4.3 The Method of Undetermined Coefficients

To solve nonhomogenous linear higher order ODE's

Exp Solve the following DE's:

① $y''' - 3y'' + 3y' - y = 4e^t$

gen. sol. is $y(t) = y_h(t) + y_p(t)$

$y_h(t) \Rightarrow$ Ch. Eq. $\Rightarrow r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

$r_1 = r_2 = r_3 = 1$

$y_1 = e^t$, $y_2 = te^t$ $y_3 = t^2e^t$

$y_h(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

$y_p(t) = A e^t t^3$ (R*) ✓ To find $A \Rightarrow$

$y_p' = At^3e^t + 3At^2e^t$

$y_p'' = At^3e^t + 3t^2Ae^t + 3At^2e^t + 6Ate^t$

$= e^t(At^3 + 6At^2 + 6At)$

$y_p''' = e^t(3At^2 + 12At + 6A) + (At^3 + 6At^2 + 6At)e^t$

Substitute y_p, y_p', y_p'', y_p''' in the nonhomogenous DE above to find $A = \frac{2}{3}$ $\Rightarrow$ $y_p(t) = \frac{2}{3}t^3 e^t$

Hence, the gen. sol. becomes

$y(t) = y_h(t) + y_p(t)$

$= c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t$

(2) $y''' + y'' = 10x^2$, $y(0) = y'(0) = y''(0) = 1$

gen. sol. is $y(x) = y_h(x) + y_p(x)$

$y_h(x) \Rightarrow$ Ch. Eq. $r^3 + r^2 = 0$ $\Rightarrow r^2(r+1) = 0$

$\Rightarrow r_1 = r_2 = 0$, $r_3 = -1$

$y_1 = 1$, $y_2 = x$, $y_3 = e^{-x}$

$$y_h(x) = c_1 + c_2 x + c_3 e^{-x}$$

$y_p(x) = (Ax^2 + Bx + c)\,x^2$ (R*)

$= Ax^4 + Bx^3 + Cx^2$

substitute y_p'' and y_p''' in the nonhomogenous DE to find

$$A = \frac{5}{6}, \quad B = -\frac{10}{3}, \quad C = 10$$

Hence, the gen. sol. becomes:

$y(x) = y_h(x) + y_p(x)$

$y(x) = c_1 + c_2 x + c_3 e^{-x} + \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$

Using the IC's $\Rightarrow$ we find $c_1 = 20$

$c_2 = -18$

$c_3 = -19$

$$y(x) = 20 - 18x - 19e^{-x} + \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$$

(3) $y^{(4)} + 8y'' + 16y = 2\sin t - 3\cos t$

gen. sol. is $y(t) = y_h(t) + y_p(t)$

$y_h(t) \Rightarrow$ ch. Eq. $r^4 + 8r^2 + 16 = 0$

$(r^2+4)(r^2+4) = 0$

$r_{1,2} = \pm 2i \quad , \quad r_{3,4} = \pm 2i \qquad \lambda = 0, \mu = 2$

$y_1 = \cos 2t \; , \; y_2 = \sin 2t \; , \; y_3 = t\cos 2t \; , \; y_4 = t\sin 2t$

$y_h(t) = c_1 \cos 2t + c_2 \sin 2t + c_3 t\cos 2t + c_4 t \sin 2t$

$y_p(t) = A\sin t + B\cos t$ (R*)

substitute $y_p, y_p'', y_p^{(4)}$ in the nonhomogenous DE above

to find $A = \frac{2}{9}$ and $B = -\frac{1}{3}$

Hence, $y(t) = y_h(t) + y_p(t)$

$$y(t) = c_1\cos 2t + c_2 \sin 2t + c_3 t\cos 2t + c_4 t \sin 2t + \frac{2}{9}\sin t - \frac{1}{3}\cos t$$

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(4) $y^{(4)} + 8y'' + 16y = 2\sin 2t - 3\cos 2t$

$y_h(t)$ is as above but $y_p(t) = (A\cos 2t + B\sin 2t)t^2$ (R*)

substitute $y_p, y_p'', y_p^{(4)}$ above to find $A = -\frac{1}{16}$, $B = \frac{3}{32}$

$$y(t) = y_h(t) + y_p(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t - \frac{1}{16}t^2\cos 2t + \frac{3}{32}t^2\sin 2t$$

Exp Find $y_p(h)$ for the DE

$$y''' - 4y' = h + 3\cos h + e^{-2h}$$

$y_h(h) \Rightarrow$ ch. Eq. $r^3 - 4r = 0 \quad \Rightarrow r(r^2-4) = 0$

$r_1 = 0,\ r_2 = 2,\ r_3 = -2$

$$y_1 = 1,\quad y_2 = e^{2h},\quad y_3 = e^{-2h}$$

$$y_h(h) = c_1 + c_2 e^{2h} + c_3 e^{-2h}$$

$y_p(h) = y_{p_1}(h) + y_{p_2}(h) + y_{p_3}(h)$

$y_{p_1}(h) = (Ah + B)h$

$y_{p_2}(h) = C\cos h + D\sin h$

$y_{p_3}(h) = E e^{-2h} h$

$y_p(h) = Ah^2 + Bh + C\cos h + D\sin h + Ehe^{-2h}$

To find $A, B, C, D, E \Rightarrow$

$$y_p' = 2Ah + B - C\sin h + D\cos h - 2Ehe^{-2h} + Ee^{-2h}$$
$$y_p'' = 2A - C\cos h - D\sin h + 4Ehe^{-2h} - 2Ee^{-2h} - 2Ee^{-2h}$$
$$y_p''' = C\sin h - D\cos h - 8Eh e^{-2h} + 4Ee^{-2h} + 8Ee^{-2h}$$

$$y_p''' - 4y_p' = h + 3\cos h + e^{-2h}$$

$$C\sin h - D\cos h - 8Ehe^{-2h} + 12Ee^{-2h} - 4\big(2Ah + B - C\sin h + D\cos h - 2Ehe^{-2h} + Ee^{-2h}\big) = h + 3\cos h + e^{-2h}$$

$-8A = 1 \Rightarrow A = -\frac{1}{8}$

$-4B = 0 \Rightarrow B = 0$

$C + 4C = 0 \Rightarrow C = 0$

$-D - 4D = 3 \Rightarrow D = -\frac{3}{5}$

$-8E + 8E = 0 \Rightarrow 0 = 0$ ✓

$12E - 4E = 1 \Rightarrow E = \frac{1}{8}$

$$\Rightarrow y_p(h) = -\frac{1}{8}h^2 - \frac{3}{5}\sin h + \frac{1}{8}he^{-2h}$$

Exp Find the particular solution $y_p(t)$ for the following DE (Don't Evaluate Coefficients)

$$y^{(iv)} + 2y''' + 2y'' = 3e^{x} + 2xe^{-x} + e^{-x}\sin x$$

$y_h(x) \Rightarrow$ Ch. Eq. $r^4 + 2r^3 + 2r^2 = 0$

$$r^2(r^2 + 2r + 2) = 0$$

$r_1 = r_2 = 0$ and $r_{3,4} = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm \sqrt{4}\, i}{2}$

$= -1 \pm i$ $\quad \lambda = -1$, $\mu = 1$

$y_1 = 1$, $y_2 = x$, $y_3 = e^{-x}\cos x$, $y_4 = e^{-x}\sin x$

$$y_h(x) = c_1 + c_2 x + c_3 e^{-x}\cos x + c_4 e^{-x}\sin x$$

$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + y_{p_3}(x)$ $\quad$ R* ✓

$y_{p_1}(x) = Ae^{x}$ ✓

$y_{p_2}(x) = (Bx + C)e^{-x}$ ✓

$y_{p_3}(x) = (D\cos x + E\sin x)e^{-x}x$ ✓

$$y_p(x) = Ae^{x} + (Bx + C)e^{-x} + xe^{-x}(D\cos x + E\sin x)$$