6.2 Solving IVP's Using LT

To solve Initial Value Problems using Laplace Transform we still need to find $L\{y'\}, L\{y''\}, \dots, L\{y^{(n)}\}$

Th 6.2.1

Assume $f(t)$ is cont. and $f'(t)$ is PC on $0 \le t \le b$.
Then

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

That is
$$F'(s) = s\,F(s) - f_0$$

Proof

$$L\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)\,dt = \lim_{b\to\infty} \int_0^{b} e^{-st} f'(t)\,dt$$

Since f' is PC on $0 \le t \le b \Rightarrow$
f' is cont. on the sub-intervals $0 < t_1 < t_2 < \dots < t_n = b$

$$L\{f'(t)\} = \lim_{b\to\infty}\left[\int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \dots + \int_{t_{n-1}}^{b=t_n} e^{-st} f'(t)\,dt\right]$$

$u = e^{-st}$ $\quad dv = f'(t)$

$du = -se^{-st}$ $\quad v = f(t)$

$$= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \dots + e^{-st} f(t)\Big|_{t_{n-1}}^{b}\right.$$

$$\left. + s\left(\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \dots + \int_{t_{n-1}}^{b} e^{-st} f(t)\,dt\right)\right]$$

since f is cont. on $[0, b] \Rightarrow$

$$L\{f'(t)\} = \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^{b} + s\int_0^{b} e^{-st} f(t)\,dt\right]$$

$$= \lim_{b\to\infty}\left(e^{-sb} f(b) - e^{0} f(0)\right) + s\int_0^{\infty} e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = 0 - f(0) + s\, L\{f(t)\}$$
$$= s\,F(s) - f_0$$

Exp Show that

$$L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0)$$

Since $L\{y'\} = s\,L\{y\} - y(0)$

$$\Rightarrow \quad L\{y''\} = s\,L\{y'\} - y'(0)$$
$$= s\left(s\,L\{y\} - y(0)\right) - y'(0)$$
$$= s^2 L\{y\} - s\,y(0) - y'(0)$$

Similarly one can show that

$$L\{y^{(n)}\} = s^n L\{y\} - s^{n-1} y(0) - s^{n-2} y'(0) - \dots - s\,y^{(n-2)}(0) - y^{(n-1)}(0)$$

Exp ① $L\{y'\} = s\,L\{y\} - y(0)$

② $L\{y''\} = s^2 L\{y\} - s\,y(0) - y'(0)$

③ $L\{y'''\} = s^3 L\{y\} - s^2 y(0) - s\,y'(0) - y''(0)$

④ $L\{y^{(4)}\} = s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0)$

$\vdots$

Note $L\{y\} = Y(s)$, $L\{y'\} = Y'(s)$, ...

Exp Use Laplace Transform to solve the IVP:

$$y'' - y' - 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$$

$$L\{y''\} - L\{y'\} - L\{2y\} = L\{0\}$$

$$\left(s^2 L\{y\} - s\,y(0) - y'(0)\right) - \left(s\,L\{y\} - y(0)\right) - 2L\{y\} = \frac{0}{s}$$

$$(s^2 - s - 2)\,L\{y\} - s + 1 = 0$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

The unknow is $y(t)$ and not $Y(s)$ so we take inverse

$$y(t) = L^{-1}\left(Y(s)\right) = L^{-1}\left(\frac{s-1}{s^2-s-2}\right) = L^{-1}\left(\frac{s-1}{(s-2)(s+1)}\right)$$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s+1}\right) \quad \text{where} \quad A = \frac{\boxed{2}-1}{\boxed{2}+1} = \frac{1}{3}$$

$$B = \frac{\boxed{-1}-1}{\boxed{-1}-2} = \frac{2}{3}$$

$$= L^{-1}\left(\frac{\frac{1}{3}}{s-2}\right) + L^{-1}\left(\frac{\frac{2}{3}}{s+1}\right)$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Note that $r^2 - r - 2 = 0$

$$(r-2)(r+1) = 0$$

$$r_1 = 2 \,,\; r_2 = -1$$

$$y_1 = e^{2t} \,,\; y_2 = e^{-t}$$

$$y(t) = c_1 e^{2t} + c_2 e^{-t}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

To find c_1 and c_2 $\Rightarrow$

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - c_2 = 0$$

$$c_1 = \frac{1}{3}$$

$$c_2 = \frac{2}{3}$$

Exp Use LT to solve the IVP:

$$y'' + y = \sin 2t \;,\quad y(0) = 2 \;,\quad y'(0) = 1$$

$$L\{y''\} + L\{y\} = L\{\sin 2t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = \frac{2}{s^2+4}$$

$$L\{y\}(s^2+1) - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s)\,(s^2+1) = 2s + 1 + \frac{2}{s^2+4} = \frac{2s^3 + s^2 + 8s + 4 + 2}{s^2+4}$$

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)}$$

$$y(t) = L^{-1}\left(\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}\right) = L^{-1}\left(\frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}\right)$$

using partial fraction $\Rightarrow a = 2,\ b = \frac{5}{3},\ c = 0,\ d = \frac{-2}{3}$

$$= L^{-1}\left(\frac{2s+\frac{5}{3}}{s^2+1}\right) + L^{-1}\left(\frac{-\frac{2}{3}}{s^2+4}\right)$$

$$= 2\,L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= \underbrace{2\cos t + \frac{5}{3}\sin t}_{y_h(t)} \underbrace{- \frac{1}{3}\sin 2t}_{y_p(t)}$$

$y_h(t) : \Rightarrow r^2 + 1 = 0 \quad \Rightarrow r_{1,2} = \pm i \quad \Rightarrow y_1 = \cos t \,,\ y_2 = \sin t$

$\Rightarrow y_h(t) = c_1 \cos t + c_2 \sin t$

$y_p(t) = A\sin 2t + B\cos 2t \quad \Rightarrow$ substitute y_p, y_p'' in the DE $\Rightarrow A = -\frac{1}{3}$, $B = 0$

$= -\frac{1}{3}\sin 2t$

Exp Use Laplace Transform to solve the IVP:

$$y^{(4)} - y = 0 \ , \ y(0) = y''(0) = y'''(0) = 0 \ , \ y'(0) = 1$$

$$L\{y^{(4)}\} - L\{y\} = L\{0\}$$

$$s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - L\{y\} = \frac{0}{s}$$

$$(s^4 - 1) L\{y\} - s^2 = 0$$

$$L\{y\} = \frac{s^2}{s^4 - 1}$$

$$y(t) = L^{-1}\left(\frac{s^2}{s^4 - 1}\right) = L^{-1}\left(\frac{s^2}{(s^2-1)(s^2+1)}\right)$$

using partial fraction $\Rightarrow$

$a = 0$

$b = \frac{1}{2}$

$c = 0$

$d = \frac{1}{2}$

$$= L^{-1}\left(\frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}\right)$$

$$= L^{-1}\left(\frac{\frac{1}{2}}{s^2-1}\right) + L^{-1}\left(\frac{\frac{1}{2}}{s^2+1}\right)$$

$$= \frac{1}{2}\sinh t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(\frac{e^t - e^{-t}}{2}\right) + \frac{1}{2}\sin t$$

$$= \frac{1}{4}e^t - \frac{1}{4}e^{-t} + \frac{1}{2}\sin t$$

Note that $r^4 - 1 = 0 \Rightarrow (r^2-1)(r^2+1) = 0 \Rightarrow (r-1)(r+1)(r^2+1) = 0$

$\Rightarrow r_1 = 1, \ r_2 = -1, \ r_{3,4} = \pm i$

$\Rightarrow y_1 = e^t, \ y_2 = e^{-t}, \ y_3 = \cos t, \ y_4 = \sin t$

$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \Rightarrow c_1 = \frac{1}{4}, c_2 = \frac{-1}{4}, c_3 = 0, c_4 = \frac{1}{2}$

(A)
Remark

To find Laplace Transform for product of two functions (one is exponensial), we use shifting.

1st shifting

$$L\{e^{at} f(t)\} = F(s-a)$$

where $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Hence, $e^{at} f(t) = L^{-1}(F(s-a))$

Proof

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a)$$

Exp Find ① $L\{e^{2t} \sin t\} = F(s-2) = \frac{1}{(s-2)^2+1}$

where $F(s) = L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

$$= \frac{1}{s^2-4s+4+1}$$

$$= \frac{1}{s^2-4s+5}$$

② $L\{e^{-3t} \cos et\} = F(s+3) = \frac{s+3}{(s+3)^2+e^2}$

where $F(s) = L\{f(t)\} = L\{\cos et\}$

$$= \frac{s}{s^2+e^2}$$

(3) $L\{e^t t^2\} = F(s-1) = \frac{2}{(s-1)^3}$

where $F(s) = L\{f(t)\} = L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

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(4) $L^{-1}\left(\frac{s}{(s-2)^2+9}\right) = L^{-1}\left(\frac{s-2+2}{(s-2)^2+9}\right)$

$= L^{-1}\left(\frac{s-2}{(s-2)^2+3^2}\right) + \frac{2}{3} L^{-1}\left(\frac{3}{(s-2)^2+3^2}\right)$

$= e^{2t}\cos 3t + \frac{2}{3} e^{2t}\sin 3t$

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(5) $L^{-1}\left(\frac{2s-1}{s^2+2s+5}\right) = L^{-1}\left(\frac{2s-1}{(s+1)^2+4}\right) = L^{-1}\left(\frac{2(s+1)-3}{(s+1)^2+4}\right)$

$= 2 L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - \frac{3}{2} L^{-1}\left(\frac{2}{(s+1)^2+4}\right)$

$= 2e^{-t}\cos 2t - \frac{3}{2} e^{-t}\sin 2t$

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(6) $L^{-1}\left(\frac{-10}{(s+1)^3}\right) = -5 L^{-1}\left(\frac{2!}{(s+1)^3}\right) = -5e^{-t}t^2$

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(7) $L^{-1}\left(\frac{s}{s^2+s-1}\right) = L^{-1}\left(\frac{s}{(s+\frac{1}{2})^2-\frac{5}{4}}\right) = L^{-1}\left(\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right)$

$= L^{-1}\left(\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right) - \frac{1}{\sqrt{5}} L^{-1}\left(\frac{\frac{\sqrt{5}}{2}}{(s+\frac{1}{2})^2-\frac{5}{4}}\right)$

$= e^{-\frac{1}{2}t}\cosh\frac{\sqrt{5}}{2}t - \frac{1}{\sqrt{5}} e^{-\frac{1}{2}t}\sinh\frac{\sqrt{5}}{2}t$

(8) inverse transform of $H(s) = \dfrac{4s-10}{s^2-6s+10}$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{4s-10}{(s-3)^2+1}\right) = L^{-1}\left(\frac{4(s-3)+2}{(s-3)^2+1}\right)$$

$$= 4L^{-1}\left(\frac{s-3}{(s-3)^2+1}\right) + 2L^{-1}\left(\frac{1}{(s-3)^2+1}\right)$$

$$= 4e^{3t}\cos t + 2e^{3t}\sin t$$

Exp Solve the following IVP using Laplace Transform:

$$y'' - 8y' + 25y = 0 \quad , \quad y(0) = 0 \; , \; y'(0) = 6$$

$$L\{y''\} - 8L\{y'\} + 25L\{y\} = L\{0\}$$

$$s^2L\{y\} - s\,y(0) - y'(0) - 8\left(sL\{y\} - y(0)\right) + 25L\{y\} = 0$$

$$(s^2 - 8s + 25)L\{y\} - 6 = 0$$

$$L\{y\} = \frac{6}{s^2-8s+25}$$

$$y(t) = L^{-1}(L\{y\}) = L^{-1}\left(\frac{6}{(s-4)^2+9}\right) = 2L^{-1}\left(\frac{3}{(s-4)^2+9}\right)$$

$$= 2e^{4t}\sin 3t$$

(B)
Remark
To find Laplace Transform for product of two functions (one is poly. t^n), we use derivatives.

Exp show that
$$L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

where $F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Proof

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$F'(s) = \frac{dF}{ds} = -\int_0^\infty t\, e^{-st} f(t)\,dt = (-1)^1 L\{t^1 f(t)\}$$

$$F''(s) = \frac{d^2F}{ds^2} = \int_0^\infty t^2 e^{-st} f(t)\,dt = (-1)^2 L\{t^2 f(t)\}$$

$$\vdots$$

$$F^{(n)}(s) = \frac{d^nF}{ds^n} = (-1)^n L\{t^n f(t)\}$$

$$\frac{1}{(-1)^n} F^{(n)}(s) = L\{t^n f(t)\}$$

$$(-1)^n F^{(n)}(s) = L\{t^n f(t)\}$$

Exp Find Laplace Transform of

① $h(t) = t \sin t$

$$H(s) = \mathcal{L}\{t \sin t\} = (-1)^{1} F'(s) = (-1)(-1)(s^2+1)^{-2}(2s) = \frac{2s}{(s^2+1)^2}$$

where $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1} = (s^2+1)^{-1}$

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② $h(t) = \mathcal{L}\{t^2 \sin t\}$

$$H(s) = \mathcal{L}\{t^2 \sin t\} = (-1)^2 F''(s) = F''(s) = \frac{6s^2-2}{(s^2+1)^3}$$

where $F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1} = (s^2+1)^{-1}$

$$F'(s) = (-1)(s^2+1)^{-2}(2s)$$
$$= (-2s)(s^2+1)^{-2}$$

$$F''(s) = (-2s)(-2)(s^2+1)^{-3}(2s) + (s^2+1)^{-2}(-2)$$

$$= (s^2+1)^{-2}\left(8s^2(s^2+1)^{-1} + (-2)\right)$$

$$= \frac{-2}{(s^2+1)^2}\left(1 - \frac{4s^2}{s^2+1}\right)$$

$$= \left(\frac{-2}{(s^2+1)^2}\right)\left(\frac{s^2+1-4s^2}{s^2+1}\right)$$

$$= \frac{6s^2-2}{(s^2+1)^3}$$

③ $h(t) = t^3 e^{4t}$

solution 1 $\quad H(s) = L\{h(t)\} = L\{t^3 e^{4t}\} = (-1)^3 F'''(s)$

where $F(s) = L\{f(t)\}$

$= L\{e^{4t}\}$

$= \dfrac{1}{s-4}$

$= (s-4)^{-1}$

$F'(s) = -(s-4)^{-2}$

$F''(s) = 2(s-4)^{-3}$

$F'''(s) = -6(s-4)^{-4}$

$= \dfrac{-6}{(s-4)^4}$

$H(s) = (-1)^3 F'''(s)$

$= -F'''(s)$

$= \dfrac{6}{(s-4)^4}$

solution 2 $\quad H(s) = L\{h(t)\} = L\{e^{4t} t^3\} = F(s-4)$

where $F(s) = L\{f(t)\} = L\{t^3\}$

$= \dfrac{3!}{s^4}$

$= \dfrac{6}{(s-4)^4}$

④ $h(t) = 2t\, e^{-t} \cosh t$

$H(s) = L\{h(t)\} = L\left\{2t\, e^{-t}\left(\dfrac{e^{t} + e^{-t}}{2}\right)\right\} = L\{t e^{-t}(e^{t} + e^{-t})\}$

$= L\{t + t e^{-2t}\} = L\{t\} + L\{t e^{-2t}\}$

$= \dfrac{1}{s^2} + (-1)^1 F'(s)$

$= \dfrac{1}{s^2} + \dfrac{1}{(s+2)^2}$

where $F(s) = L\{e^{-2t}\}$

$= \dfrac{1}{s+2}$

$F'(s) = \dfrac{-1}{(s+2)^2}$

Exp Find $L\{t\, e^{t} \cos 3t\}$

$$L\{t\, e^{t} \cos 3t\} = (-1)^{1} F'(s) = -F'(s) = \frac{(s-1)^2 - 9}{[(s-1)^2 + 9]^2}$$

where $F(s) = L\{e^{t} \cos 3t\}$

$$= G(s-1) \quad \text{where } G(s) = L\{\cos 3t\}$$

$$= \frac{s-1}{(s-1)^2 + 9} \qquad = \frac{s}{s^2 + 9}$$

$$F'(s) = \frac{((s-1)^2 + 9)(1) - (s-1)(2)(s-1)}{[(s-1)^2 + 9]^2}$$

$$= \frac{9 - (s-1)^2}{((s-1)^2 + 9)^2}$$