5.1 Review of Power Series

- In this chapter we will learn how to find power series solution for some 2^{nd} order linear DE's

- The reason for that because some of these DE's could be with non constant coefficients.

Exp Solve the DE: $y'' + y = 0$

Ch. Eq. $r^2 + 1 = 0$

$r_{1,2} = \pm i$, $\lambda = 0$, $\mu = 1$

$y_1(x) = \cos x$ and $y_2(x) = \sin x$

gen. sol. $\Rightarrow$ $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$y(x) = c_1 \cos x + c_2 \sin x$

$$\underbrace{y(x)}_{\text{Power Series Solution}} = c_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + c_2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Fundemental Power Series Solutions about $x_0 = 0$

Remark The Power Series Solution about x_0 (center) for a given DE has the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

Question: Why Power Series Solution?

Answer: Exp Solve the DE: $y'' + xy = 0$

we can not use ch1, nor ch2 (since it is not missing x and y), nor ch3 (since it is not constant coefficients, nor Euler DE, nor ch4 ...

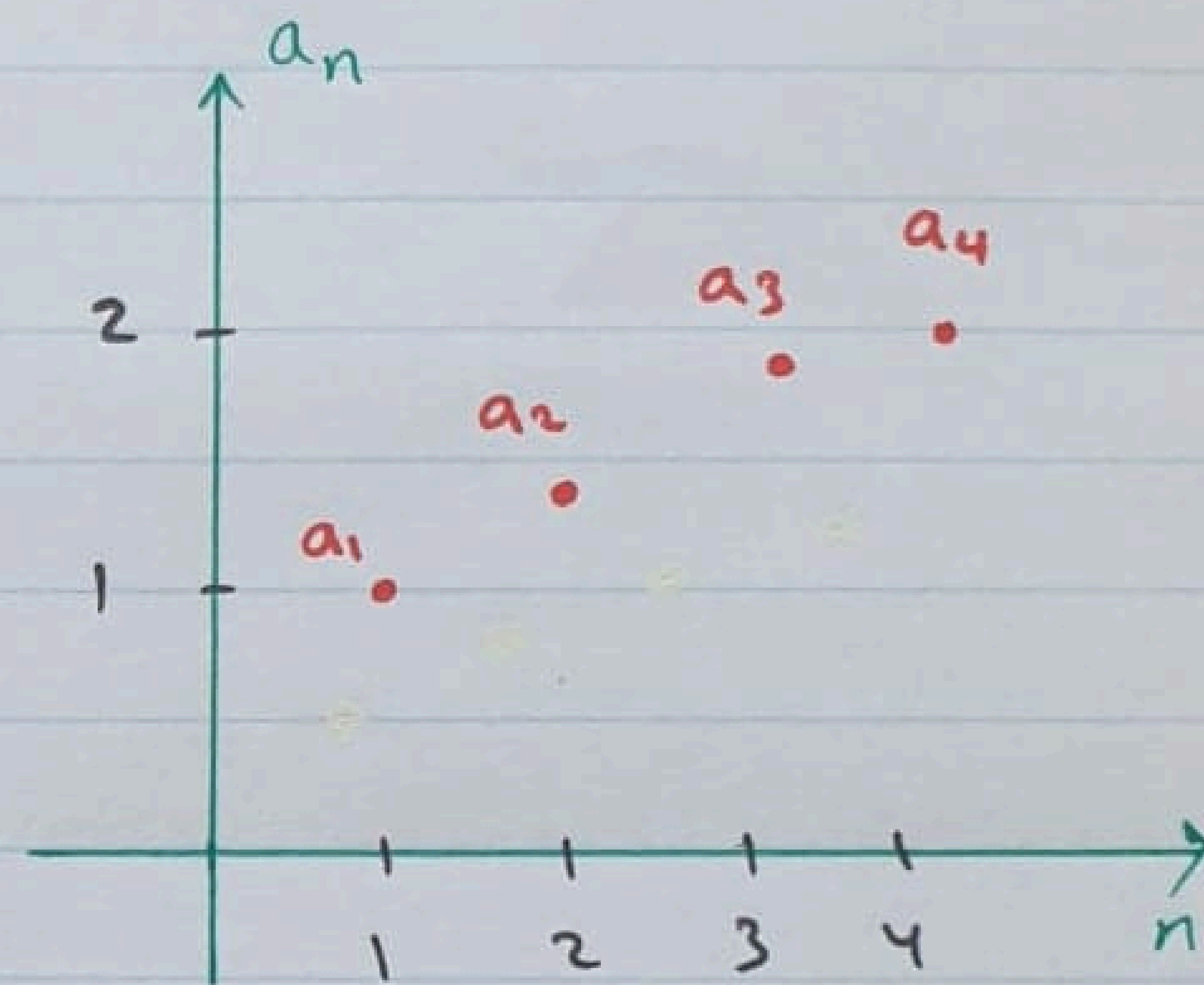
so we need ch5

Review of Sequences:

Exp $a_n = \sqrt{n}$, $n = 1, 2, 3, \dots$

$a_1 = \sqrt{1} = 1$
$a_2 = \sqrt{2}$
$a_3 = \sqrt{3}$
$\vdots$

The sequence diverges since $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{n} = \infty$

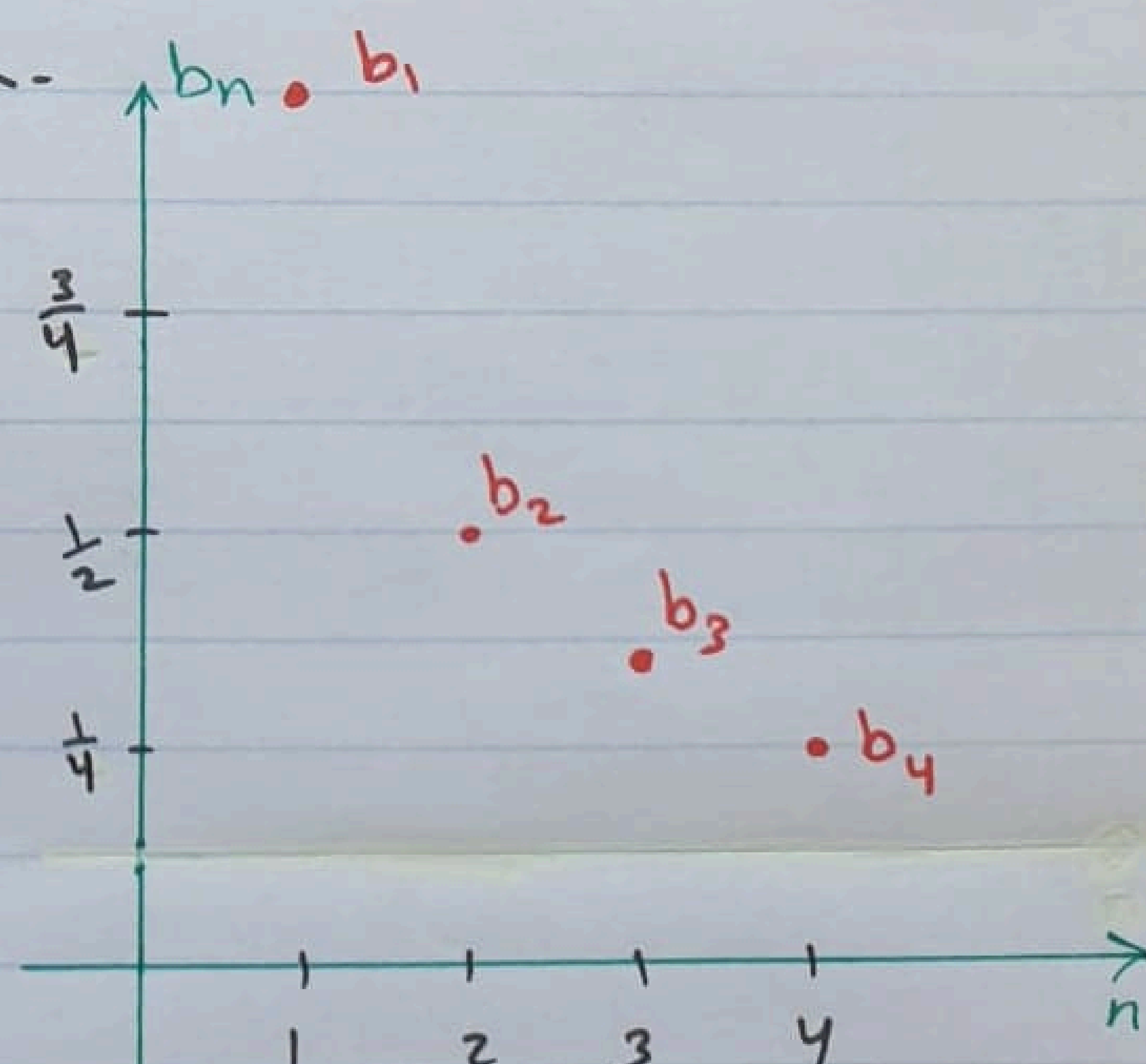


Exp $b_n = \frac{1}{n}$, $n = 1, 2, 3, \dots$

$b_1 = 1$
$b_2 = \frac{1}{2}$
$b_3 = \frac{1}{3}$
$\vdots$

The sequence converges to 0 since

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$



- Recall Taylor Series Expansion for an infinitly many differentiable function $f(x)$ about the point x_0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

- when $x_0 = 0$ Taylor Series is called Maclurine Series

- e^x, $\sin x$, $\cos x$ are examples of analytic functions since they have Taylor Series Expansion everywhere " at any point x_0 "

- $f(x) = \frac{1}{x}$ is analytic everywhere except at $x = 0$

- To solve DE's using the idea of finding power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{about } x_0 \quad \Rightarrow$$

we need to check the convergence of this power series solution $\Rightarrow$ so we may apply Ratio Test (RT) as follows:

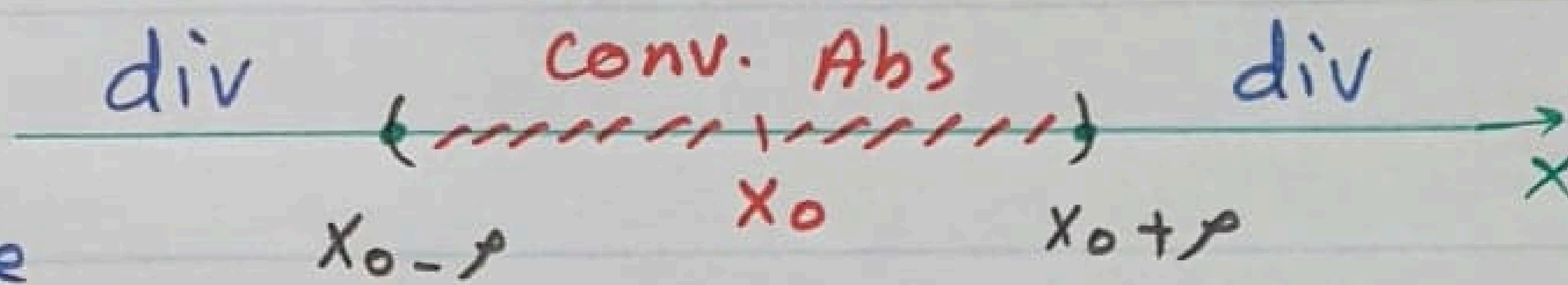
Assume $\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L$, where $b_n = a_n (x-x_0)^n$

1. If $L < 1$, then the power series converges

2. If $L > 1$, then the power series diverges

3. If $L = 1$, then the test fails

The power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ will converge absolutely for every x belongs to the interval $|x - x_0| < \rho$



ρ : Radius of Convergence

IC : Interval of Convergence

we check the endpoints for conditional Convergence.

Exp Find ρ and IC for the following power series:

[1] $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n \quad \Rightarrow x_0 = 2$

Apply RT $\Rightarrow$ $L = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} (n+1)(x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right|$

$$= |x-2| \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = |x-2| (1)$$

$$= |x-2| < 1$$

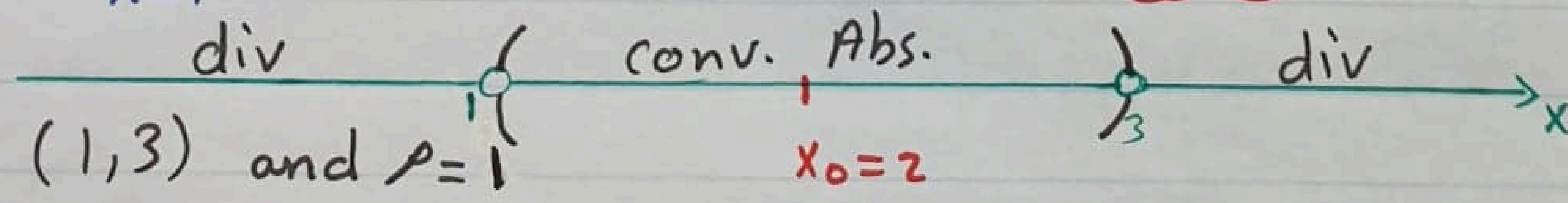
$$-1 < x-2 < 1$$

$$1 < x < 3$$

The power series converges Absolutely on $(1, 3)$

- when $x = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} n (1-2)^n = \sum_{n=1}^{\infty} (-1)^{2n+1} n$
- when $x = 3 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} n (3-2)^n = \sum (-1)^{n+1} n \longrightarrow$

which diverges by n^{th} term test as $\lim_{n\to\infty} n \neq 0$



Hence, $IC = (1,3)$ and $\rho = 1$

$\boxed{2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n} \Rightarrow x_0 = -1$

Apply RT $\Rightarrow$ $\displaystyle L = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(x+1)^n} \right|$

$$= \frac{|x+1|}{2} \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = \frac{|x+1|}{2}(1) < 1$$

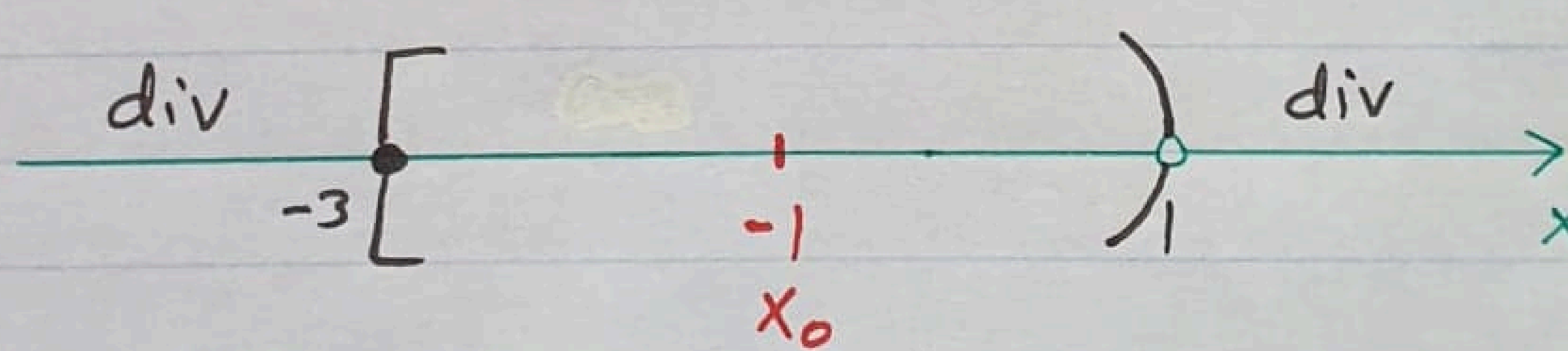
$$|x+1| < 2$$

$$-2 < x+1 < 2$$

$$-3 < x < 1$$

The power series converges Abs. on $(-3, 1)$

- when $x = -3$ $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-3+1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ convergent Alternating Series
- when $x = 1$ $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(1+1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent Harmonic series



Hence, $IC = [-3, 1)$ and $\rho = 2$

$\Rrightarrow$ The power series in this Exp

Converges conditionally at $x = -3$ $\Rightarrow$ This means

The power series converges at $x = -3$ but not Absolutely.

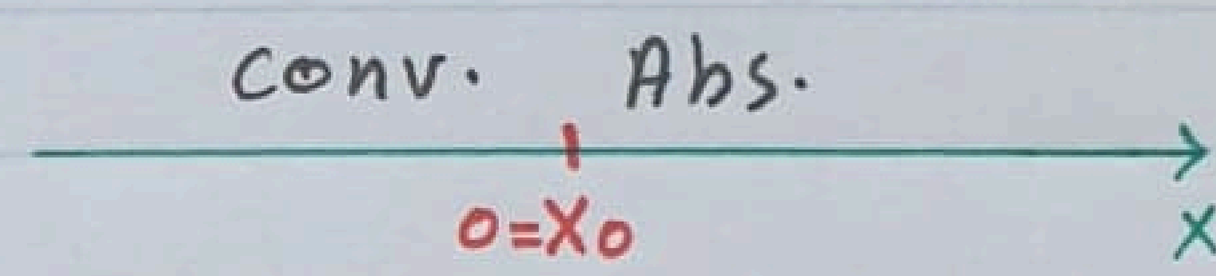
$\boxed{3}$ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\Rightarrow x_0 = 0$

Apply RT $\Rightarrow L = \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$= |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| (0) = 0 < 1$ ✓

Hence, this power series converges Abs. for every x

$IC = \mathbb{R} = (-\infty, \infty)$ with $\rho = \infty$



Note that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ "Maclurine Series of e^x"

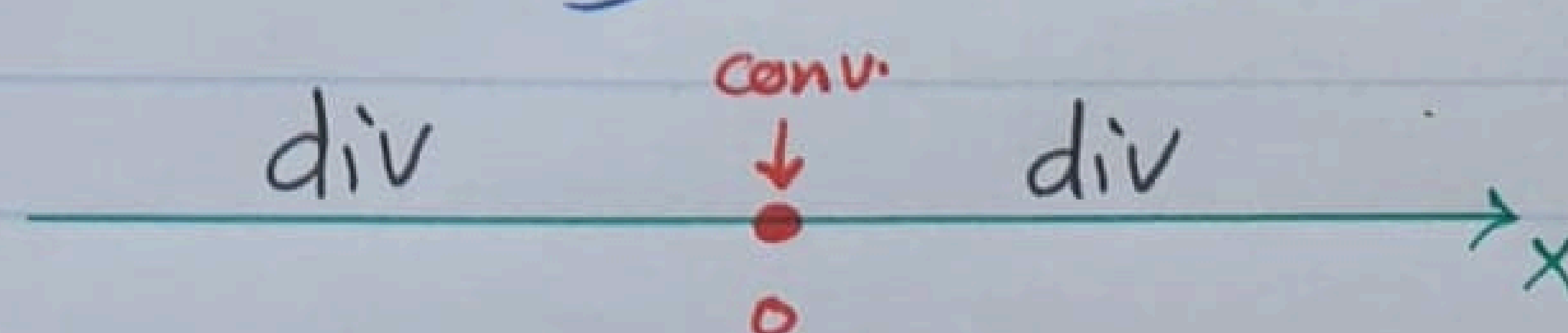
$\boxed{4}$ $\sum_{n=0}^{\infty} n!\, x^n$ $\Rightarrow x_0 = 0$

Apply RT $\Rightarrow \lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$

$= |x| \lim_{n\to\infty} (n+1) = \infty > 1$ if $x \neq 0$
and so it diverges

If $x = 0 \Rightarrow \sum_{n=0}^{\infty} n!\, 0^n = 0 < 1$ and so it converges

Hence, $\sum_{n=0}^{\infty} n!\, x^n$ diverges for every $x \in \mathbb{R} \setminus \{0\}$



$\Rightarrow \rho = 0$ and the power series converges only at $x = 0$

Derivatives of the power series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n (x - x_0)^{n-1} = a_1 + 2 a_2 (x - x_0) + \cdots$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)\, a_n (x - x_0)^{n-2} = 2 a_2 + 3(2)\, a_3 (x - x_0) + \cdots$$

Shifting Index :

It is not important which index we use in the upper and lower limits of the sum. That is

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{k=0}^{\infty} a_k (x - x_0)^k = \sum_{m=-1}^{\infty} a_{m+1} (x - x_0)^{m+1}$$

$$= \sum_{n=10}^{\infty} a_{n-10} (x - x_0)^{n-10}$$

Exp Rewrite the following power series involving the power of $(x-2)^n$

① $$\sum_{n=0}^{\infty} a_n (x-2)^{n+1} = \sum_{n=1}^{\infty} a_{n-1} (x-2)^n$$

② $$\sum_{n=0}^{\infty} n\, a_n (x-2)^{3+n} = \sum_{n=3}^{\infty} (n-3)\, a_{n-3} (x-2)^n$$

③ $$\sum_{k=5}^{\infty} k(k-1)(k-2)(x-2)^{k-3} = \sum_{n=2}^{\infty} (n+3)(n+2)(n+1)(x-2)^n$$