## 5.2 Series Solution Near an Ordinary Point $x_0$ 
"Part I"

- Given the DE
$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \cdots (*)$$
where $P, Q, R$ are polynomials

- Note that $(*)$ is $2^{nd}$ order linear homogeneous DE with variable coefficients

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**Def** • The DE $(*)$ has an Ordinary Point $x_0$ iff $P(x_0) \neq 0$

• The DE $(*)$ has Singular Point $z_0$ iff $P(z_0) = 0$

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**Exp** ① The DE $(x^2-4)\,y'' + (\sin x)\,y' - e^x y = 0$
has two singular points $\Rightarrow$
$$P(x) = x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = 2 \text{ or } x = -2$$

All other points are ordinary "$\mathbb{R} \setminus \{-2, 2\}$"

② $(\ln x)\,y'' - x y' + y = 0$
has only on singular point $\Rightarrow$
$$P(x) = \ln x = 0$$
$$x = 1$$

All other points are ordinary "$\mathbb{R} \setminus \{1\}$"

③ $y'' - e^x y' + y = 0$
has no singular points $\Rightarrow$ All points are ordinary

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- Assume $x_0$ is an Ordinary Point (OP) for the DE $(*)$:

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

- Hence, $P(x_0) \neq 0$.

- Let $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$

- Note that $p(x)$ and $q(x)$ are well-defined at the OP $x_0$. Moreover, $p(x)$ and $q(x)$ are analytic at $x_0$. That is, $p(x)$ and $q(x)$ have Taylor Series Expansion about the OP $x_0$:

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

- Now divide the DE $(*)$ by $P(x)$ $\Rightarrow$

$(*')$ $$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = y_0, \; y'(x_0) = y_0'$$

where $p(x)$ and $q(x)$ are cont. on an open interval $I$ about $x_0$

- By Th3.2.1 $\Rightarrow$ $\exists$ a unique solution $y(x)$ satisfies the IVP $(*')$ on $I$.

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In this section we will find a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

about the OP $x_0$ for the DE $(*)$.

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- To find two independent power series solutions $y_1(x)$ and $y_2(x)$ $\Rightarrow$ we write the coefficients $a_2, a_3, a_4, \dots$ interms of $a_0$ or $a_1$ so that the power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \dots$$
$$= a_0\, y_1(x) + a_1\, y_2(x)$$

Then we check $w(y_1(x), y_2(x))(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$

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<u>Exp</u> Find a series solution $\sum_{n=0} a_n x^n$ for the DE

$$y'' + y = 0 \quad , \quad x \in \mathbb{R}$$

- Comparing $\sum_{n=0}^{\infty} a_n x^n$ with $\sum_{n=0} a_n (x-x_0)^n \Rightarrow x_0 = 0$ is an OP since $P(x) = 1$ and so all points are ordinary

- Our power series solution is then given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$
$$\Rightarrow y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)\, x^{n-2}$$

- Substitue $y''$ and $y$ in the DE $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n \, x^{n-2} + \sum_{n=0}^{\infty} a_n \, x^{n} = 0$$

not          same power

① same power
② same index

shifting index

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^{n} + \sum_{n=0}^{\infty} a_n \, x^{n} = 0$$

same index

$$\sum_{n=0}^{\infty} \left[ (n+2)(n+1)\, a_{n+2} + a_n \right] x^{n} = 0$$

Comparing the coefficients of $x^n$ $\Rightarrow$

$$(n+2)(n+1)\, a_{n+2} + a_n = 0 \quad , \quad n = 0, 1, 2, \dots$$

Recurrence Relation (RR)

$$\boxed{a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \ , \ n = 0, 1, 2, \dots}$$

We use RR to write $a_2, a_3, \dots$ interms of $a_0$ and $a_1$ $\Rightarrow$

$$n=0 \Rightarrow a_2 = \frac{-a_0}{(1)(2)} = \frac{-a_0}{2!}$$

$$n=1 \Rightarrow a_3 = \frac{-a_1}{(2)(3)} = \frac{-a_1}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{-a_2}{(3)(4)} = \frac{-\frac{-a_0}{2!}}{(3)(4)} = \frac{a_0}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{-a_3}{(4)(5)} = -\frac{-\frac{a_1}{3!}}{(4)(5)} = \frac{a_1}{5!}$$

$$n=4 \Rightarrow a_6 = \frac{-a_4}{(5)(6)} = -\frac{\frac{a_0}{4!}}{(5)(6)} = -\frac{a_0}{6!}$$

$\vdots$

The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 - \frac{a_0}{6!} x^6 + \cdots$$

$$= a_0 \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] + a_1 \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right]$$

$$= a_0 \; y_1(x) + a_1 \; y_2(x)$$

$$= a_0 \cos x + a_1 \sin x$$

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Note that the power series solutions $y_1(x)$ and $y_2(x)$ are L. Indep. since

$$W(y_1(x), y_2(x))(x_0) = W(y_1(x), y_2(x))(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence, they form fondemental series solutions

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Note also that this DE : $y'' + y = 0$ can be easily solved as follows: Ch. Eq. $r^2 + 1 = 0$

$$r_{1,2} = \pm i \qquad \lambda = 0, \quad \mu = 1$$

$$y_1(x) = \cos x \quad \text{and} \quad y_2(x) = \sin x$$

Hence, the gen. sol. is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 \cos x + c_2 \sin x$$

Exp Find two indep. power series solutions in the power of x for the DE $y'' - xy = 0$

- The series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n$ so $x_0 = 0$ is OP since $P(x) = 1$ and so all points are ordinary
- $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$
- Substitute $y''$ and $y$ in the DE above $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n \underset{\uparrow \text{not}}{x^{n-2}} - \sum_{n=0}^{\infty} a_n \underset{\uparrow \text{not same power}}{x^{n+1}} = 0$$

✓ ① same power
✓✓ ② same index

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

not same index

$$(2)(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2 a_2 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$\boxed{a_2 = 0}$ and $\boxed{a_{n+2} = \dfrac{a_{n-1}}{(n+1)(n+2)}, \quad n = 1, 2, 3, \ldots}$ (RR) Recurrence Relation we use RR to write $a_2, a_3, \dots$ interms of $a_0$ and $a_1 \Rightarrow$

$$n=1 \Rightarrow a_3 = \frac{a_0}{(2)(3)} = \frac{a_0}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{a_1}{(3)(4)} = \frac{2a_1}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{a_2}{(4)(5)} = 0$$

$$n=4 \Rightarrow a_6 = \frac{a_3}{(5)(6)} = \frac{\frac{a_0}{(2)(3)}}{(5)(6)} = \frac{a_0}{(2)(3)(5)(6)} = \frac{4a_0}{6!}$$

$$n=5 \Rightarrow a_7 = \frac{a_4}{(6)(7)} = \frac{a_1/(3)(4)}{(6)(7)} = \frac{a_1}{(3)(4)(6)(7)} = \frac{10a_1}{7!}$$

$$n=6 \Rightarrow a_8 = \frac{a_5}{(7)(8)} = 0$$

$\vdots$

The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cancel{a_2 x^2} + a_3 x^3 + a_4 x^4 + \cancel{a_5 x^5} + a_6 x^6 + \dots$$

$$= a_0 + a_1 x + 0 + \frac{a_0}{3!}x^3 + \frac{2a_1}{4!}x^4 + 0 + \frac{4a_0}{6!}x^6 + \frac{10a_1}{7!}x^7 + \dots$$

$$= a_0\left[1 + \frac{x^3}{3!} + \frac{4x^6}{6!} + \dots\right] + a_1\left[x + \frac{2x^4}{4!} + \frac{10x^7}{7!} + \dots\right]$$

$$= a_0 \; y_1(x) + a_1 \; y_2(x)$$

$y_1(x)$ and $y_2(x)$ are the two indep. power series solutions since

$$W(y_1(x), y_2(x))(x_0) = W(y_1(x), y_2(x))(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

Hence, they also form Fundemental series solutions.

Exp Find Fundemental series solutions for the DE: $y'' - xy = 0$ about $x_0 = 1$

- $P(x) = 1$ never zero so all points are ordinary $\Rightarrow x_0 = 1$ is OP
- The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

- substitute $y''$ and $y$ in the DE $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \left((x-1)+1\right) \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n \underset{\uparrow \text{not}}{(x-1)^{n-2}} - \sum_{n=0}^{\infty} a_n \underset{\uparrow \text{not}}{(x-1)^{n+1}} - \sum_{n=0}^{\infty} a_n \underset{\uparrow}{(x-1)^n} = 0$$

① same power
② same index

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0$$

$$(2)(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n (x-1)^n = 0$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) a_{n+2} - a_{n-1} - a_n \right] (x-1)^n = 0$$

Comparing coefficients $\Rightarrow$

$$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$$

$$\boxed{a_{n+2} = \frac{a_{n-1} + a_n}{(n+1)(n+2)}, \quad n = 1,2,3,\ldots} \text{ — (RR) Recurrence Relation}$$

We use RR to write the coefficients $a_3, a_4, a_5, \ldots$ interms of $a_0$ and $a_1$ $\Rightarrow$

$$n=1 \Rightarrow a_3 = \frac{a_0 + a_1}{(2)(3)} = \frac{a_0}{3!} + \frac{a_1}{3!}$$

$$n=2 \Rightarrow a_4 = \frac{a_1 + a_2}{(3)(4)} = \frac{2a_1}{4!} + \frac{a_0/2}{(3)(4)} = \frac{2a_1}{4!} + \frac{a_0}{4!}$$

$$n=3 \Rightarrow a_5 = \frac{a_2 + a_3}{(4)(5)} = \frac{a_0/2 + a_0/3! + a_1/3!}{(4)(5)} = \frac{4a_0}{5!} + \frac{a_1}{5!}$$

$\vdots$

The series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + \cdots$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \left(\frac{a_0}{3!} + \frac{a_1}{3!}\right)(x-1)^3 + \left(\frac{2a_1}{4!} + \frac{a_0}{4!}\right)(x-1)^4 + \cdots$$

$$= a_0\left[1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + \cdots\right] + a_1\left[(x-1) + \frac{(x-1)^3}{3!} + \frac{2(x-1)^4}{4!} + \cdots\right]$$

$$= a_0 \; y_1(x) + a_1 \; y_2(x)$$

$$W(y_1(x), y_2(x))(1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$y_1$ and $y_2$ are L. Indep. Thus, they form fundemental set of solutions.

Exp Find power series solution for the DE

$$(1-x^2)y'' - 2xy' + 6y = 0 \quad \text{about } x_0 = 0$$

• $P(x) = 1 - x^2 = 0 \iff x = \pm 1$ singular Points

Hence, $x_0 = 0$ is OP

• The series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

• substitute $y'', y', y$ in the DE

$$\Rightarrow y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n \underset{\uparrow}{x^{n-2}} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

✓ same power

✓✓ same index

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} x^n - \sum_{n=2 \to n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=1 \to n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 6 a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2n a_n + 6a_n\right]x^n = 0$$

$$-a_n(n^2 - n + 2n - 6)$$

$$-a_n(n^2 + n - 6)$$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_n(n-2)(n+3)\right]x^n = 0$$

$\Downarrow$

$$\boxed{a_{n+2} = \frac{(n-2)(n+3)\,a_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, \dots} \rightarrow \text{Recurrence Relation (RR)}$$

We use RR to write $a_2, a_3, a_4, \dots$ interms of $a_0$ and $a_1$ as follow $\Rightarrow$

$$n=0 \Rightarrow a_2 = \frac{(-\cancel{2})(3)\,a_0}{(1)\cancel{(2)}} = -3a_0$$

$$n=1 \Rightarrow a_3 = \frac{(-1)(\cancel{4}^{2})\,a_1}{\cancel{(2)}(3)} = -\frac{2}{3}a_1$$

$$n=2 \Rightarrow a_4 = 0$$

$$n=3 \Rightarrow a_5 = \frac{(1)(\cancel{6}^{3})\,a_3}{(\cancel{4}_{2})(5)} = \frac{3\left(-\frac{2}{3}a_1\right)}{(2)(5)} = -\frac{1}{5}a_1$$

$$n=4 \Rightarrow a_6 = 0$$

$$n=5 \Rightarrow a_7 = \frac{(\cancel{3})(\cancel{8}^{4})\,a_5}{(\cancel{6}_{3})(7)} = \frac{4\left(-\frac{1}{5}a_1\right)}{7} = -\frac{4}{35}a_1$$

$$n=6 \Rightarrow a_8 = 0$$

$$\vdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cancel{a_4 x^4}^{\,0} + a_5 x^5 + \cancel{a_6 x^6}^{\,0} + \dots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \dots$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{2}{3}a_1 x^3 - \frac{1}{5}a_1 x^5 - \frac{4}{35}a_1 x^7 + \dots$$

$$= a_0\left[1 - 3x^2\right] + a_1\left[x - \frac{2}{3}x^3 - \frac{1}{5}x^3 - \frac{4}{35}x^7 + \dots\right]$$

$$= a_0\, y_1(x) + a_1\, y_2(x)$$