$y_2(x) = e^{\lambda x} \sin \mu x = e^{-2x} \sin x$$

gen. sol. $y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$

Taylor Series

- If $f(x)$ is infinitly many differentiable, then Taylor expansion of $f(x)$ about $x=a$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- Special case when $a=0$ " Maclurin Series "

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(a) + f'(a)\,x + \frac{f''(a)}{2!} x^2 + \cdots$$

Exp Derive Euler Formula $e^{ix} = \cos x + i \sin x$

Recall Maclurine Series of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$i = \sqrt{-1}$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$\vdots$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \cos x + i \sin x$$

Exp Rewrite $e^{2+\frac{\pi}{2}i}$ as $a+bi$

$$e^{2+\frac{\pi}{2}i} = e^2 e^{\frac{\pi}{2}i} = e^2\left[\cos\tfrac{\pi}{2} + i \sin\tfrac{\pi}{2}\right] = e^2[0+i]$$

$$= e^2 i \qquad a=0, \ b=e^2$$

Exp $e^{-i\theta} = \cos\theta - i\sin\theta$ show this form of Euler Formula

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Exp Use Euler Formula to write $e^{1-\frac{\pi}{3}i}$ in the form of $a+bi$

$$e^{1-\frac{\pi}{3}i} = e\, e^{-\frac{\pi}{3}i} = e\left(\cos\tfrac{\pi}{3} - i\sin\tfrac{\pi}{3}\right) = e\left(\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right) = \frac{e}{2} - \frac{e\sqrt{3}}{2}i$$