4.2 Homogenous LDE with Constant Coefficients (for Higher order)

The solution of this kind of DE's follow similarly to the solution of 2^{nd} OLH DE's with CC.

Exp Find the gen. sol. of

① $y^{(4)} - y = 0$, $y(0) = \frac{7}{2}$, $y'(0) = -4$, $y''(0) = \frac{5}{2}$, $y'''(0) = -2$

ch. Eq $r^4 - 1 = 0$

$(r^2-1)(r^2+1) = 0$

$(r-1)(r+1)(r^2+1) = 0$

$r_1 = 1$, $r_2 = -1$, $r_{3,4} = \pm i$

$\lambda = 0, \mu = 1$

$y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cos x$, $y_4 = \sin x$ $\Rightarrow$ gen. sol. is

$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

To find c_1, c_2, c_3, c_4 we use IC's

$y'(x) = c_1 e^x - c_2 e^{-x} - c_3 \sin x + c_4 \cos x$

$y''(x) = c_1 e^x + c_2 e^{-x} - c_3 \cos x - c_4 \sin x$

$y'''(x) = c_1 e^x - c_2 e^{-x} + c_3 \sin x - c_4 \cos x$

$$\left.\begin{aligned} \tfrac{7}{2} &= c_1 + c_2 + c_3 \\ -4 &= c_1 - c_2 + c_4 \\ \tfrac{5}{2} &= c_1 + c_2 - c_3 \\ -2 &= c_1 - c_2 - c_4 \end{aligned}\right\} \Rightarrow \quad c_1 = 0, \; c_2 = 3, \; c_3 = \frac{1}{2}, \; c_4 = -1$$

Hence, the gen. sol. becomes

$y(x) = 3e^{-x} + \frac{1}{2}\cos x - \sin x$

② $y^{(4)} + y''' - 7y'' - y' + 6y = 0$

Ch. Eq. $r^4 + r^3 - 7r^2 - r + 6 = 0$

$\pm 1, \pm 2$
$\pm 3, \pm 6$

$(r^2 - 1)(r^2 + r - 6) = 0$

- $r_1 = 1$ is root $\Rightarrow$ $r - 1$ is factor
- $r_2 = -1$ is root $\Rightarrow$ $r_2 + 1$ is factor
- Hence, $r^2 - 1$ is factor

$(r-1)(r+1)(r-2)(r+3) = 0$

$$
\begin{array}{r}
r^2 + r - 6 \\
r^2 - 1 \,\overline{\big)\, \cancel{r^4} + r^3 - 7r^2 - r + 6} \\
-\cancel{r^4} + r^2 \\
\hline
\cancel{r^3} - 6r^2 - r + 6 \\
-\cancel{r^3} + r \\
\hline
-6r^2 + 6 \\
+6r^2 \mp 6 \\
\hline
0
\end{array}
$$

$r_1 = 1 \Rightarrow y_1 = e^{x}$
$r_2 = -1 \Rightarrow y_2 = e^{-x}$
$r_3 = 2 \Rightarrow y_3 = e^{2x}$
$r_4 = -3 \Rightarrow y_4 = e^{-3x}$

Hence, the gen. sol. is

$$y(x) = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x}$$

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③ $y^{(iv)} + 2y'' + y = 0$

Ch. Eq. $r^4 + 2r^2 + 1 = 0$

$(r^2+1)(r^2+1) = 0$

$r_{1,2} = \pm i \ , \ r_{3,4} = \pm i$ $\qquad \lambda = 0, \ \mu = 1$

$y_1 = e^{\lambda x} \cos \mu x = \cos x$
$y_2 = e^{\lambda x} \sin \mu x = \sin x$

$y_3 = x \cos x$
$y_4 = x \sin x$ $\qquad$ Hence, the gen. sol. is

$$y(x) = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

(4) $y^{(4)} + 2y''' - 13y'' - 14y' + 24y = 0$

$y(0) = 1,\ y'(0) = -1,\ y''(0) = 0,\ y'''(0) = -1$

Ch. Eq $\quad r^4 + 2r^3 - 13r^2 - 14r + 24 = 0$

$r_1 = 1$ is root $\Rightarrow r-1$ is factor
$r_2 = -2$ is root $\Rightarrow r+2$ is factor
$\Rightarrow (r-1)(r+2)$ is factor
r^2+r-2

$$
\begin{array}{r}
r^2 + r - 12 \\
r^2+r-2 \,\big)\, r^4 + 2r^3 - 13r^2 - 14r + 24 \\
-r^4 \mp r^3 \pm 2r^2 \\
\hline
r^3 - 11r^2 - 14r + 24 \\
-r^3 \mp r^2 \pm 2r \\
\hline
-12r^2 - 12r + 24 \\
+12r^2 + 12r \mp 24 \\
\hline
0
\end{array}
$$

$(r^2+r-2)(r^2+r-12) = 0$
$(r-1)(r+2)(r+4)(r-3) = 0$

$r_1 = 1 \Rightarrow y_1 = e^{t}$
$r_2 = -2 \Rightarrow y_2 = e^{-2t}$
$r_3 = -4 \Rightarrow y_3 = e^{-4t}$
$r_4 = 3 \Rightarrow y_4 = e^{3t}$

gen. sol. $\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{-4t} + c_4 e^{3t}$

$y'(t) = c_1 e^{t} - 2c_2 e^{-2t} - 4c_3 e^{-4t} + 3c_4 e^{3t}$
$y''(t) = c_1 e^{t} + 4c_2 e^{-2t} + 16c_3 e^{-4t} + 9c_4 e^{3t}$
$y'''(t) = c_1 e^{t} - 8c_2 e^{-2t} - 48c_3 e^{-4t} + 27c_4 e^{3t}$

$$
\left.\begin{array}{l}
1 = c_1 + c_2 + c_3 + c_4 \\
-1 = c_1 - 2c_2 - 4c_3 + 3c_4 \\
0 = c_1 + 4c_2 + 16c_3 + 9c_4 \\
-1 = c_1 - 8c_2 - 48c_3 + 27c_4
\end{array}\right\} \Rightarrow
\begin{array}{l}
c_1 \approx 0.4 \\
c_2 \approx 0.9 \\
c_3 \approx -0.2 \\
c_4 \approx -0.1
\end{array}
$$

$y(t) = 0.4e^{t} + 0.9e^{-2t} - 0.2e^{-4t} - 0.1e^{3t}$

(5) $y^{(4)} + y'' = 0$

Ch. Eq. $r^4 + r^2 = 0 \Rightarrow r^2(r^2+1) = 0 \Rightarrow r_1 = r_2 = 0$

$r_{3,4} = \pm i$

$\lambda = 0,\ \mu = 1$

$$\left.\begin{array}{l} y_1 = e^{rt} = e^{0t} = 1 \\ y_2 = t\,y_1 = t(1) = t \\ y_3 = e^{\lambda t}\cos\mu t = \cos t \\ y_4 = e^{\lambda t}\sin\mu t = \sin t \end{array}\right\} \Rightarrow \text{gen. sol. is}$$

$$y(t) = c_1 + c_2 t + c_3\cos t + c_4 \sin t$$

(6) $y^{(iv)} - y''' - y'' + y' = 0$

Ch. Eq. $r^4 - r^3 - r^2 + r = 0$

$r(r^3 - r^2 - r + 1) = 0$

$r[r^2(r-1) - (r-1)] = 0$

$r(r-1)(r^2-1) = 0$

$r(r-1)(r-1)(r+1) = 0$

$r_1 = 0 \Rightarrow y_1 = 1$

$r_2 = r_3 = 1 \Rightarrow y_2 = e^t$

$\Rightarrow y_3 = t e^t$

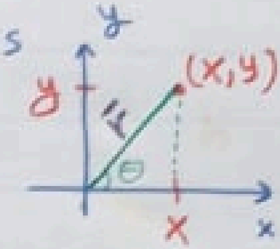
$r_4 = -1 \Rightarrow y_4 = e^{-t}$

gen. sol. $\Rightarrow y(t) = c_1 + c_2 e^t + c_3 t e^t + c_4 e^{-t}$

Exp Express the following complex numbers in the form Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

① $-1+\sqrt{3}\,i$

Recall that the length of the complex number $z = x+iy$ is $\bar{r} = |z| = \sqrt{x^2+y^2}$



$$\bar{r} = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$x=-1$, $y=\sqrt{3}$

$x = \bar{r}\cos\theta$ and $y = \bar{r}\sin\theta$

$-1 = 2\cos\theta$ and $\sqrt{3} = 2\sin\theta$

$\frac{-1}{2} = \cos\theta$ and $\frac{\sqrt{3}}{2} = \sin\theta$

$$\theta = \frac{2\pi}{3} + 2\pi m$$

$m = 0, \pm1, \pm2, \ldots$

Note that any complex number

$$x+iy = \bar{r}\cos\theta + i\,\bar{r}\sin\theta = \bar{r}(\cos\theta + i\sin\theta) = \bar{r}\,e^{i\theta}$$

Hence, $-1+\sqrt{3}\,i = \bar{r}\,e^{i\theta} = 2\,e^{i\left(\frac{2\pi}{3}+2\pi m\right)}$

$$= 2\left[\cos\left(\frac{2\pi}{3}+2\pi m\right) + i\sin\left(\frac{2\pi}{3}+2\pi m\right)\right]$$

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② $-3 \Rightarrow -3 = -3+0i \Rightarrow x=-3, \ y=0$

$\bar{r} = \sqrt{9+0} = \sqrt{9} = 3$

$x = \bar{r}\cos\theta$	$y = \bar{r}\sin\theta$
$-3 = 3\cos\theta$	$0 = \sin\theta$
$-1 = \cos\theta$	$0 = \sin\theta$

$$\theta = \pi + 2\pi m$$

$m = 0, \pm1, \pm2, \ldots$

$$-3 = \bar{r}\,e^{i\theta} = 3\,e^{i(\pi+2\pi m)}$$

$$= 3\left[\cos(\pi+2\pi m) + i\sin(\pi+2\pi m)\right]$$

Exp Show that $(1+i)^8 = 16$

Exp Solve the DE: $y^{(4)} + y = 0$

ch. Eq. $r^4 + 1 = 0 \Rightarrow r^4 = -1 \Rightarrow r = (-1)^{\frac{1}{4}}$
$= (-1+0i)^{\frac{1}{4}}$

$-1 + 0i = \bar{r}\, e^{i\Theta}$
$= e^{i(\pi + 2\pi m)}$

$x = -1$ and $y = 0$
$\bar{r} = \sqrt{1+0} = 1$
$x = \bar{r}\cos\Theta$ and $y = \bar{r}\sin\Theta$
$-1 = \cos\Theta$ and $0 = \sin\Theta$
$\Theta = \pi + 2\pi m$
$m = 0, \pm 1, \pm 2, \dots$

Hence, $r = (-1+0i)^{\frac{1}{4}}$
$= \left[e^{i(\pi + 2\pi m)}\right]^{\frac{1}{4}}$
$= e^{i\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)}$
$= \cos\left(\frac{\pi}{4} + \frac{\pi m}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)$

The four roots are r_1 when $m=0$, r_2 when $m=1$, r_3 when $m=2$, r_4 $m=3$

$m=0 \Rightarrow r_1 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m=1 \Rightarrow r_2 = e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$m=2 \Rightarrow r_3 = e^{i\left(\frac{\pi}{4}+\pi\right)} = \cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right) = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$m=3 \Rightarrow r_4 = e^{i\left(\frac{\pi}{4}+\frac{3\pi}{2}\right)} = \cos\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{3\pi}{2}\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

The four roots are: $r_{1,4} = \underbrace{\frac{1}{\sqrt{2}}}_{\lambda_1} \pm \underbrace{\frac{1}{\sqrt{2}}}_{\mu_1} i$ and $r_{2,3} = \underbrace{\frac{-1}{\sqrt{2}}}_{\lambda_2} \pm \underbrace{\frac{1}{\sqrt{2}}}_{\mu_2} i$

The gen. sol. is

$$y(t) = c_1 e^{\frac{1}{\sqrt{2}}t}\cos\left(\frac{1}{\sqrt{2}}t\right) + c_2 e^{\frac{1}{\sqrt{2}}t}\sin\left(\frac{1}{\sqrt{2}}t\right) + c_3 e^{-\frac{1}{\sqrt{2}}t}\cos\left(\frac{1}{\sqrt{2}}t\right) + c_4 e^{-\frac{1}{\sqrt{2}}t}\sin\left(\frac{1}{\sqrt{2}}t\right)$$

Exp solve the DE: $y''' + 8y = 0$

solution 1 ch. Eq. $r^3 + 8 = 0$

$r = -2$ is root $\Rightarrow r+2$ is factor

$r^3 + 8 = 0$

$(r+2)(r^2 - 2r + 4) = 0$

$r_1 = -2$ and $r_{2,3} = \dfrac{2 \pm \sqrt{4-16}}{2}$

$= \dfrac{2 \pm \sqrt{12}\, i}{2}$

$= 1 \pm \sqrt{3}\, i$

$$
\begin{array}{r|l}
 & r^2 - 2r + 4 \\
\hline
r+2 & r^3 + 8 \\
 & -r^3 \mp 2r^2 \\
\hline
 & -2r^2 + 8 \\
 & +2r^2 + 4r \\
\hline
 & 4r + 8 \\
 & -4r \mp 8 \\
\hline
 & 0
\end{array}
$$

$y_1 = e^{-2t}$ and $y_2 = e^{t} \cos\sqrt{3}t$ and $y_3 = e^{t} \sin\sqrt{3}t$

gen. sol. $y(t) = c_1 e^{-2t} + c_2 e^{t} \cos\sqrt{3}t + c_3 e^{t} \sin\sqrt{3}t$

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solution 2 $r^3 + 8 = 0 \Rightarrow r^3 = -8 \Rightarrow r = (-8)^{\frac{1}{3}} = (-8 + 0i)^{\frac{1}{3}}$

$r = (-8+0i)^{\frac{1}{3}} = \left(\bar{r}\, e^{i\theta}\right)^{\frac{1}{3}}$

$= \left[8\, e^{i(\pi + 2\pi m)}\right]^{\frac{1}{3}}$

$= 2\, e^{i\left(\frac{\pi}{3} + \frac{2\pi m}{3}\right)}$

$= 2\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{3} m\right) + i \sin\left(\frac{\pi}{3} + \frac{2\pi}{3} m\right)\right]$

$x = -8$ and $y = 0$

$\bar{r} = \sqrt{64 + 0} = 8$

$x = \bar{r} \cos\theta$ and $y = \bar{r} \sin\theta$

$-1 = \cos\theta$ and $0 = \sin\theta$

$\theta = \pi + 2\pi m$

$m = 0, \pm 1, \pm 2, \ldots$

when $m = 0 \Rightarrow r_2 = 2\left[\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right] = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

$m = 1 \Rightarrow r_1 = 2\left[\cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i \sin\pi\right] = 2(-1 + 0i) = -2$

$m = 2 \Rightarrow r_3 = 2\left[\cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

gen. sol. $\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{t} \cos\sqrt{3}t + c_3 e^{t} \sin\sqrt{3}t$