6.1 Laplace Transform

In this Chapter we will learn how to solve some DE's using new technique called Laplace Transform.

Question: Why Laplace Transform?

Answer:
- → To solve nonhomogeneous DE's for y_h and y_p in different way
- → To solve some DE's with discontinuity factors or external factors or forces
- → To solve some Integral Equations

Recall that the Improper Integral of $f(t)$ defined on an unbounded interval $[a, \infty)$ is defined by

$$\int_a^\infty f(t)\,dt = \lim_{b\to\infty} \int_a^b f(t)\,dt$$

This Improper Integral converges if ① $\int_a^b f(t)\,dt$ exists

and ② limit exists

To guarantee ①, we assume $f(t)$ Piecewise Continuous (PC)

Def The function $f(t)$ is PC on interval $I = (\alpha, \beta)$ if I can be partioned into small K sub-intervals

$$\alpha = t_0 < t_1 < \cdots < t_K = \beta$$

s.t:

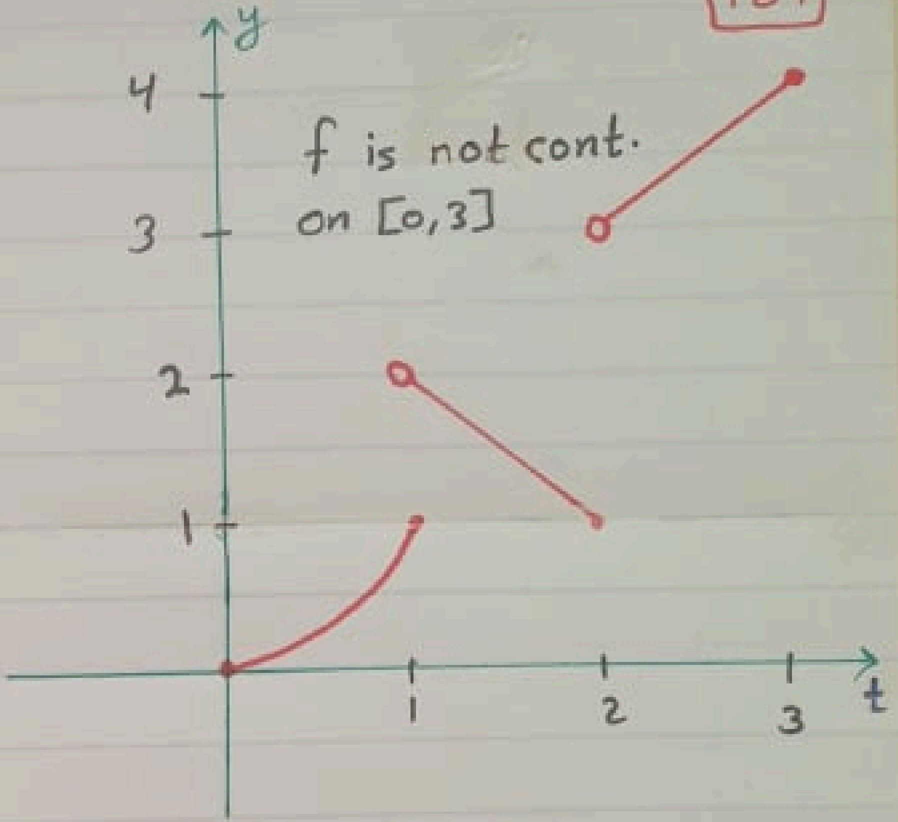
① $f(t)$ is cont. on each sub-interval and

② $f(t)$ has finite limit at the boundary of each sub-interval

Exp Show that

$$f(t)=\begin{cases} t^2, & 0\le t\le 1\\ 3-t, & 1<t\le 2\\ t+1, & 2<t\le 3\end{cases}$$

is PC on $[0,3]$



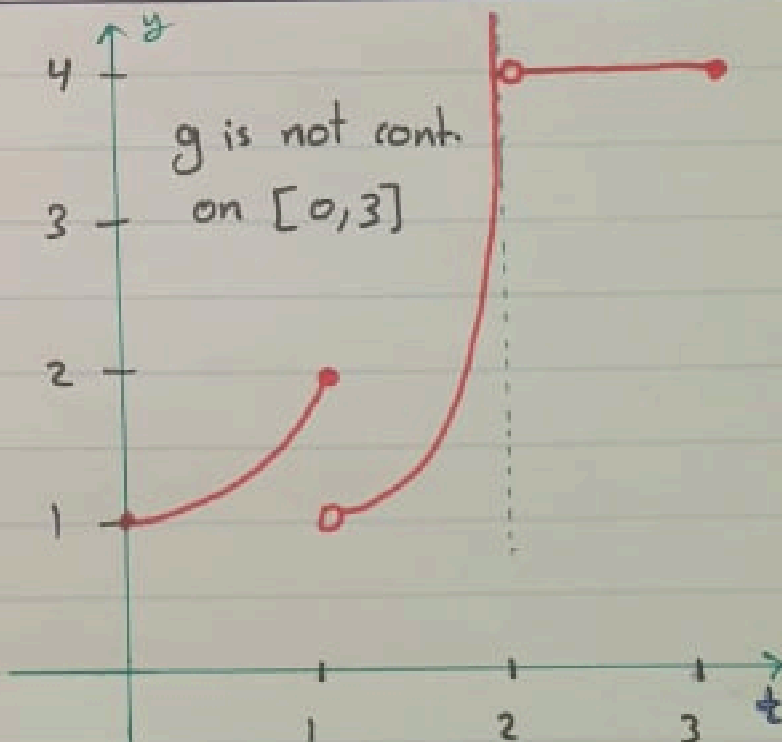
- f is cont. on each sub-interval: $(0,1)$, $(1,2)$, $(2,3)$
- $\lim\limits_{t\to0^+} f(t)=0$, $\lim\limits_{t\to1^-} f(t)=1$, $\lim\limits_{t\to1^+} f(t)=2$, $\lim\limits_{t\to2^-} f(t)=1$, $\lim\limits_{t\to2^+} f(t)=3$, $\lim\limits_{t\to3^-} f(t)=4$ } All finite

---

Exp Show that

$$g(t)=\begin{cases} t^2+1, & 0\le t\le 1\\ \dfrac{1}{2-t}, & 1<t< 2\\ 4, & 2<t\le 3\end{cases}$$

is not PC on $[0,3]$



- g is cont. on each subinterval $(0,1)$, $(1,2)$, $(2,3)$
- But $\lim\limits_{t\to2^-} g(t)=\infty$

Def The Laplace Transform of the function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s), \quad s \in \mathbb{R}^+$$

f is PC

$$\xrightarrow[f(t)]{\text{input}} \boxed{\mathcal{L}\{f(t)\}} \xrightarrow[F(s)]{\text{output}}$$ Laplace Transform (LT)

$$\xleftarrow[f(t)]{\text{output}} \boxed{\mathcal{L}^{-1}(F(s))} \xleftarrow[F(s)]{\text{input}}$$ Inverse Transform

---

Exp Find Laplace Transform of the following functions:

① $f(t) = c$, c is constant

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{c\} = \int_0^\infty e^{-st} c \, dt$$

$$= c \lim_{b\to\infty} \int_0^b e^{-st} dt = c \lim_{b\to\infty} \frac{-1}{s} e^{-st} \Big|_0^b$$

$$= \frac{-c}{s} \lim_{b\to\infty} \left[e^{-sb} - e^{0} \right] = \frac{-c}{s} \left[0 - 1 \right]$$

$$= \frac{c}{s}$$

$$= F(s)$$

Hence, $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{c}{s}\right) = c = f(t)$

Exp $L\{2\} = \frac{2}{s}$

$L\{\pi\} = \frac{\pi}{s}$

$L\{\sqrt{5}\} = \frac{\sqrt{5}}{s}$

$L^{-1}\left(\frac{e}{s}\right) = e$

$L^{-1}\left(\frac{\sqrt{3}}{2s}\right) = L^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{s}\right) = \frac{\sqrt{3}}{2}$

---

② $f(t) = t$

$$F(s) = L\{f(t)\} = L\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \lim_{b\to\infty} \int_0^b t\, e^{-st}\, dt$$

$$\begin{array}{lll} t & \xrightarrow{(+)} & e^{-st} \\ 1 & \xrightarrow{(-)} & -\frac{1}{s} e^{-st} \\ 0 & & \frac{1}{s^2} e^{-st} \end{array}$$

$$= \lim_{b\to\infty} \left[\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]\Bigg|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{s}\frac{b}{e^{sb}} - \frac{1}{s^2} e^{-sb} - \left(0 - \frac{1}{s^2}\right)\right]$$

$$= 0 - 0 + \frac{1}{s^2}$$

$$\lim_{b\to\infty} \frac{b}{e^{sb}} = \lim_{b\to\infty} \frac{1}{s e^{sb}} = 0$$

$$= \frac{1}{s^2}$$

Hence, $L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2}\right) = t = f(t)$

---

③ $f(t) = t^2$

$$F(s) = L\{f(t)\} = L\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt$$

$$= \lim_{b\to\infty} \int_0^b t^2 e^{-st}\, dt$$

t^2	$\xrightarrow{(+)}$	e^{-st}
$2t$	$\xrightarrow{(-)}$	$\frac{-1}{s} e^{-st}$
2	$\xrightarrow{(+)}$	$\frac{1}{s^2} e^{-st}$
0		$\frac{-1}{s^3} e^{-st}$

$$= \lim_{b\to\infty} \left[\frac{-t^2}{s} e^{-st} - \frac{2t}{s^2} e^{-st} - \frac{2}{s^3} e^{-st} \right]\Bigg|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-1}{s} \frac{b^2}{e^{sb}} - \frac{2}{s^2} \frac{b}{e^{sb}} - \frac{2}{s^3} e^{-sb} - \left(0 - 0 - \frac{2}{s^3}\right) \right]$$

$$= 0 - 0 - 0 + \frac{2}{s^3}$$

$$= \frac{2}{s^3}$$

Hence, $L^{-1}(F(s)) = L^{-1}\left(\frac{2}{s^3}\right) = t^2 = f(t)$

One can show that if $f(t) = t^n$ then $F(s) = \frac{n!}{s^{n+1}}$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Hence, $L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

Exp $L\{t\} = L\{t^1\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Exp $$L^{-1}\left(\frac{4}{s^3}\right) = \frac{4}{2!} L^{-1}\left(\frac{2!}{s^3}\right) = 2t^2$$

$$L^{-1}\left(\frac{7}{s^6}\right) = \frac{7}{5!} L^{-1}\left(\frac{5!}{s^6}\right) = \frac{7}{5!} t^5$$

$$L^{-1}\left(\frac{2s^2-4s}{s^3}\right) = L^{-1}\left(\frac{2}{s} - \frac{4}{s^2}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 4L^{-1}\left(\frac{1}{s^2}\right)$$
$$= 2(1) - (4)t$$
$$= 2 - 4t$$

---

(4) $f(t) = c_1 f_1(t) + c_2 f_2(t)$

$$F(s) = L\{f(t)\} = L\{c_1 f_1(t) + c_2 f_2(t)\}$$
$$= \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$
$$= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$
$$= c_1 F_1(s) + c_2 F_2(s)$$

Hence, $$L^{-1}\left(c_1 F_1(s) + c_2 F_2(s)\right) = c_1 L^{-1}\left(F_1(s)\right) + c_2 L^{-1}\left(F_2(s)\right)$$
$$= c_1 f_1(t) + c_2 f_2(t)$$

Exp $$L\{2 + 3t^4\} = L\{2\} + 3L\{t^4\}$$
$$= \frac{2}{s} + 3\,\frac{4!}{s^5}$$

(5) $f(t) = e^{at}$, $a \in \mathbb{R}$

$$F(s) = L\{f(t)\} = L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{(a-s)t}\, dt = \lim_{b \to \infty} \int_0^b e^{-(s-a)t}\, dt$$

$$= \lim_{b \to \infty} \frac{-1}{s-a} e^{-(s-a)t} \Big|_0^b$$

$$= \frac{1}{s-a} \lim_{b \to \infty} \left[- e^{-(s-a)b} + 1 \right]$$

$$= \frac{1}{s-a}$$

Hence, $L^{-1}\left(\frac{1}{s-a}\right) = L^{-1}\left(L\{e^{at}\}\right) = e^{at} = f(t)$

Exp $L\{e^{2t}\} = \frac{1}{s-2}$

$$L\{e^{-et}\} = \frac{1}{s--e} = \frac{1}{s+e}$$

$$L\left\{\frac{1}{e^{5t}}\right\} = L\{e^{-5t}\} = \frac{1}{s+5}$$

$$L^{-1}\left(\frac{2}{s-\sqrt{\pi}}\right) = 2L^{-1}\left(L\{e^{\sqrt{\pi}t}\}\right) = 2e^{\sqrt{\pi}t}$$

$$L^{-1}\left(\frac{s-2}{s^2-4}\right) = L^{-1}\left(\frac{s-2}{(s-2)(s+2)}\right) = L^{-1}\left(\frac{1}{s+2}\right) = L^{-1}\left(L\{e^{-2t}\}\right) = e^{-2t}$$

---

⑥ $f(t) = \sin at$

$$F(s) = L\{f(t)\} = L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st} \sin at \, dt$$

$$\begin{array}{lcl} e^{-st} & \xrightarrow{(+)} & \sin at \\ -se^{-st} & \xrightarrow{(-)} & -\frac{1}{a}\cos at \\ s^2 e^{-st} & \xrightarrow{\int} & \frac{-1}{a^2}\sin at \end{array}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{a} e^{-st} \cos at - \frac{s}{a^2} e^{-st} \sin at \right]_0^b - \frac{s^2}{a^2} \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[(0 - 0) - \left(\frac{-1}{a} - 0\right) \right] - \frac{s^2}{a^2} F(s)$$

$$F(s) + \frac{s^2}{a^2} F(s) = \frac{1}{a}$$

$$F(s)\left(1 + \frac{s^2}{a^2}\right) = \frac{1}{a} \Rightarrow F(s)\left(\frac{s^2+a^2}{a^2}\right) = \frac{1}{a}$$

$$F(s) = \frac{a}{s^2+a^2}$$

Hence, $L^{-1}\left(\frac{a}{s^2+a^2}\right) = L^{-1}\left(L\{\sin at\}\right) = \sin at$

Exp $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{\sin \sqrt{2}t\} = \frac{\sqrt{2}}{s^2+2}$

$$L^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2} L^{-1}\left(\frac{2}{s^2+4}\right) = \frac{1}{2} L^{-1}\left(L\{\sin 2t\}\right) = \frac{1}{2}\sin 2t$$

$$L^{-1}\left(\frac{3}{s^2+7}\right) = \frac{3}{\sqrt{7}} L^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{3}{\sqrt{7}} L^{-1}\left(L\{\sin\sqrt{7}t\}\right) = \frac{3}{\sqrt{7}} \sin\sqrt{7}\, t$$

⑦ $f(t) = \cos at$

$$F(s) = L\{f(t)\} = L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt = \frac{s}{s^2+a^2}$$

we do same work as in ⑥

Hence, $L^{-1}\left(\frac{s}{s^2+a^2}\right) = L^{-1}\left(L\{\cos at\}\right) = \cos at$

Exp $L\{\cos 4t\} = \frac{s}{s^2+16}$

$L\{3\cos 2t\} = 3 \frac{s}{s^2+4} = \frac{3s}{s^2+4}$

$L\{\cos e\} = \frac{\cos e}{s}$ since $\cos e$ is number

$$L^{-1}\left(\frac{6s}{s^2+1}\right) = 6\, L^{-1}\left(L\{\cos t\}\right) = 6\cos t$$

$$L^{-1}\left(\frac{2s-4}{s^2+3}\right) = L^{-1}\left(\frac{2s}{s^2+3}\right) - L^{-1}\left(\frac{4}{s^2+3}\right)$$

$$= 2\, L^{-1}\left(\frac{s}{s^2+3}\right) - \frac{4}{\sqrt{3}} L^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)$$

$$= 2\, L^{-1}\left(L\{\cos\sqrt{3}\,t\}\right) - \frac{4}{\sqrt{3}} L^{-1}\left(L\{\sin\sqrt{3}\,t\}\right)$$

$$= 2\cos\sqrt{3}\,t - \frac{4}{\sqrt{3}}\sin\sqrt{3}\,t$$

Exp $L\{2\sin 3t - 10t^2 + 5e^{-3t} + \pi + 2\cos\sqrt{7}\,t\}$

$$= 2\left(\frac{3}{s^2+9}\right) - 10\left(\frac{2!}{s^3}\right) + 5\left(\frac{1}{s+3}\right) + \frac{\pi}{s} + 2\left(\frac{s}{s^2+7}\right)$$

Exp Find Laplace Inverse of $F(s) = \dfrac{1}{s^2 - 5s + 6}$

$$L^{-1}\left(F(s)\right) = L^{-1}\left(\frac{1}{s^2 - 5s + 6}\right) = L^{-1}\left(\frac{1}{(s-2)(s-3)}\right)$$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s-3}\right)$$

using Partial Fraction

$$A = \frac{1}{\boxed{2} - 3} = -1$$

$$B = \frac{1}{\boxed{3} - 2} = 1$$

$$= L^{-1}\left(\frac{-1}{s-2} + \frac{1}{s-3}\right)$$

$$= -L^{-1}\left(L\{e^{2t}\}\right) + L^{-1}\left(L\{e^{3t}\}\right)$$

$$= -e^{2t} + e^{3t} = f(t)$$

Exp Find Laplace Inverse of $G(s) = \dfrac{15 - s}{s^2 + 5s}$

$$g(t) = L^{-1}\left(G(s)\right) = L^{-1}\left(\frac{15 - s}{s^2 + 5s}\right) = L^{-1}\left(\frac{15 - s}{s(s+5)}\right)$$

$$= L^{-1}\left(\frac{A}{s} + \frac{B}{s+5}\right)$$

using Partial Fraction

$$A = \frac{15 - \boxed{0}}{\boxed{0} + 5} = 3$$

$$B = \frac{15 - \boxed{-5}}{\boxed{-5}} = \frac{+20}{-5} = -4$$

$$= L^{-1}\left(\frac{3}{s} + \frac{-4}{s+5}\right)$$

$$= L^{-1}\left(\frac{3}{s}\right) + -4\, L^{-1}\left(\frac{1}{s - -5}\right)$$

$$= 3 + -4\, e^{-5t}$$

(8) $f(t) = \sinh at$

$$F(s) = L\{f(t)\} = L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\,\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}$$

$$= \frac{a}{s^2-a^2}$$

Hence, $L^{-1}\left(\frac{a}{s^2-a^2}\right) = L^{-1}\left(L\{\sinh at\}\right) = \sinh at$

(9) $f(t) = \cosh at$

$$F(s) = L\{f(t)\} = L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\,\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2} = \frac{s}{s^2-a^2}$$

Hence, $L^{-1}\left(\frac{s}{s^2-a^2}\right) = L^{-1}\left(L\{\cosh at\}\right) = \cosh at$

Exp Find $h(t)$ if $H(s) = \frac{2s-12}{s^2-6}$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{2s-12}{s^2-6}\right) = 2L^{-1}\left(\frac{s}{s^2-6}\right) - \frac{12}{\sqrt{6}}L^{-1}\left(\frac{\sqrt{6}}{s^2-6}\right)$$

$$= 2\cosh\sqrt{6}\,t - \frac{12}{\sqrt{6}}\sinh\sqrt{6}\,t$$

Exp Find Laplace Transform of

(A) $f(t) = 2\sinh 7t \Rightarrow F(s) = (2)\dfrac{7}{s^2-49} = \dfrac{14}{s^2-49}$

(B) $r(t) = 1 - 3\cosh t \Rightarrow R(s) = \dfrac{1}{s} - \dfrac{3s}{s^2-1}$

(C) $h(t) = \begin{cases} 1 & , 0 \le t \le 1 \\ 5 & , \quad t = 1 \\ 0 & , \quad t > 1 \end{cases}$

$$H(s) = L\{h(t)\} = \int_0^\infty e^{-st}\, h(t)\, dt$$

$$= \int_0^1 e^{-st}(1)\, dt + \int_1^1 e^{-st}(5)\, dt + \int_1^\infty e^{-st}(0)\, dt$$

$$= -\frac{1}{s} e^{-st}\Big|_0^1 + 0 + 0$$

$$= -\frac{1}{s}\left(e^{-s} - e^{0}\right)$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

(D) $s(t) = -2 + t^3 - e^{-t} + 2\sin\frac{t}{2} - \cos 8t + \sinh\sqrt{10}t$

$$S(s) = L\{s(t)\} = L\{-2\} + L\{t^3\} - L\{e^{-t}\} + 2L\{\sin\tfrac{t}{2}\} - L\{\cos 8t\} + L\{\sinh\sqrt{10}t\}$$

$$= \frac{-2}{s} + \frac{3!}{s^4} - \frac{1}{s+1} + (2)\frac{\frac{1}{2}}{s^2+\frac{1}{4}} - \frac{s}{s^2+64} + \frac{\sqrt{10}}{s^2-10}$$

Summary

$$L\{f(t)\} = F(s) \Rightarrow L^{-1}(F(s)) = f(t)$$

$$L\{c\} = \frac{c}{s} \Rightarrow L^{-1}\left(\frac{c}{s}\right) = c$$

$$L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$$

$$L\{e^{at}\} = \frac{1}{s-a} \Rightarrow L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

$$L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$$

$$L\{y'\} = s\,Y(s) - y_0$$

$$L\{y''\} = s^2\,Y(s) - s\,y_0 - y_0'$$

$$L\{y'''\} = s^3\,Y(s) - s^2 y_0 - s\,y_0' - y_0''$$