3.6 Variation of Parameters Method

Recall that we can find the particular solution $y_p(t)$ for 2^{nd} order linear homogenous DE:

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad * \quad \text{using}$$

[A] the method of undetermined coefficients (3.5) when $p(t)$ and $q(t)$ are constants and $g(t)$ is sin / cos / poly. / exp

[B] the method of Variation of Parameters when $p(t)$ and $q(t)$ are other than constants and $g(t)$ is other than sin / cos / poly. / exp

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Th 3.6.1. Assume $p(t), q(t), g(t)$ are cont. functions on an open interval I for the nonhomogenous DE $*$.

- If y_1 and y_2 form fundemental solutions for the corresponding homogenous DE:

$$y'' + p(t)y' + q(t)y = 0$$

then the particular solution $y_p(t)$ is given by

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

where

$$v_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}\,dt \quad \text{and} \quad v_2(t) = \int \frac{y_1(t)g(t)\,dt}{W(y_1, y_2)(t)}$$

- Furthermore, the gen. sol. of the nonhomogenous DE $*$ is given by

$$y(t) = y_h(t) + y_p(t)$$
$$= c_1 y_1(t) + c_2 y_2(t) + v_1(t)y_1(t) + v_2(t)y_2(t)$$

Exp solve the DE : $y'' - 4y' + 4y = \dfrac{e^{2x}}{x}$, $x > 0$

The gen. sol. is $y(x) = y_h(x) + y_p(x)$

nonhomogenous
$g(x) = \frac{1}{x} e^{2x}$
3.6 ✓

• $y_h(x)$: $y'' - 4y' + 4y = 0$

$$r^2 - 4r + 4 = 0$$
$$(r-2)(r-2) = 0$$
$$r_1 = r_2 = 2 \quad \Rightarrow \quad y_1(x) = e^{2x} , \quad y_2(x) = x e^{2x}$$

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$
$$= c_1 e^{2x} + c_2 x e^{2x}$$

• $y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = \cancel{2x e^{4x}} + e^{4x} - \cancel{2x e^{4x}} = e^{4x}$$

$$v_1(x) = -\int \frac{y_2 g}{W} dx = -\int \frac{x e^{2x} \frac{1}{x} e^{2x}}{e^{4x}} dx = -\int dx = -x + K_1$$

$$v_2(x) = \int \frac{y_1 g}{W} dx = \int \frac{e^{2x} \frac{1}{x} e^{2x}}{e^{4x}} dx = \int \frac{dx}{x} = \ln x + K_2$$

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$
$$= (-x + K_1) e^{2x} + (\ln x + K_2) x e^{2x}$$

• gen. sol. $y(x) = y_h(x) + y_p(x)$

$$= c_1 e^{2x} + c_2 x e^{2x} + (-x + K_1) e^{2x} + (\ln x + K_2) x e^{2x}$$

$d_1 = c_1 + K_1$

$d_2 = c_2 + K_2 - 1$

$$= d_1 e^{2x} + d_2 x e^{2x} + x \ln x \, e^{2x} \rightarrow y_p$$

Exp Find $y_p(t)$ for the DE :

$$t^2 y'' - 3t y' + 3y = 12 t^4 \quad , t > 0$$

nonhomogenous

3.6

$g(t) = 12t^2$

- $y_h(t) \Rightarrow \quad t^2 y'' - 3t y' + 3y = 0$

Euler DE with $\alpha = -3$ and $\beta = 3$

$$r^2 + (\alpha - 1) r + \beta = 0$$
$$r^2 - 4r + 3 = 0$$
$$(r-3)(r-1) = 0$$
$$r_1 = 3 \; , \; r_2 = 1 \quad \Rightarrow y_1(t) = t^3 \; , \; y_2(t) = t$$

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$= c_1 t^3 + c_2 t$$

- $W(t^3, t)(t) = \begin{vmatrix} t^3 & t \\ 3t^2 & 1 \end{vmatrix} = t^3 - 3t^3 = -2t^3$

- $y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$

$$v_1(t) = -\int \frac{y_2 g}{W} dt = -\int \frac{t(12t^2)}{-2t^3} dt = \int 6 dt = 6t + k_1$$

$$v_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{t^3 (12t^2)}{-2t^3} dt = -\int 6t^2 dt = -2t^3 + k_2$$

$$y_p(t) = v_1(t) y_1(t) + v_2(t) y_2(t)$$
$$= (6t + k_1) t^3 + (-2t^3 + k_2) t$$
$$= \underbrace{4t^4}_{\to y_p} + \underbrace{k_1 t^3 + k_2 t}_{\to y_h}$$

We can use 3.5 also

$y_p(t) = (At^2 + Bt + C) t^4$

substitute y_p, y_p', y_p''

above $\Rightarrow$ $A = B \cancel{= C} = 0$

and $C = 4 \Rightarrow y_p(t) = 4t^4$

Exp Solve the DE: $y'' + y = \tan t$

nonhomogeneous
$g(t) = \tan t$
3.6 ✓

• $y_h(t) \Rightarrow y'' + y = 0$
$r^2 + 1 = 0$
$r_{1,2} = \pm i \Rightarrow y_1(t) = \cos t, \quad y_2(t) = \sin t$
$\lambda = 0, \ \mu = 1$

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$
$$= c_1 \cos t + c_2 \sin t$$

• $W(\cos t, \sin t)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t - -\sin^2 t = 1$

• $V_1(t) = -\int \frac{y_2 g}{W} dt = -\int \frac{\sin t \tan t}{1} dt = -\int \frac{\sin^2 t}{\cos t} dt$
$$= -\int \frac{1-\cos^2 t}{\cos t} dt = \int (\cos t - \sec t)\, dt$$
$$= \sin t - \ln|\sec t + \tan t| + K_1$$

• $V_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{\cos t \tan t}{1} dt = \int \sin t \, dt = -\cos t + K_2$

• $y_p(t) = V_1(t)\, y_1(t) + V_2(t)\, y_2(t)$
$$= (\sin t - \ln|\sec t + \tan t| + K_1)\cos t + (-\cos t + K_2)\sin t$$
$$= (-\ln|\sec t + \tan t| + K_1)\cos t + K_2 \sin t$$

• gen. sol. $y(t) = y_h(t) + y_p(t)$
$$= c_1 y_1 + c_2 y_2 + V_1 y_1 + V_2 y_2$$

$d_1 = c_1 + K_1$
$d_2 = c_2 + K_2$

$$y(t) = c_1 \cos t + c_2 \sin t + (-\ln|\sec t + \tan t| + K_1)\cos t + K_2 \sin t$$
$$= d_1 \cos t + d_2 \sin t - \cos t \ln|\sec t + \tan t|$$

Exercises :

Solve the DE's:

A $$x^2 y'' + x y' - y = x \ln x \quad , x>0$$

B $$x^2 y'' - 3x y' + 4y = x^2 , \quad x>0$$

C If $y_1(t) = e^t$ is solution for the DE

$$t y'' - (1+t) y' + y = t^2 e^{2t} , \quad t>0$$

Find $y_p(t)$.