3.4 Repeated Roots; Reduction of Order Method

Exp Solve the IVP: $y'' + 2y' + y = 0$, $y(0) = y'(0) = 1$ (2nd OLHCC)

Ch. Eq
$$r^2 + 2r + 1 = 0$$
$$(r+1)(r+1) = 0$$
$$r_1 = r_2 = r = -1$$

$$y_1(t) = e^{rt} = e^{-t}$$
$$y_2(t) = t e^{rt} = t e^{-t}$$

gen. sol.
$$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{-t} + c_2 t e^{-t}$$
$$y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$$

$$y(0) = c_1 + 0 = 1 \Rightarrow c_1 = 1$$
$$y'(0) = -c_1 + c_2 - 0 = 1 \Rightarrow c_2 = 2$$

$$y(t) = e^{-t} + 2t e^{-t}$$

$\lim_{t\to\infty} y(t) = 0$

Exp Find Fundemental solutions of Exp above.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = -te^{-2t} + e^{-2t} - -te^{-2t} = e^{-2t} \neq 0$$

Hence, $y_1 = e^{-t}$ and $y_2 = te^{-t}$ are L. Indep.

$\Rightarrow \{e^{-t}, te^{-t}\}$ form fundemental Set of solutions

Reduction of Order Method (ROM)

Given $y_1(t)$ solution for the 2^{nd} order linear homogeneous DE:

$$y'' + p(t)y' + q(t)y = 0 \quad \text{—} (2)$$

How to find 2^{nd} independent solution $y_2(t)$?

We use ROM to reduce the order of (2) as follow:

- Assume $\boxed{y_2(t) = v(t)\,y_1(t)}$ — $\boxed{C}$ is solution for (2)

$\Rightarrow y_2' = v y_1' + y_1 v' \qquad \Rightarrow y_2'' = v y_1'' + y_1' v' + y_1' v' + v'' y_1$

$$= v y_1'' + 2 y_1' v' + y_1 v''$$

- Substitute y_2, y_2', y_2'' in (2) $\Rightarrow$

$$v y_1'' + 2y_1' v' + y_1 v'' + p(t)(v y_1' + y_1 v') + q(t) v(t) y_1 = 0$$

$$y_1 v'' + (2y_1' + p(t) y_1) v' + v\underbrace{(y_1'' + p(t) y_1' + q(t) y_1)}_{\text{zero since } y_1 \text{ solves } (2)} = 0$$

$$y_1 v'' + (2y_1' + p(t)y_1) v' = 0$$

Let $\boxed{F = v'}$ — $\boxed{B}$ $\Rightarrow F' = v''$

$$\boxed{y_1 F' + (2y_1' + p(t) y_1) F = 0} \quad \text{—} (A)$$

First solve (A) for F
then solve (B) for V
then solve (C) for y_2

"Note that can be solved using B^* since it is 1^{st} order linear"

Exp Given $y_1(x) = \frac{1}{x}$ is a solution for the DE

Euler DE

$$x^2 y'' + 3x y' + y = 0 \quad , \quad x > 0$$

Use ROM to find a second independent solution.

- We write the DE in the form of ② ⇒

$$y'' + \frac{3}{x} y' + \frac{1}{x^2} y = 0 \qquad \Rightarrow p(x) = \frac{3}{x}$$

- $y_1(x) = \frac{1}{x} \quad \Rightarrow \quad y_1' = -\frac{1}{x^2}$

- First solve Ⓐ ⇒ $y_1 F' + (2y_1' + p(x)\, y_1) F = 0$

$$\frac{1}{x} F' + \left(\frac{-2}{x^2} + \frac{3}{x}\frac{1}{x}\right) F = 0$$

$$\frac{1}{x} F' + \frac{1}{x^2} F = 0 \qquad \Rightarrow F' + \frac{1}{x} F = 0$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = e^{\ln x} = x$$

$$F(x) = \frac{1}{\mu}\left[\int \mu g \, dx + c\right] = \frac{1}{x}\left[\int x(0)\, dx + c\right] = \frac{c}{x}$$

- Then solve Ⓑ ⇒ $F = v'$

$$\int \frac{c}{x} = \int v' \qquad \Rightarrow v = c \ln x + d$$

- Then solve Ⓐ ⇒ $y_2(x) = v(x)\, y_1(x)$

gen. sol.

$$= (c \ln x + d)\frac{1}{x}$$

$$= c\left(\frac{\ln x}{x}\right) + d\left(\frac{1}{x}\right)$$

$\frac{\ln x}{x} \to y_2$, $\frac{1}{x} \to y_1$

$$y_2(x) = \frac{\ln x}{x}$$

Exp Use ROM to show that if $y_1(t)$ is solution to the DE ②:

$$y'' + p(t)y' + q(t)y = 0$$

then the 2^{nd} independent solution is given by

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

- First solve Ⓐ for F: $y_1 F' + (2y_1' + p(t)y_1)F = 0$

$$\frac{F'}{F} + \left(2\frac{y_1'}{y_1} + p(t)\right) = 0 \quad \Rightarrow \int \frac{F'}{F} = \int -\left(2\frac{y_1'}{y_1} + p(t)\right)$$

$$\ln|F| = -2\ln|y_1| - \int p(t)\,dt + d$$

$$|F| = e^{\ln|y_1|^{-2}} \, e^{-\int p(t)dt} \, e^{d}$$

$$F = \pm e^{d} \frac{1}{y_1^2} e^{-\int p(t)dt} = \frac{c\, e^{-\int p(t)dt}}{y_1^2(t)}$$

$(c = \pm e^{d})$

- Then solve Ⓑ for $V \Rightarrow V' = F$

$$V = \int F\,dt = \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

- Then solve Ⓒ for $y_2 \Rightarrow y_2(t) = y_1(t)\, V(t)$

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\, dt$$

Remark: $\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2} \Rightarrow \frac{y_2}{y_1} = \int \frac{W}{y_1^2}\, dt \Rightarrow y_2 = y_1 \int \frac{W}{y_1^2}\, dt$

Exp Find second independent solution for the DE

$$2t^2 y'' + 3t y' - y = 0, \quad t > 0$$ (Euler DE)

if $y_1(t) = \frac{1}{t}$ is solution.

First we write the DE in the form of ②

$$y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0 \quad \Rightarrow p(t) = \frac{3}{2t}$$

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt} = c\, e^{-\int \frac{3}{2}\frac{1}{t}\,dt} = c\, e^{-\frac{3}{2}\ln|t|}$$

$$= c\, e^{-\frac{3}{2}\ln t} = c\, t^{-\frac{3}{2}}$$

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2)(t)}{y_1^2(t)}\,dt$$

$$= \frac{1}{t} \int \frac{c\, t^{-\frac{3}{2}}}{\frac{1}{t^2}}\,dt = \frac{1}{t} \int c\, t^{2-\frac{3}{2}}\,dt$$

$$= c\,\frac{1}{t} \int t^{\frac{1}{2}}\,dt = \frac{c}{t}\left[\frac{2}{3} t^{\frac{3}{2}} + d\right]$$

$$= c_1 \sqrt{t} + c_2 \frac{1}{t}$$

$\sqrt{t} \to y_2(t)$, $\quad \frac{1}{t} \to y_1(t)$

$c_1 = \frac{2}{3} c$

$c_2 = cd$

Exp Find y_2 if $y_1 = t^3$ solves $t^2 y'' - 6y = 0$ (Euler DE)