Recall from 5.2 that the series solution about the OP x_0 has the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

for a given DE of the form

$$\boxed{P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0} \quad *$$

where $P(x)$, $Q(x)$, $R(x)$ are polynomials.

Question what happen if $P(x), Q(x), R(x)$ not all poly. ?

Answer: It will be hard to find series solution as in 5.2 procedure. $\Rightarrow$

Exp' Find series solution of power x for the DE

$$(x+1)\, y'' - \ln(e + x^2)\, y' - 2y = 0$$

Note that $Q(x) = -\ln(e + x^2)$ is not poly.

series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ since $x_0 = 0$ is OP.
since $P(0) = 1 \neq 0$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

substitute y, y', y'' in the DE $\Rightarrow$

$$(x+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \underbrace{- \ln(e + x^2) \sum n a_n x^{n-1}}_{\text{Problem}} - 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

We need new method to solve Exp'

Question Given the IVP:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad , \ y(x_0) = y_0 \ , \ y'(x_0) = y'_0$$

where x_0 is an OP and $P(x), Q(x), R(x)$ are functions having all derivative at x_0.

show that if $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$

is a power series solution to this IVP about the OP x_0, then the coefficients $a_0, a_1, a_2, \dots, a_m, \dots$ are given by

$$\boxed{a_m = \frac{y^{(m)}(x_0)}{m!} \ , \quad m = 0, 1, 2, \dots}$$

Answer:

$$a_0 = \frac{y(x_0)}{0!} = y_0 \quad ✓$$

$$a_1 = \frac{y'(x_0)}{1!} = y'_0 \quad ✓$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

$$\vdots$$

$$y^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-(m-1))\, a_n (x-x_0)^{n-m}$$

$$y^{(m)}(x) = m(m-1)(m-2)\cdots(m-m+1)\,a_m + \sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-(m-1))\,a_n\,(x-x_0)^{n-m}$$

$$y^{(m)}(x_0) = m!\,a_m + 0 \quad \Rightarrow \quad a_m = \frac{y^{(m)}(x_0)}{m!}$$

Exp Given the IVP:

$$(x+1)y'' - \ln(e+x^2)\,y' - 2y = 0, \quad y(0)=1,\ y'(0)=1$$

Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series solution of this IVP, find the first four terms.

$x_0 = 0$ is OP since $P(0) = 1 \neq 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$a_0 = y_0 = 1$ $\quad a_1 = y'_0 = 1$ $\quad \frac{3}{2}$ $\quad \frac{1}{3}$

$$a_2 = \frac{y''(x_0)}{2!} = \frac{y''(0)}{2} = \frac{3}{2}$$

$$a_3 = \frac{y'''(x_0)}{3!} = \frac{y'''(0)}{6} = \frac{2}{6} = \frac{1}{3}$$

$$(x+1)y'' - \ln(e+x^2)\,y' - 2y = 0$$

$$(0+1)\,y''(0) - \ln(e+0^2)\,y'(0) - 2y(0) = 0$$

$$y''(0) - \ln e\ y'(0) - 2y(0) = 0$$

$$y''(0) - (1)(1) - 2(1) = 0$$

$$\boxed{y''(0) = 3}$$

To find $y'''(0) \Rightarrow$

$$(x+1)y''' + y'' - \ln(e+x^2)\,y'' - \frac{2x}{e+x^2}\,y' - 2y' = 0$$

$$y'''(0) + y''(0) - \ln e\ y''(0) - 0 - 2y'(0) = 0$$

$$y'''(0) + 3 - (1)(3) - 2(1) = 0 \quad \Rightarrow \quad \boxed{y'''(0) = 2}$$

The 1st four terms: $\left\{1,\ x,\ \frac{3}{2}x^2,\ \frac{1}{3}x^3\right\}$

Exp Given the IVP :

$$y'' + xy' + y = 0 \quad , y(0) = 1, \; y'(0) = 0$$

1 Suppose $y = \phi(x)$ is solution to this IVP.
Find $\phi''(0)$, $\phi'''(0)$, $\phi^{(4)}(0)$

- $y''(0) + (0)\,y'(0) + y(0) = 0$
$y''(0) + 0 + 1 = 0 \quad \Rightarrow y''(0) = \phi''(0) = -1$

- To find $\phi'''(0)$ we derive $\Rightarrow y''' + xy'' + y' + y' = 0$
$y'''(0) + 0 + 2y'(0) = 0$
$y'''(0) = \phi'''(0) = -2y'_0 = 0$

- To find $\phi^{(4)}(0)$ we derive $\Rightarrow y'''' + xy''' + y'' + 2y'' = 0$
$y^{(4)}(0) + 0 + 3y''(0) = 0$
$y^{(4)}(0) = \phi^{(4)}(0) = -3y''(0) = 3$

---

2 Find the 1st three nonzero terms of the power series solution about $x_0 = 0$

- $x_0 = 0$ is OP since $P(x) = 1$ never zero

- The power series solution is $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$a_0 = y_0 = 1$ and $a_1 = y'_0 = 0$

$a_2 = \frac{y''(0)}{2!} = \frac{-1}{2}$ and $a_3 = \frac{y'''(0)}{3!} = 0$

$a_4 = \frac{y^{(4)}(0)}{4!} = \frac{3}{24} = \frac{1}{8}$

$y(x) = 1 + 0 - \frac{1}{2}x^2 + 0 + \frac{x^4}{8} + \dots$

Hence, the 1st three nonzero terms
$\left\{1, -\frac{1}{2}x^2, \frac{x^4}{8}\right\}$

Exp solve the IVP:

$$y'' - 4e^{2x}y' - (3x^2 + 2x + 5)y = 0 \;, \quad y(0) = y_0, \quad y'(0) = y_0'$$

• $P(x) = 1$ never zero ⇒ all points are ordinary
⇒ $x_0 = 0$ is OP

• The series solution about the OP $x_0 = 0$ is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$a_0 = y_0, \quad a_1 = y_0', \quad a_2 = \frac{y_0''}{2!}, \quad a_3 = \frac{y_0'''}{3!}, \quad a_4 = \frac{y_0^{(4)}}{4!}$$

• To find a_2 ⇒ $y''(0) - 4e^{0}y'(0) - (5)y(0) = 0$

$$y''(0) - 4(1)y_0' - 5y_0 = 0$$

$$y''(0) - 4a_1 - 5a_0 = 0 \Rightarrow y''(0) = 4a_1 + 5a_0$$

$$a_2 = \frac{y''(0)}{2!} = \frac{4a_1 + 5a_0}{2} = 2a_1 + \frac{5}{2}a_0$$

• To find a_3 ⇒ $y''' - 4e^{2x}y'' - 8e^{2x}y' - (3x^2 + 2x + 5)y' - (6x + 2)y = 0$

$$y'''(0) - 4y''(0) - 8y'(0) - (5)y'(0) - (2)y(0) = 0$$

$$y'''(0) - 4(4a_1 + 5a_0) - 8a_1 - 5a_1 - 2a_0 = 0$$

$$y'''(0) - 29a_1 - 22a_0 = 0$$

$$\Rightarrow y'''(0) = 29a_1 + 22a_0$$

$$a_3 = \frac{y'''(0)}{3!} = \frac{29a_1 + 22a_0}{6} = \frac{29}{6}a_1 + \frac{11}{3}a_0$$

$$y(x) = a_0 + a_1 x + \left(2a_1 + \frac{5}{2}a_0\right)x^2 + \left(\frac{29}{6}a_1 + \frac{11}{3}a_0\right)x^3 + \cdots$$

$$= a_0\left(1 + \frac{5}{2}x^2 + \frac{11}{3}x^3 + \cdots\right) + a_1\left(x + 2x^2 + \frac{29}{6}x^3 + \cdots\right)$$

$$= a_0\, y_1(x) + a_1\, y_2(x)$$

Th (5.3.1)

If x_0 is an OP for the DE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \text{---} \ *$$

where

$$p(x) = \frac{Q(x)}{P(x)} \quad \text{and} \quad q(x) = \frac{R(x)}{P(x)} \text{ are analytic at } x_0,$$

then the general solution of the DE $*$ is given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0\, y_1(x) + a_1\, y_2(x)$$

where

- a_0 and a_1 are arbitrary constants
- y_1 and y_2 are two power series solution which are analytic at x_0
- the series solutions y_1 and y_2 form fundemental set of solution

Furthermore, the radius of convergence for each power series solution $y_1(x)$ and $y_2(x)$ is given by

$$\rho = \min\{\rho_1, \rho_2\}$$

where

ρ_1 is the radius of convergence for the power series of $p(x)$

and

ρ_2 is the radius of convergence for the power series of $q(x)$

Remark: Th (5.3.1) provides strategy to find ρ for power series solution $y(x) = \sum a_n (x - x_0)^n$ for a given DE about OP x_0 without solving the DE

Remark[2] ① If $P(x), Q(x), R(x)$ are all poly.
then we can find ρ_1 and ρ_2 straight forward for $p(x)$ and $q(x)$.

② If $P(x), Q(x), R(x)$ are not all poly.
then first we find Taylor series for $p(x)$ and $q(x)$ then find ρ_1 and ρ_2

Exp Determine a lower bound for the radius of convergence ρ of the series solution of

① $y'' - xy = 0$ about $x_0 = 1$

$\left.\begin{array}{l} P(x) = 1 \\ Q(x) = 0 \\ R(x) = -x \end{array}\right\} \Rightarrow$ all poly.

$P(x) = 1$ never zero
all points are ordinary
$x_0 = 1$ is OP

$$p(x) = \frac{Q(x)}{P(x)} = \frac{0}{1} = 0 \quad \text{is analytic everywhere} \Rightarrow \rho_1 = \infty$$

$$q(x) = \frac{R(x)}{P(x)} = \frac{-x}{1} = -x \quad \text{is analytic everywhere} \Rightarrow \rho_2 = \infty$$

Hence, the radius of convergence ρ for the series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ is $\min\{\rho_1, \rho_2\} = \infty$

by Th 5.3.1

---

② $(x^2+3x)y'' + y' + y = 0$ about $x_0 = -1$

$\left.\begin{array}{l} P(x) = x^2 + 3x \\ Q(x) = 1 \\ R(x) = 1 \end{array}\right\}$ All poly.

$P(x) = x(x+3) = 0$

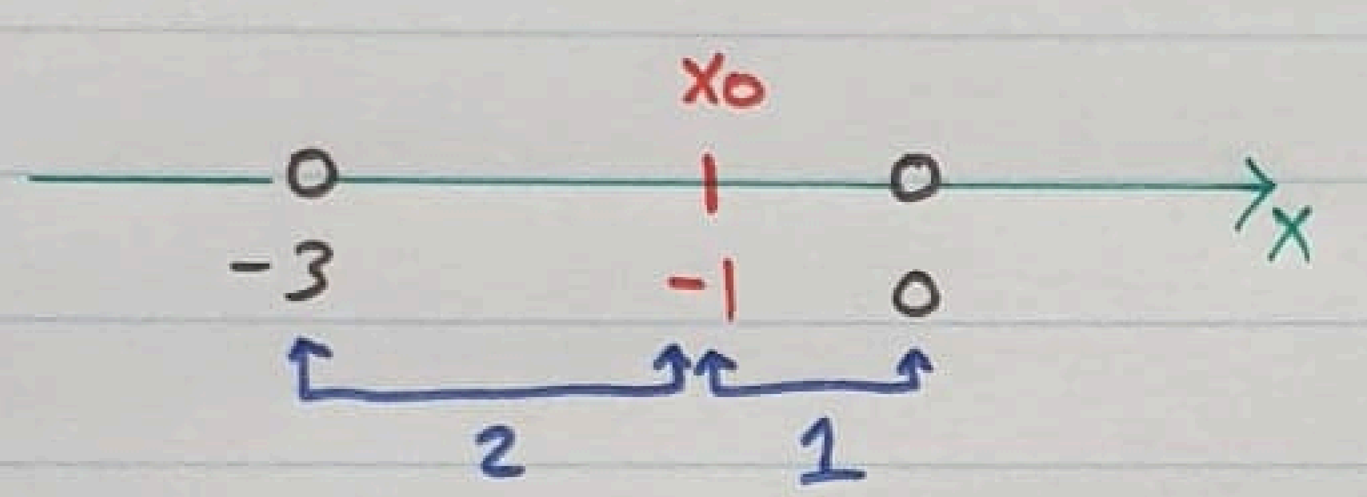
$x = 0,\ x = -3$

Singlur Points

$x_0 = -1$ is OP

$\left.\begin{array}{l} p(x) = \dfrac{Q(x)}{P(x)} = \dfrac{1}{x(x+3)} \\ \\ q(x) = \dfrac{R(x)}{P(x)} = \dfrac{1}{x(x+3)} \end{array}\right\} \Rightarrow$ are analytic everywhere except at $x = 0$ and $x = -3$

p(x)



$\Rightarrow \rho_1 = 1$ for $p(x)$

$\rho_2 = 1$ for $q(x)$

Hence, the radius of convergence for the series solution $y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$ is $\rho = \min\{\rho_1, \rho_2\} = 1$ by Th5.3.1

---

③ $(1+x^2)y'' + 2xy' + 4x^2 y = 0$ about $x_0 = 0$

$x_0 = \frac{1}{2}$

$\left.\begin{array}{l} P(x) = 1 + x^2 \\ Q(x) = 2x \\ R(x) = 4x^2 \end{array}\right\}$ All poly.

$P(x) = 1 + x^2 = 0$

$x = \pm i$

Singular Points

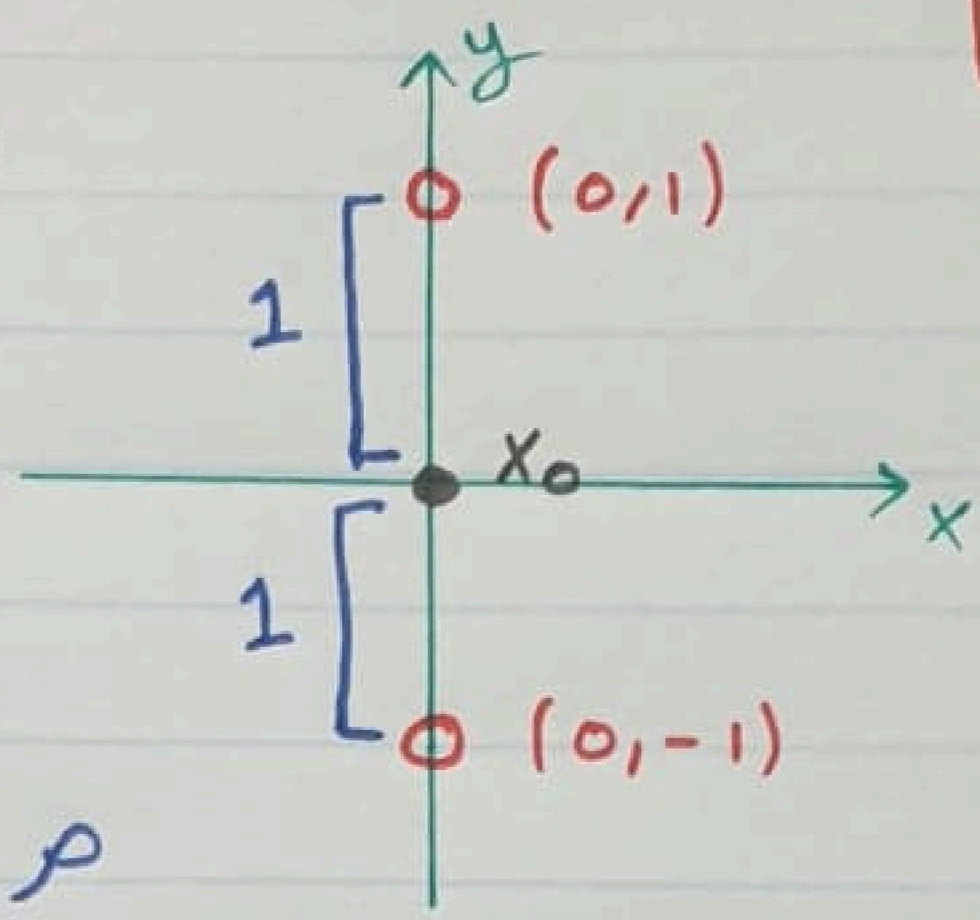
$x_0 = 0$ and $x_0 = \frac{1}{2}$ are OP

$\left.\begin{array}{l} p(x) = \dfrac{2x}{1+x^2} \\ \\ q(x) = \dfrac{4x^2}{1+x^2} \end{array}\right\} \Rightarrow$ are analytic everywhere except at $x = \pm i = 0 \pm i$ comparing with $z = x + yi$

$(0,1)$ or $(0,-1)$

$\boxed{x_0 = 0}$

$p(x)$



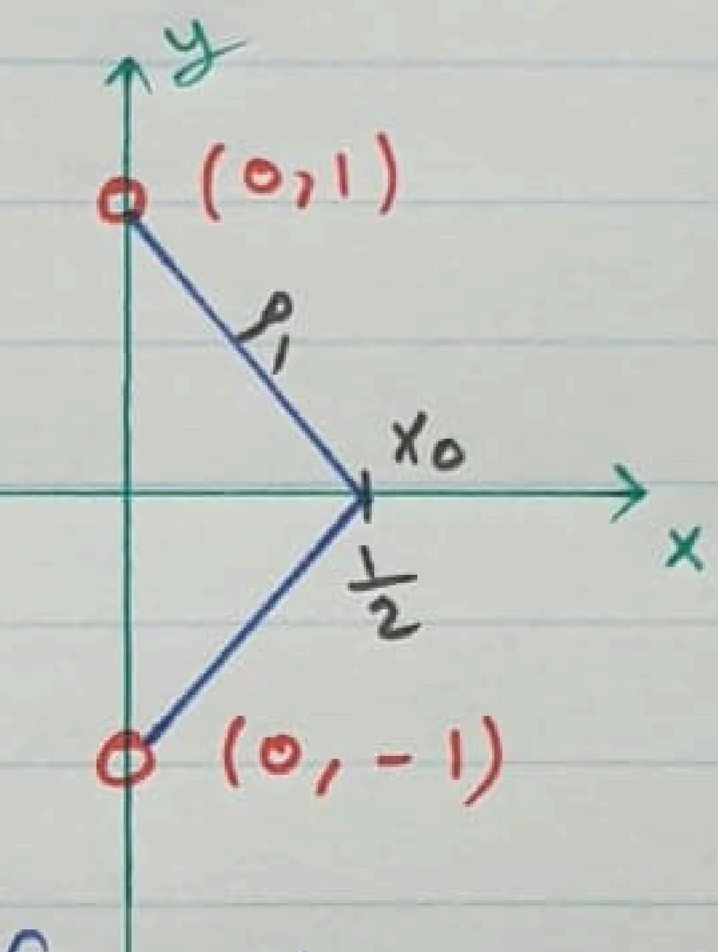
$\rho_1 = 1$ for $p(x)$

$\rho_2 = 1$ for $q(x)$ "similar"

Hence, the radius of convergence ρ for the series solution $\sum a_n x^n$ is $\rho = \min\{1, 1\} = 1$

$\boxed{x_0 = \frac{1}{2}}$

$p(x)$



$\rho_1 = \sqrt{\left(\frac{1}{2}\right)^2 + (1)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ for $p(x)$

$\rho_2 = \frac{\sqrt{5}}{2}$ for $q(x)$ "similar"

Hence, the radius of convergence ρ for the series solution $\sum a_n (x - \frac{1}{2})^n$ is $\min\{\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\} = \frac{\sqrt{5}}{2}$

---

4 $(1 + x^2)y'' + (1 + x^2)y' + y = 0$ about $x_0 = 0$

$$\left.\begin{aligned} P(x) &= 1 + x^2 \\ Q(x) &= 1 + x^2 \\ R(x) &= 1 \end{aligned}\right\} \text{all poly}$$

$P(x) = 0 \Rightarrow 1 + x^2 = 0 \Rightarrow$
$x = \pm i$ Singular Points

$x_0 = 0$ is OP

$p(x) = 1$ is analytic everywhere $\Rightarrow \rho_1 = \infty$

$q(x) = \frac{1}{1+x^2}$ is analytic everywhere except at $x = \pm i = 0 \pm i$

$\Rightarrow \rho_2 = 1$ by part 3 $\quad (0,1), (0,-1)$

Hence, the radius of convergence for the series solution $\sum a_n x^n$ is $\rho = \min\{1, \infty\} = 1$

---

⑤ $x(x^2-2x+2)y'' + xy' + (x^2-2x+2)y = 0$ about $x_0 = 2$

$P(x) = x(x^2-2x+2)$
$Q(x) = x$
$R(x) = x^2-2x+2$
} All poly

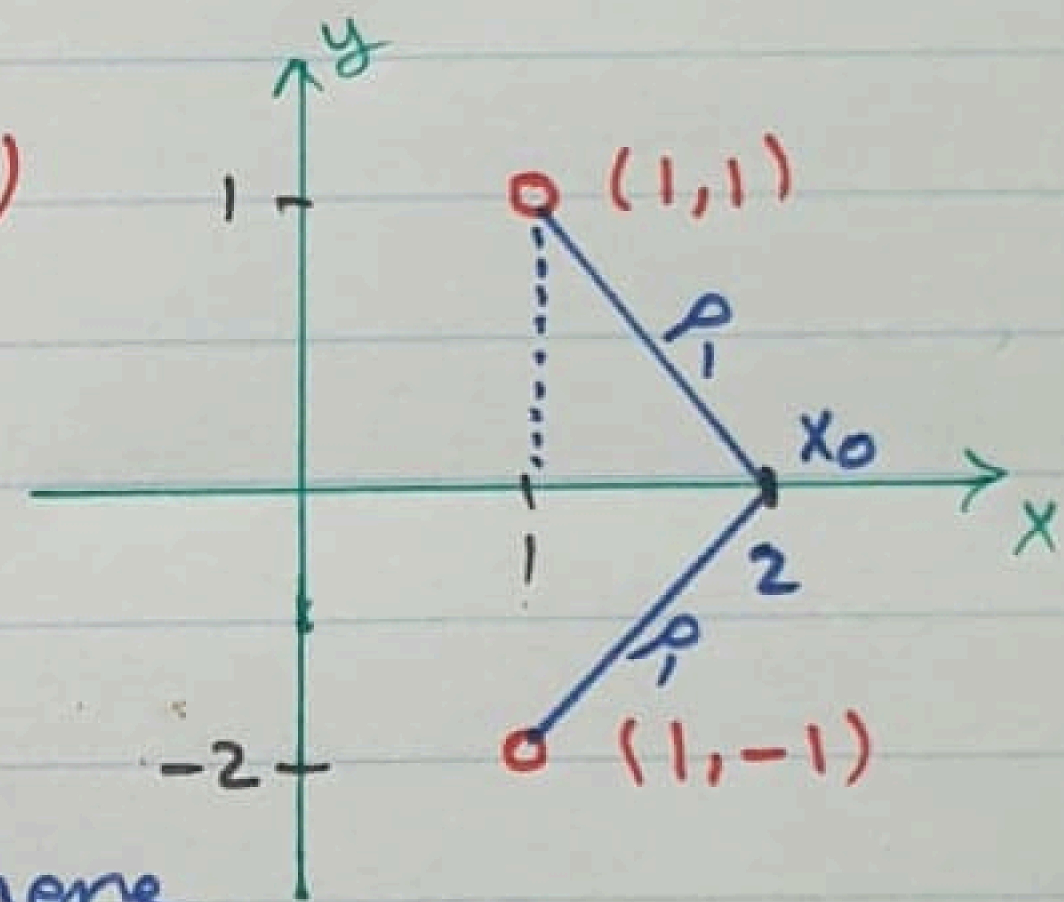
$P(x) = 0$
$x(x^2-2x+2) = 0$
$x = 0\ ,\ x = \frac{2 \pm \sqrt{4-4(2)}}{2}$
$= 1 \pm i$

$x_0 = 2$ is OP

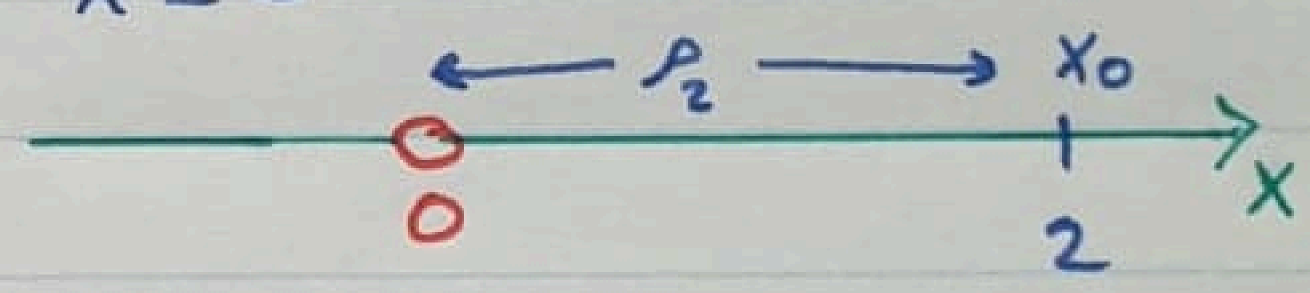
$p(x) = \frac{1}{x^2-2x+2}$ is analytic every where except at $x = 1 \pm i$

$(1,1)\ ,\ (1,-1)$

$\rho_1 = \sqrt{1^2+1^2} = \sqrt{2}$ for $p(x)$



$q(x) = \frac{1}{x}$ is analytic everywhere except at $x = 0$



$\rho_2 = 2$ for $q(x)$

Hence, the radius of convergence for the series solution $\sum a_n (x-2)^n$ is

$$\rho = \min\{\rho_1, \rho_2\} = \min\{\sqrt{2}, 2\} = \sqrt{2}$$

Now we will consider an example when $P(x)$, $Q(x)$, $R(x)$ are not all poly.

⑥ $y'' + (\sin x)\, y' + (1+x^2)\, y = 0$ about $x_0 = 0$

$\left.\begin{array}{l} P(x) = 1 \\ Q(x) = \sin x \\ R(x) = 1 + x^2 \end{array}\right\}$ Not all Poly.

$P(x) = 1$ never zero
$\Rightarrow$ all points are ordinary
$\Rightarrow x_0 = 0$ is OP

$p(x) = \dfrac{\sin x}{1} = \sin x$ which is analytic every where $\Rightarrow \rho_1 = \infty$

$q(x) = \dfrac{1+x^2}{1} = 1 + x^2$ which is analytic every where $\Rightarrow \rho_2 = \infty$

Hence, the radius of convergence ρ for the series solution $\sum a_n x^n$ is ∞

Basically we find Taylor series expansion for $\sin x$ about $x_0 = 0$
" Maclurine Series " $\Rightarrow$

$$p(x) = \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \text{Apply RT}$$

$\Rightarrow \rho_1 = \infty$

same for $q(x) = 1 + x^2$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots$$

$$= (1) + (0)x + \frac{2}{2} x^2 + 0 + 0 + \cdots$$

$$= 1 + x^2$$

$f(x) = 1 + x^2$
$f'(x) = 2x$
$f''(x) = 2$
$f'''(x) = 0$
$f^{(4)}(x) = 0$
$\vdots$

(7) $(x^2+1)y'' + xy' + \frac{1}{x-2}y = 0$ about $x_0 = 1$

Multiply all terms by $x-2$

$$(x-2)(x^2+1)y'' + x(x-2)y' + y = 0$$

$P(x) = (x-2)(x^2+1)$
$Q(x) = (x-2)x$
$R(x) = 1$
} All poly.

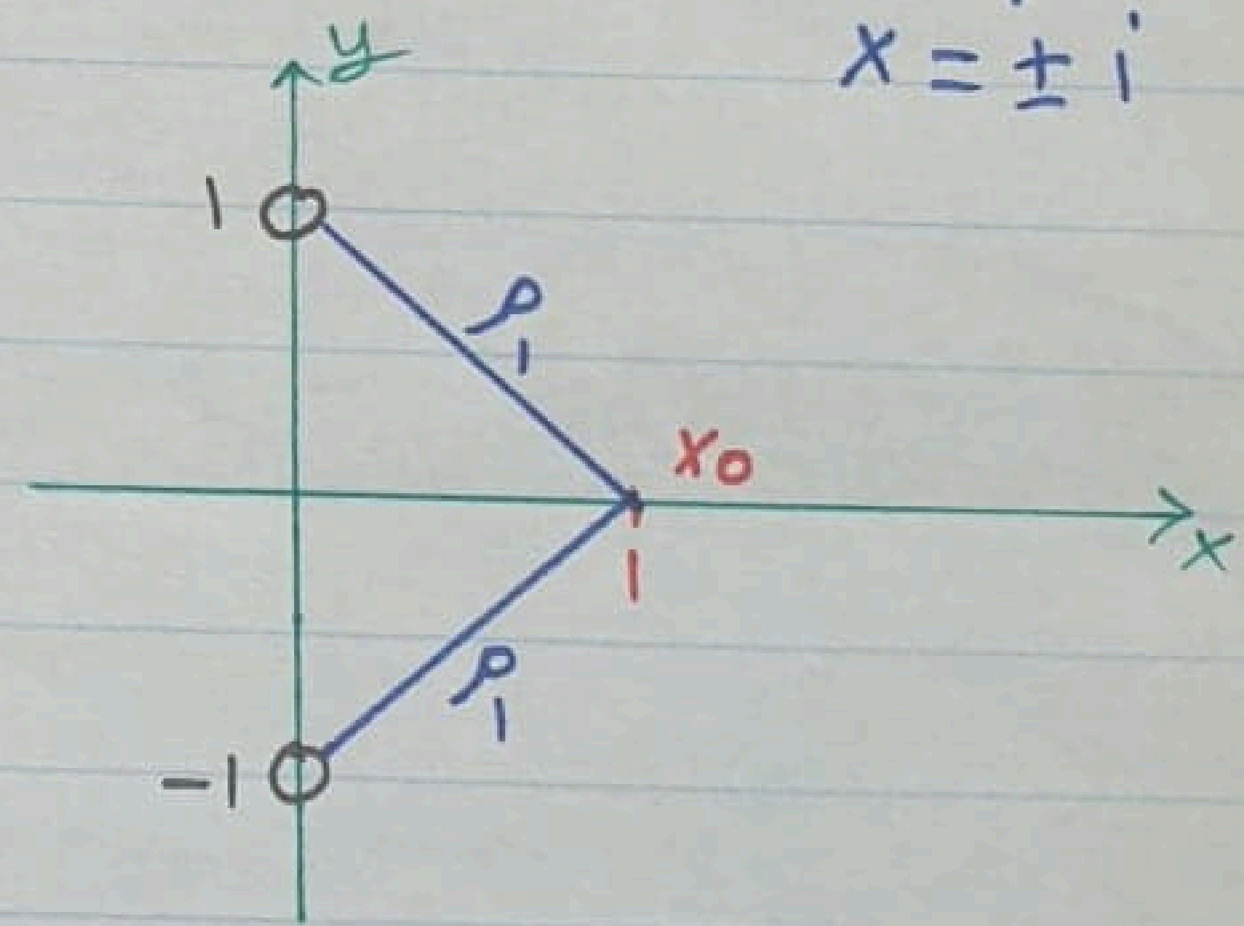
$P(x) = 0 \iff (x-2)(x^2+1) = 0 \iff x = 2,\ x = \pm i$

Singular Points

$(0,1), (0,-1)$

$\Rightarrow$ $x_0 = 1$ is OP

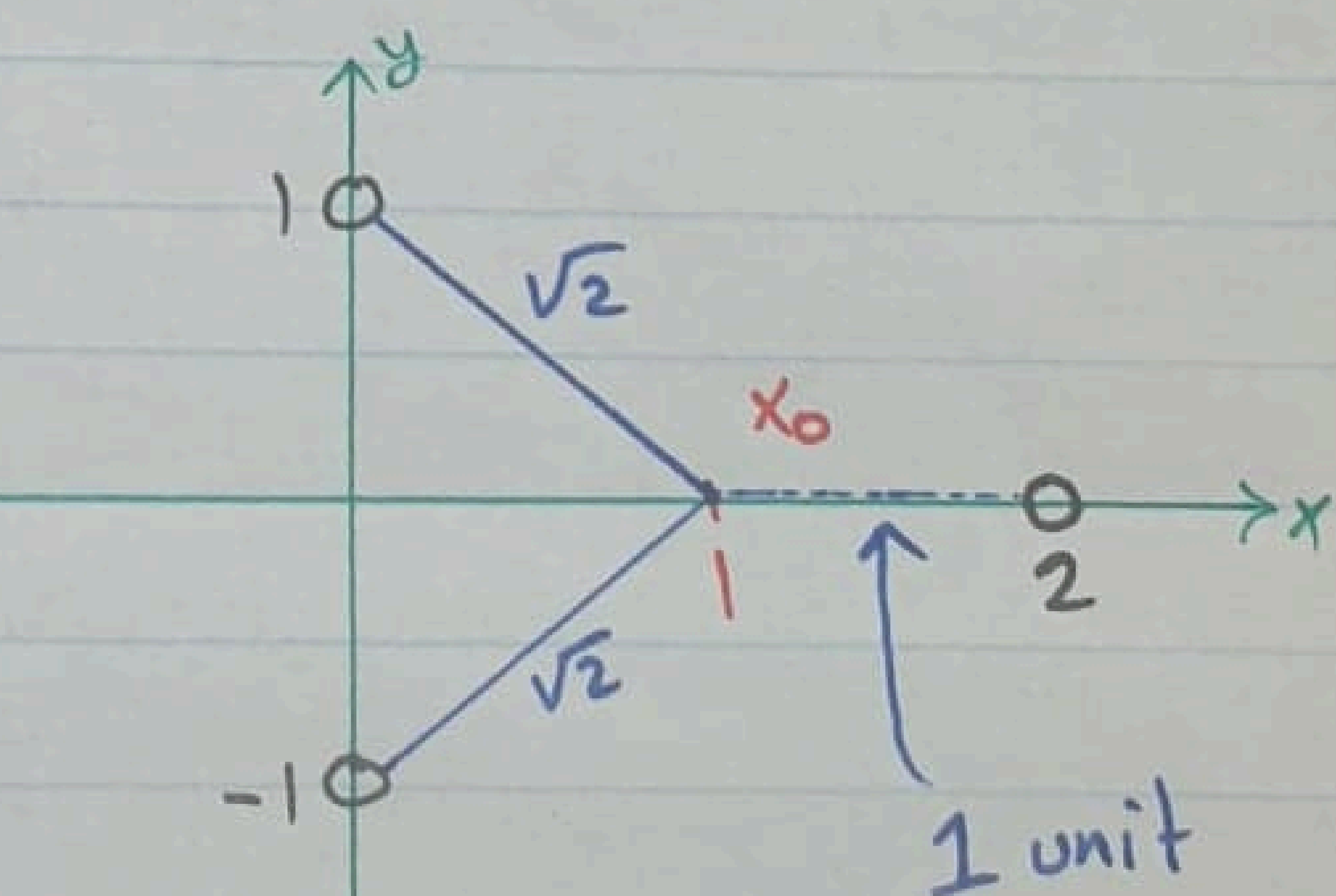
$p(x) = \frac{Q(x)}{P(x)} = \frac{x}{x^2+1}$ is analytic every where except at $x = \pm i$



$$\rho_1 = \sqrt{1^2+1^2} = \sqrt{2}$$

$q(x) = \frac{R(x)}{P(x)} = \frac{1}{(x-2)(x^2+1)}$

is analytic every where except $x = 2$ and $x = \pm i$



$$\rho_2 = \min\{1, \sqrt{2}\} = 1$$

- Hence, the radius of convergence for the power series solution $\sum a_n (x-1)^n$ is $\rho = \min\{\rho_1, \rho_2\}$

$= \min\{\sqrt{2}, 1\}$

$= 1$