Ch 4: Higher Order linear ODE's

4.1 The n^{th} order linear ODE has the general form:

$$y^{(n)} + p_{n-1}(t)\, y^{(n-1)} + p_{n-2}(t)\, y^{(n-2)} + \cdots + P_1(t)\, y' + P_0(t)\, y = g(t) \quad (1)$$

- The gen. sol. of (1) is

$$y(t) = y_h(t) + y_p(t)$$

$$= c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + y_p(t)$$

- To find the constants $c_1, c_2, \ldots, c_n$ we need n initial conditions:

$$y(t_0) = y_0,\quad y'(t_0) = y_0',\ \ldots,\ y^{(n-1)}(t_0) = y_0^{(n-1)} \quad (2)$$

Remark: The theory for 2^{nd} order linear ODE fits perfectly well with the n^{th} order linear ODE

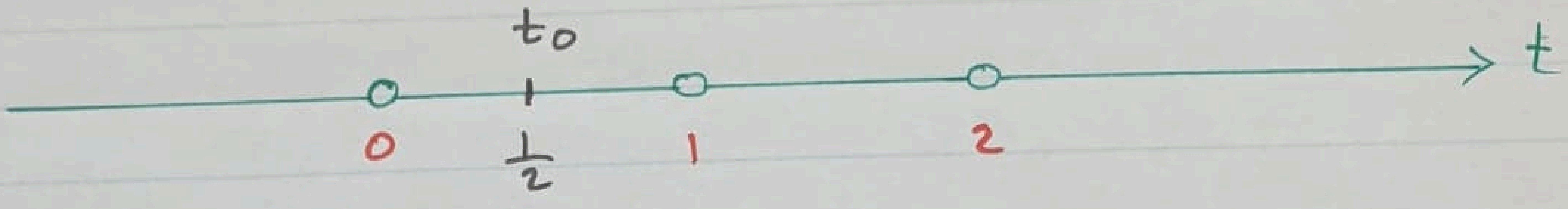
Th Assume $P_0, P_1, \cdots, P_{n-1}$ are cont. on an open interval I containing t_0. Then $\exists$ a unique solution $y(t) = \phi(t)$ satisfying (1) and (2) on I.

Exp Find the largest interval in which the solution of the IVP:

$$\ln|t-1|\, y^{(4)} + (t+1)\, y^{(3)} - y = \cos t$$

$$y(\tfrac{1}{2}) = 5,\ y'(\tfrac{1}{2}) = 3,\ y''(\tfrac{1}{2}) = \tfrac{1}{2},\ y'''(\tfrac{1}{2}) = 4 \text{ is valid}$$

$$y^{(4)} + \frac{t+1}{\ln|t-1|}\,y''' - \frac{1}{\ln|t-1|}\,y = \frac{\cos t}{\ln|t-1|}$$



$P_3(t), P_2(t), P_1(t), P_0(t)$ are all cont. on $\mathbb{R} \setminus \{0,1,2\}$

Since $t_0 = \frac{1}{2} \in (0,1) \Rightarrow I = (0,1)$

Remark. If $y_1, y_2, \dots, y_n$ are solutions for the homogenuous DE:

$$y^{(n)} + P_{n-1}(t)\,y^{(n-1)} + P_{n-2}(t)\,y^{(n-2)} + \dots + P_1(t)\,y' + P_0(t)\,y = 0 \quad \text{--- (3)}$$

with IC's as given in (2) then the gen. sol. is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$$

- To find $c_1, c_2, \dots, c_n \Rightarrow$ we use IC's from (2)

$$c_1 y_1(t_0) + c_2 y_2(t_0) + \dots + c_n y_n(t_0) = y_0$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) + \dots + c_n y_n'(t_0) = y_0'$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \dots + c_n y^{(n-1)}(t_0) = y_0^{(n-1)}$$

- For $c_1, c_2, \dots, c_n$ to make sense we must have $W(y_1, y_2, \dots, y_n)(t_0) \neq 0$, where the

$$W(y_1, y_2, \dots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

- If $w(y_1, y_2, \dots, y_n)(t) \neq 0$ then $y_1, y_2, \dots, y_n$ are Linearly Independent

and since they are solutions for the DE 3 $\Rightarrow$

$\{y_1, y_2, \dots, y_n\}$ form fondemental set of solutions

- If Y_1 and Y_2 are solutions for the nonhomogenuous DE 1 then $Y_1 - Y_2$ is solution for the homogenuous DE 3

Exp show that $\{1, t, t^3\}$ form fondemental set of solutions for the DE: $-t^2 y''' + t y'' = 0$, $t \neq 0$.

- First we show $1, t, t^3$ are solutions $\Rightarrow$

$y_1 = 1 \Rightarrow y_1' = y_1'' = y_1''' = 0 \qquad \Rightarrow -t^2 y_1''' + t y_1'' = 0$

$y_2 = t \Rightarrow y_2' = 1$ and $y_2'' = y_2''' = 0 \qquad \Rightarrow -t^2 y_2''' + t y_2'' = 0$

$y_3 = t^3 \Rightarrow y_3' = 3t^2,\ y_3'' = 6t,\ y_3''' = 6 \Rightarrow$

$$-t^2 y_3''' + t y'' = -t^2(6) + t(6t) = 0$$

- Now we show $1, t, t^3$ are Linearly Independent $\Rightarrow$

$$w(1, t, t^3)(t) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t \neq 0 \text{ since } t \neq 0$$

Hence, y_1, y_2, y_3 are L. Indep. $\Rightarrow$ $\{1, t, t^3\}$ form fondemental set of solutions

Note that $\begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = (1)\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} - (t)\begin{vmatrix} 0 & 3t^2 \\ 0 & 6t \end{vmatrix} + (t^3)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 6t$