Solving Linear Nonhomogenous DE's of order 2

We will learn two methods to solve 2^{nd} order linear nonhomogenous DE's :

A (section 3.5): The Method of Undetermined Coefficients

B (Section 3.6): The Variation of Parameter Method "More General" $y'' + p(t)y' + q(t)y = g(t)$

A 3.5 The Method of Undetermined Coefficients

We use this method solve 2^{nd} order linear nonhomogenous DE's of the form:

$$\boxed{a y'' + b y' + c y = g(t)} \quad (1)$$

where $g(t)$ is one of the following functions:

1. exponential
2. polynomial
3. Sin or Cos
4. multiple or addition of 1, 2, 3

and "a, b, c constant".

Remark : In section 3.6 we use The Variation of Parameter Method to solve 2^{nd} order linear nonhomogenous DE's of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where $g(t)$ is other than 1, 2, 3, 4 and $p(t)$, $q(t)$ are functions.

Question: How do we use The Method of Undetermined Coefficients to solve the DE ①:

$$\boxed{a y'' + b y' + c y = g(t)}$$

Answer: The gen. sol. of ① is

$$\boxed{y(t) = y_h(t) + y_p(t)}$$

where

$y_h(t)$: is the homogenous solution obtained by solving the corresponding homogenous DE of ①

$$a y'' + b y' + c y = 0 \quad \text{using } \underline{\text{Ch. Eq}}$$

$ar^2 + br + c = 0$

Find r_1 and r_2

Find y_1 and y_2

$$\boxed{y_h(t) = c_1 y_1(t) + c_2 y_2(t)}$$

$y_p(t)$: is the particular solution which depends on the form of $g(t)$:

1. If $g(t) = c e^{rt}$ then we let $y_p(t) = A e^{rt}$
   Then substitute y_p, y_p', y_p'' in ① to find A

2. If $g(t) = c_n t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0$ then we let $y_p(t) = A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$

   Then substitute y_p, y_p', y_p'' in ① to find the constants $A_n, A_{n-1}, \dots, A_1, A_0$

3) If $g(t) = c_1 \sin rt$ or
$g(t) = c_2 \cos rt$ or
$g(t) = c_1 \sin rt + c_2 \cos rt$ then we let

$$y_p(t) = A_1 \sin rt + A_2 \cos rt$$

Then substitute y_p, y_p', y_p'' in ① to find A_1, A_2

Remark* • The form of the particular solution $y_p(t)$ must be independent of the form of the homogenous solution $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

• If y_p is part of y_h then we multiply y_p by t or t^2 or t^3 ... depending on the case.

Exp Solve the following DE's:

① $y'' - 5y' + 6y = 3e^{4t}$

non hom. ⇒ we can apply 3.5

gen. sol. $y(t) = y_h(t) + y_p(t)$

• To find $y_h(t)$ ⇒ we solve $y'' - 5y' + 6y = 0$

$$r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r_1 = 2 , \quad r_2 = 3$$
$$y_1(t) = e^{2t} , \quad y_2(t) = e^{3t}$$

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t) \Leftarrow$$
$$= c_1 e^{2t} + c_2 e^{3t}$$

- To find $y_p(t) \Rightarrow$ Let $y_p(t) = A e^{4t}$ → Indep. from y_1 and y_2 R* ✓

$$y_p' = 4A e^{4t}$$
$$y_p'' = 16A e^{4t}$$

substitute y_p, y_p', y_p'' in the nonhomogenous DE to find A:

$$y_p'' - 5y_p' + 6y_p = 3e^{4t}$$

$$16A e^{4t} - 5(4A e^{4t}) + 6(A e^{4t}) = 3e^{4t}$$

$$16A - 20A + 6A = 3$$

$$2A = 3 \Rightarrow A = \frac{3}{2}$$

$$\Rightarrow \boxed{y_p(t) = \frac{3}{2} e^{4t}}$$

gen. sol. $\Rightarrow y(t) = y_h(t) + y_p(t)$

$$= c_1 y_1 + c_2 y_2 + y_p$$

$$= c_1 e^{2t} + c_2 e^{3t} + \frac{3}{2} e^{4t}$$

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(2) $y'' - 5y' + 6y = 10 e^{3x}$ nonhomo. $\Rightarrow$ we can apply 3.5

gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$= c_1 e^{2x} + c_2 e^{3x} + y_p(x)$$

$y_p(x) = xA e^{3x}$ R* ✓

$$y_p' = 3Ax e^{3x} + A e^{3x}$$
$$y_p'' = 3A e^{3x} + 9Ax e^{3x} + 3A e^{3x}$$
$$= 6A e^{3x} + 9Ax e^{3x}$$

$\Rightarrow$ To find A we substitute y_p, y_p', y_p'' in the nonhomogenuous DE

$$y_p'' - 5y_p' + 6y_p = 10e^{3x}$$

$$6Ae^{3x} + 9Axe^{3x} - 5(3Axe^{3x} + Ae^{3x}) + 6(xAe^{3x}) = 10e^{3x}$$

$$6A - 5A = 10 \qquad\qquad 9A - 15A + 6A = 0$$

$$A = 10 \qquad\qquad 0 = 0$$

Hence, $y_p(x) = 10xe^{3x}$ and the gen. sol. becomes

$$y(x) = c_1e^{2x} + c_2e^{3x} + 10xe^{3x}$$

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③ $y'' - 5y' + 6y = 18x^2$ nonhomo. $\Rightarrow$ we can apply 3.5

gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1y_1(x) + c_2y_2(x) + y_p(x)$$

$$= c_1e^{2x} + c_2e^{3x} + y_p(x)$$

$y_p(x) = Ax^2 + Bx + C$ R* ✓

$y_p' = 2Ax + B$

$y_p'' = 2A$

substitute y_p, y_p', y_p'' in the nonhomo. DE

$$y_p'' - 5y_p' + 6y_p = 18x^2$$

$$2A - 5(2Ax + B) + 6(Ax^2 + Bx + C) = 18x^2$$

$6A = 18 \Rightarrow A = 3$

$-10A + 6B = 0 \Rightarrow 6B = 30 \Rightarrow B = 5$

$2A - 5B + 6C = 0 \Rightarrow 6 - 25 + 6C = 0 \Rightarrow C = \frac{19}{6}$

Hence, $y_p(x) = Ax^2 + Bx + C$
$= 3x^2 + 5x + \frac{19}{6}$

and the gen. sol. $y(x) = c_1 e^{2x} + c_2 e^{3x} + 3x^2 + 5x + \frac{19}{6}$

Exp Find the particular solution of the following DE's:

① $y'' + y' = 10t^2$

non homo. $\Rightarrow$ we can apply 3.5

First we find $y_h(t) \Rightarrow y'' + y' = 0$

$r^2 + r = 0$
$r(r+1) = 0$
$r_1 = 0, \quad r_2 = -1$
$y_1(t) = 1, \quad y_2(t) = e^{-t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$
$= c_1 + c_2 e^{-t}$

$y_p(t) = (At^2 + Bt + c)t$ R* ✓
$= At^3 + Bt^2 + ct$

$y_p'(t) = 3At^2 + 2Bt + c$
$y_p''(t) = 6At + 2B$

$\Rightarrow$ substitute $y_p, y_p', y_p'' \Rightarrow$

$y_p'' + y_p' = 10t^2$

$6At + 2B + 3At^2 + 2Bt + C = 10t^2$

$3A = 10 \Rightarrow A = \frac{10}{3}$

$6A + 2B = 0 \Rightarrow 20 + 2B = 0 \Rightarrow B = -10$

$2B + C = 0 \Rightarrow -20 + C = 0 \Rightarrow C = 20$

$y_p(t) = At^3 + Bt^2 + ct$
$= \frac{10}{3}t^3 - 10t^2 + 20t$

② $y'' + y' - 6y = 10\cos 3x$

nonhomo. $\Rightarrow$ we can apply 3.5

First we find $y_h(x) \Rightarrow$ $y'' + y' - 6y = 0$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r_1 = -3, \quad r_2 = 2$$

$$y_1(x) = e^{-3x}, \quad y_2(x) = e^{2x}$$

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^{-3x} + c_2 e^{2x}$$

$y_p(x) = A\cos 3x + B\sin 3x$ (R*)

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

$\Rightarrow$ substitute $\Rightarrow$

$$y_p'' + y_p' - 6y_p = 10\cos 3x$$

$$-9A\cos 3x - 9B\sin 3x - 3A\sin 3x + 3B\cos 3x - 6A\cos 3x - 6B\sin 3x = 10\cos 3x$$

$$-9A + 3B - 6A = 10 \Rightarrow \boxed{3B - 15A = 10}$$

$$-9B - 3A - 6B = 0 \Rightarrow \boxed{15B + 3A = 0}$$

$$A = -\frac{50}{78}, \quad B = \frac{10}{78}$$

$$y_p(x) = A\cos 3x + B\sin 3x = -\frac{50}{78}\cos 3x + \frac{10}{78}\sin 3x$$

Exp Find y_p "Don't Evaluate Coefficients"

① $y'' + y = 5\sin x$

$y_h(x)$: $\Rightarrow r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \Rightarrow \lambda = 0$, $\mu = 1$

$$y_1(x) = e^{\lambda x}\cos \mu x = \cos x$$
$$y_2(x) = e^{\lambda x}\sin \mu x = \sin x$$

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 \cos x + c_2 \sin x$$

$$y_p(x) = (A\sin x + B\cos x)x$$
$$= Ax\sin x + Bx\cos x$$

R*

② $y'' + y = 5x\sin x$

$$y_h(x) = c_1\cos x + c_2 \sin x$$

$$y_p(x) = (Ax + B)(C\cos x + D\sin x)x$$
$$= (Ax^2 + Bx)(C\cos x + D\sin x)$$

R*

③ $y'' + y = \sin 5x$

$$y_h(x) = c_1\cos x + c_2\sin x$$

$$y_p(x) = A\sin 5x + B\cos 5x$$

R*

④ $y'' - 2y' + y = 2e^{t} + 3$

$y_h(t) \Rightarrow \quad y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r_1 = r_2 = r = 1 \quad \Rightarrow y_1(t) = e^{t} \;,\; y_2(t) = te^{t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

$= c_1 e^{t} + c_2 t e^{t}$

$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$

$= A e^{t} t^{2} + B$ R*

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⑤ $y'' - y' = 2e^{2t} + 3t$

$y_h(t) \Rightarrow \quad y'' - y' = 0$

$r^2 - r = 0$

$r(r-1) = 0$

$r_1 = 0 \;,\; r_2 = 1$

$y_1(t) = 1 \quad , \; y_2(t) = e^{t}$

$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

$= c_1 + c_2 e^{t}$

$y_p(t) = y_{p_1}(t) + y_{p_2}(t)$

$= A e^{2t} + (Bt + C)\, t$

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⑥ $y'' - y = x^2 e^x$

$y_h(x) \Rightarrow y'' - y = 0$

$r^2 - 1 = 0$

$(r-1)(r+1) = 0$

$r_1 = 1 \,,\; r_2 = -1$

$y_1(x) = e^x \,,\quad y_2(x) = e^{-x}$

$$\Rightarrow y_h(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 e^x + c_2 e^{-x}$$

$y_p(x) = (Ax^2 + Bx + C)\, e^x\, x$ R* ✓

⑦ $y'' = 3x^2$

$y_h(x) \Rightarrow y'' = 0$

$r^2 = 0$

$r_1 = r_2 = 0 \quad \Rightarrow y_1(x) = 1 \,,\; y_2(x) = x$

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

$= c_1 + c_2 x$

$y_p(x) = (Ax^2 + Bx + C)\, x^2$ R* ✓

$= Ax^4 + Bx^3 + Cx^2$

Note that we can solve Exp ⑦ as follows:

$y'' = 3x^2 \quad \Rightarrow \quad y' = x^3 + c_2$

The gen. sol. is $y(x) = \dfrac{x^4}{4} + c_2 x + c_1 = y_h + y_p$

we can find A, B, C

and conclude that $A = \frac{1}{4}$, $B = 0$, $C = 0$

Th Assume Y_1 and Y_2 are solution for the nonhomogenous DE

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad (1)$$

Then $Y_1 - Y_2$ is solution for the homogenous DE

$$\boxed{y'' + p(t)y' + q(t)y = 0} \quad (2)$$

Furthermore, if y_1 and y_2 form Fundemental set of solutions for the homogenous DE (2), then $\exists$ constants c_1 and c_2 s.t

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

Proof Since Y_1 and Y_2 solutions for (1) $\Rightarrow$

$$Y_1'' + p(t)\,Y_1' + q(t)\,Y_1 = g(t) \quad \rightarrow A$$
$$Y_2'' + p(t)\,Y_2' + q(t)\,Y_2 = g(t) \quad \rightarrow B$$

$A - B \Rightarrow$

$$(Y_1 - Y_2)'' + p(t)(Y_1 - Y_2)' + q(t)(Y_1 - Y_2) = 0$$

Thus, $Y_1 - Y_2$ is solution for (2).

Remark (A) If Y_1 and Y_2 solutions for (1) then $cY_1 - cY_2$ is solution for (2)

The second part of the proof follows trivially $\Rightarrow$ since if y_1 and y_2 are indep. $\Rightarrow \exists$ c_1 and c_2 s.t the linear combination is solution so $c_1 y_1 + c_2 y_2$ is sol. for (2) but $Y_1 - Y_2$ is solution for (2) $\Rightarrow Y_1 - Y_2 = c_1 y_1 + c_2 y_2$

by Principle of Superposition Theorem