5.4 Euler Differential Equations; Regular Singular Points

- Recall the Euler DE: $x^2 y'' + \alpha x y' + \beta y = 0$

- Note that Euler DE has a singular point at $x_0 = 0$ since $P(x) = x^2 \Rightarrow P(x) = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x_0 = 0$

- If we try to find a power series solution for Euler DE about the SP $x_0 = 0$ " $y(x) = \sum_{n=0}^{\infty} a_n x^n$ " as in 5.2, then we will find out that this is impossible.

The reason for that is due to the fact that $p(x)$ and $q(x)$ are not analytic at the SP $x_0 = 0$

- Thus, we need more information about the singularity of $p(x)$ and $q(x)$ to be not too severve

- So first we will classify the SP's into

Regular Singular Point (RSP) or
Irregular Singular Point (IRSP)

- In section 5.5 we will find power series solution in the neighborhood of a RSP for a given DE.

$$\underset{x_0 = 0}{\underset{\downarrow}{\text{RSP}}}$$

- Given a 2^{nd} order linear DE :

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \cdots \; *$$

- Assume the DE $*$ has a SP at x_0 $\quad (P(x_0)=0)$:

$\boxed{1}$ If P, Q, R are all poly., then x_0 is RSP if

$$\lim_{x \to x_0} (x - x_0)\, p(x) < \infty \quad \text{and}$$

$$\lim_{x \to x_0} (x - x_0)^2\, q(x) < \infty$$

$\boxed{2}$ If P, Q, R are functions more general than poly., then x_0 is RSP if

$$(x - x_0)\, p(x) \quad \text{and} \quad (x - x_0)^2\, q(x)$$

are analytic about x_0 (They have Taylor Series Expansion about x_0 with ρ s.t $|x - x_0| < \rho$)

Remark: If the Singular point x_0 is not regular, then x_0 is $\underbrace{\text{IRSP}}_{\text{Irregular Singular Point}}$.

Exp Determine the singular points of the following DE's and classify them into RSP or IRSP :

① $x^2 y'' + \alpha x y' + \beta y = 0$, α and β constants
"Euler DE"

$$\left.\begin{array}{l} P(x) = x^2 \\ Q(x) = \alpha x \\ R(x) = \beta \end{array}\right\} \text{All poly.} \quad \Rightarrow P(x) = 0 \iff \boxed{x_0 = 0} \text{ is SP}$$

Apply Ⅱ

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 0} x \frac{\alpha x}{x^2} = \alpha < \infty \checkmark \quad \text{and}$$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 0} x^2 \frac{\beta}{x^2} = \beta < \infty \checkmark$$

Hence, $x_0 = 0$ is RSP and so any Euler DE has a RSP at $x_0 = 0$

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② $(1 - x) y'' - 2x y' + 4y = 0$

$$\left.\begin{array}{l} P(x) = 1 - x \\ Q(x) = -2x \\ R(x) = 4 \end{array}\right\} \text{All poly.} \qquad P(x) = 0 \iff 1 - x = 0 \iff \boxed{x_0 = 1} \text{ is SP}$$

Apply Ⅱ $\Rightarrow \lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 1} (x - 1) \frac{-2x}{(1 - x)} = \lim_{x \to 1} 2x = 2 < \infty$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 1} (x - 1)^2 \frac{4}{(1 - x)} = \lim_{x \to 1} -4(x - 1) = 0 < \infty$$

Hence, $x_0 = 1$ is RSP

③ $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

$P(x) = 2x(x-2)^2$
$Q(x) = 3x$
$R(x) = (x-2)$
} All poly.

Apply ①

$P(x) = 0$
$2x(x-2) = 0$
$x_0 = 0$ and $x_0 = 2$ are SP's.

$\boxed{x_0 = 0}$

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 0} x \frac{3x}{2x(x-2)^2} = \frac{3}{2} \lim_{x \to 0} \frac{x}{(x-2)^2} = 0 < \infty$$

$$\lim_{x \to x_0} (x - x_0)^2 q(x) = \lim_{x \to 0} x^2 \frac{(x-2)}{2x(x-2)^2} = \frac{1}{2} \lim_{x \to 0} \frac{x}{x-2} = 0 < \infty$$

Hence, $x_0 = 0$ is RSP

$\boxed{x_0 = 2}$

$$\lim_{x \to x_0} (x - x_0) p(x) = \lim_{x \to 2} (x-2) \frac{3x}{2x(x-2)^2} = \frac{3}{2} \lim_{x \to 2} \frac{1}{x-2} \quad \text{DNE}$$

Hence, $x_0 = 2$ is IRSP.

since $$\lim_{x \to 2^+} \frac{1}{x-2} = \frac{1}{\text{small} +} = \infty$$

$$\lim_{x \to 2^-} \frac{1}{x-2} = \frac{1}{\text{small} -} = -\infty$$

④ $(x-\frac{\pi}{2})^2 y'' + \cos x\, y' + \sin x\, y = 0$

$P(x) = (x-\frac{\pi}{2})^2$
$Q(x) = \cos x$
$R(x) = \sin x$
} Not All Poly.

$\Leftarrow$ Apply $\boxed{2}$

$P(x) = 0$
$(x-\frac{\pi}{2})^2 = 0$
$\boxed{x_0 = \frac{\pi}{2}}$ is SP

$$(x-x_0)\,p(x) = (x-\tfrac{\pi}{2})\frac{\cos x}{(x-\frac{\pi}{2})^2} = \frac{\cos x}{x-\frac{\pi}{2}}$$

$$(x-x_0)^2 q(x) = (x-\tfrac{\pi}{2})^2 \frac{\sin x}{(x-\frac{\pi}{2})^2} = \sin x$$

} we find Taylor Series Expansion about $x_0 = \frac{\pi}{2}$

First we find Taylor series for $f(x) = \cos x$ about $x_0 = \frac{\pi}{2}$

$f(x) = \cos x \Rightarrow f(\frac{\pi}{2}) = 0$
$f'(x) = -\sin x \Rightarrow f'(\frac{\pi}{2}) = -1$
$f''(x) = -\cos x \Rightarrow f''(\frac{\pi}{2}) = 0$
$f'''(x) = \sin x \Rightarrow f'''(\frac{\pi}{2}) = 1$
$f^{(4)}(x) = \cos x \Rightarrow f^{(4)}(\frac{\pi}{2}) = 0$
$\vdots$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\tfrac{\pi}{2})^n$$

$$= f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2!}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{3!}(x-\tfrac{\pi}{2})^3 + \cdots$$

$$= 0 + (-1)(x-\tfrac{\pi}{2}) + 0 + \frac{1}{3!}(x-\tfrac{\pi}{2})^3 + 0 + \cdots$$

$$= -(x-\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \frac{(x-\frac{\pi}{2})^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

Hence, the Taylor series expansion for

$$\frac{\cos x}{x-\frac{\pi}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left(x-\frac{\pi}{2}\right)^{2n}}{(2n+1)!} = -1 + \frac{\left(x-\frac{\pi}{2}\right)^2}{3!} - \frac{\left(x-\frac{\pi}{2}\right)^4}{5!} + \cdots$$

<u>Second</u> we find Taylor series expansion for $g(x) = \sin x$ about $x_0 = \frac{\pi}{2}$

$$g(x) = \sin x \quad \Rightarrow g\left(\tfrac{\pi}{2}\right) = 1$$
$$g'(x) = \cos x \quad \Rightarrow g'\left(\tfrac{\pi}{2}\right) = 0$$
$$g''(x) = -\sin x \quad \Rightarrow g''\left(\tfrac{\pi}{2}\right) = -1$$
$$g'''(x) = -\cos x \quad \Rightarrow g'''\left(\tfrac{\pi}{2}\right) = 0$$
$$g^{(4)}(x) = \sin x \quad \Rightarrow g^{(4)}\left(\tfrac{\pi}{2}\right) = 1$$
$$\vdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{g^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x-\frac{\pi}{2}\right)^n = g\left(\tfrac{\pi}{2}\right) + g'\left(\tfrac{\pi}{2}\right)\left(x-\tfrac{\pi}{2}\right) + \frac{g''\left(\frac{\pi}{2}\right)}{2!}\left(x-\tfrac{\pi}{2}\right)^2 + \cdots$$

$$= 1 + 0 - \frac{1}{2!}\left(x-\tfrac{\pi}{2}\right)^2 + 0 + \frac{1}{4}\left(x-\tfrac{\pi}{2}\right)^4 + 0 + \cdots$$

$$= 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} - \frac{\left(x-\frac{\pi}{2}\right)^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$$

Hence, $x_0 = \frac{\pi}{2}$ is RSP