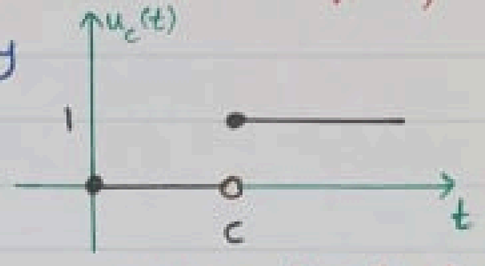
6.3 Step Functions

Def The unit step function (Heaviside Function) $u_c(t)$, $c>0$, is defined by

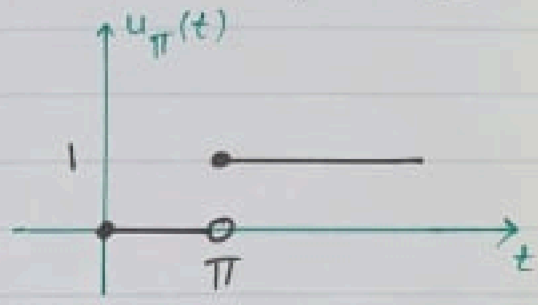
$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & c \le t \end{cases}$$



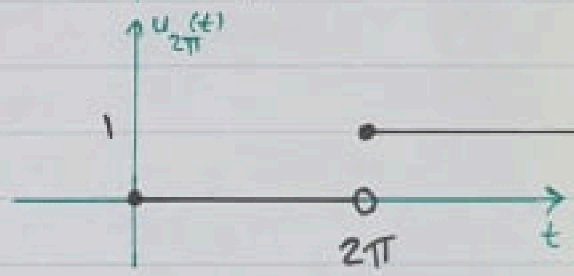
$u_c(t)$ is discont at c

Exp sketch ① $u_\pi(t)$ ② $u_{2\pi}(t)$ ③ $h(t) = u_\pi(t) - u_{2\pi}(t)$

① $$u_\pi(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t \end{cases}$$

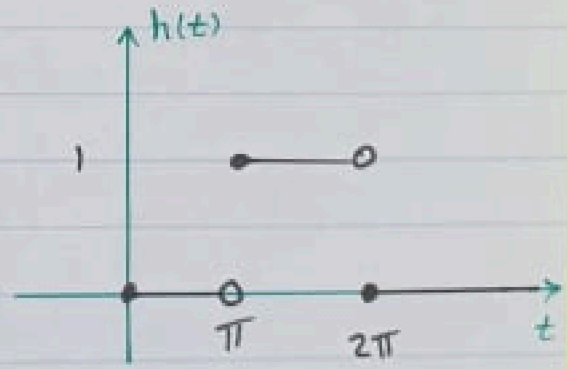


② $$u_{2\pi}(t) = \begin{cases} 0, & 0 \le t < 2\pi \\ 1, & 2\pi \le t \end{cases}$$



③ $$h(t) = u_\pi(t) - u_{2\pi}(t)$$

$$= \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & 2\pi \le t \end{cases}$$



Exp Find $f(2)$ if $f(t) = t\,u_2(t) - 3u_3(t) + 2u_1(t) + t^3 u_5(t)$

$$f(2) = (2)u_2(2) - 3u_3(2) + 2u_1(2) + (8)u_5(2)$$

$$= (2)(1) - 3(0) + 2(1) + (8)(0)$$

$$= 4$$

Exp Express the following function interms of $u_c(t)$

$$f(t)=\begin{cases} 2 & , \ 0\le t<1 \\ -1 & , \ 1\le t<2 \\ 2 & , \ 2\le t<3 \\ -1 & , \ 3\le t<4 \\ 0 & , \ 4\le t \end{cases}$$

$$f(t) = 2 + (-1-2)\,u_1(t) + (2--1)\,u_2(t) + (-1-2)\,u_3(t) + (0--1)\,u_4(t)$$

$$f(t) = 2 - 3u_1(t) + 3u_2(t) - 3u_3(t) + u_4(t)$$

$$f\left(\tfrac{5}{2}\right) = 2 - 3u_1\left(\tfrac{5}{2}\right) + 3u_2\left(\tfrac{5}{2}\right) - 3u_3\left(\tfrac{5}{2}\right) + u_4\left(\tfrac{5}{2}\right)$$

2 $= 2 - 3(1) + 3(1) - 3(0) + (0)$

$= 2$

Exp show that $L\{u_c(t)\} = \frac{e^{-cs}}{s}$

$$u_c(t) = \begin{cases} 0, & 0\le t<c \\ 1, & c\le t \end{cases}$$

$$L\{u_c(t)\} = \int_0^\infty e^{-st}\, u_c(t)\,dt$$

$$= \int_0^c e^{-st}(0)\,dt + \int_c^\infty e^{-st}(1)\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty} \frac{-1}{s} e^{-st}\Big|_c^b$$

$$= \frac{-1}{s} \lim_{b\to\infty} \left[e^{-sb} - e^{-cs}\right] = \frac{-1}{s}\left(0 - e^{-cs}\right) = \frac{e^{-cs}}{s}$$

Hence, $L^{-1}\left(\frac{e^{-cs}}{s}\right) = u_c(t)$

Exp Find

① $L\{u_3(t)\} = \frac{e^{-3s}}{s}$

② $L\{1 - u_e(t)\} = L\{1\} - L\{u_e(t)\} = \frac{1}{s} - \frac{e^{-es}}{s}$

③ $L^{-1}\left(\frac{4e^{-2s}}{7s}\right) = \frac{4}{7} L^{-1}\left(\frac{e^{-2s}}{s}\right) = \frac{4}{7} u_2(t)$

④ $L^{-1}\left(\frac{se^{-\pi s} + 6s}{s^2}\right) = L^{-1}\left(\frac{e^{-\pi s}}{s} + \frac{6}{s}\right) = L^{-1}\left(\frac{e^{-\pi s}}{s}\right) + L^{-1}\left(\frac{6}{s}\right)$

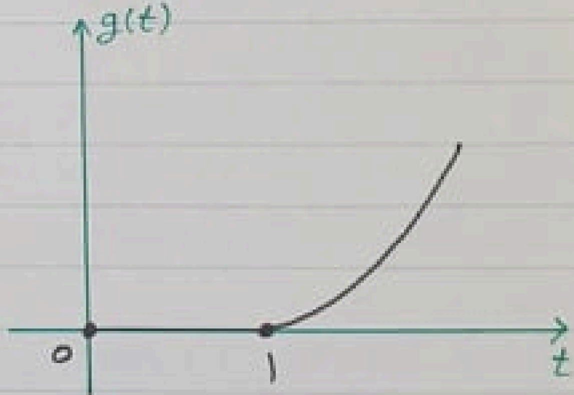
$= u_\pi(t) + 6$

Exp Let $f(t) = t^2$, $t \geq 0$.
sketch the graph of $g(t) = f(t-1)\, u_1(t)$

$g(t) = (t-1)^2\, u_1(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (t-1)^2, & 1 \leq t \end{cases}$

Note that $g(t)$ is cont.

on $[0, \infty)$



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Remark: To find Laplace Transform for product of two functions: one is $u_c(t)$ $\Rightarrow$ first we shift the other one by c to the right.

Exp Show that

$$L\{u_c(t)\, f(t-c)\} = e^{-cs}\, L\{f(t)\}$$
$$= e^{-cs}\, F(s)$$

Hence, $L^{-1}\left(e^{-cs} F(s)\right) = u_c(t)\, f(t-c)$

Proof

$$L\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\, dt$$

$$= \int_0^c e^{-st} (0) f(t-c)\, dt + \int_c^\infty e^{-st} (1) f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

$u = t - c$
$du = dt$

$$= \int_0^\infty e^{-s(u+c)} f(u)\, du$$

$t = c \Rightarrow u = 0$
$t \to \infty \Rightarrow u \to \infty$

$$= e^{-cs} \int_0^\infty e^{-su} f(u)\, du$$

$$= e^{-cs}\, L\{f(u)\}$$

$$= e^{-cs}\, F(s)$$

Exp Find ① $L\{u_2(t)\,(t-2)\}$ $\quad f(t)=t$, $f(t-2)=t-2$

$$= e^{-2s}\, L\{t\}$$

$$= e^{-2s}\,\frac{1}{s^2} = \frac{e^{-2s}}{s^2}$$

Note that $L^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u_2(t)\,(t-2)$

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② $L\{u_2(t)\,(t-1)\}$ $\quad f(t)=t$, $f(t-2)=t-2$

$$= L\{u_2(t)\,((t-2)+1)\}$$

$$= L\{u_2(t)\,(t-2) + u_2(t)\}$$

$$= L\{u_2(t)\,(t-2)\} + L\{u_2(t)\}$$

$$= \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$$

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③ $L\{u_2(t)\,t^2\}$ $\quad f(t)=t^2$ $\quad f(t)=t^2$, $f(t-2)=(t-2)^2$

$$= L\{u_2(t)\,(t-2+2)^2\}$$

$$= L\{u_2(t)\,((t-2)^2 + 4(t-2) + 4)\}$$

$$= L\{u_2(t)(t-2)^2\} + 4\,L\{u_2(t)(t-2)\} + 4\,L\{u_2(t)\}$$

$$= e^{-2s}\,L\{t^2\} + 4\,e^{-2s}\,L\{t\} + 4\,\frac{e^{-2s}}{s}$$

$$= e^{-2s}\left(\frac{2}{s^3}\right) + 4\,e^{-2s}\left(\frac{1}{s^2}\right) + 4\,\frac{e^{-2s}}{s}$$

Note that we can apply Remark ⓑ to ③ $\Rightarrow$

$$L\{t^2 u_2(t)\} = (-1)^2 F''(s) = F''(s)$$

where $F(s) = L\{u_2(t)\} = \dfrac{e^{-2s}}{s}$

$$F'(s) = \frac{(s)(-2)e^{-2s} - e^{-2s}}{s^2} = -2e^{-2s}\left(\tfrac{1}{s}\right) - \left(\tfrac{1}{s^2}\right)e^{-2s}$$

$$F''(s) = -2e^{-2s}\left(\frac{-1}{s^2}\right) + \left(\frac{1}{s}\right)(4)e^{-2s} - \left(\frac{1}{s^2}\right)(-2)e^{-2s} - e^{-2s}(-2)\left(\frac{1}{s^3}\right)$$

$$= \frac{2}{s^3}e^{-2s} + 4e^{-2s}\left(\frac{1}{s^2}\right) + 4\frac{e^{-2s}}{s}$$

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④ $$L^{-1}\left(\frac{3+e^{-7s}}{s^4}\right) = 3L^{-1}\left(\frac{1}{s^4}\right) + L^{-1}\left(\frac{e^{-7s}}{s^4}\right)$$

$$= \frac{3}{3!}L^{-1}\left(\frac{3!}{s^4}\right) + \frac{1}{3!}L^{-1}\left(\frac{3!\,e^{-7s}}{s^4}\right)$$

$$= \frac{1}{2}t^3 + \frac{1}{6}u_7(t)(t-7)^3$$

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⑤ $F(s)$ if $f(t) = \begin{cases} \sin t & , 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t-\frac{\pi}{4}) & , \frac{\pi}{4} \le t \end{cases}$

$$f(t) = \sin t + \left(\sin t + \cos(t-\tfrac{\pi}{4}) - \sin t\right)u_{\frac{\pi}{4}}(t)$$

$$= \sin t + u_{\frac{\pi}{4}}(t)\cos(t-\tfrac{\pi}{4})$$

$$F(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}L\{\cos t\} = \frac{1}{s^2+1} + \frac{s\,e^{-\frac{\pi}{4}s}}{s^2+1}$$

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Exp Find Laplace Transform of $f(t) = u_{\pi}(t) \sin t$

$$F(s) = L\{f(t)\} = L\{u_{\pi}(t) \sin t\} = L\{u_{\pi}(t) \sin(t - \pi + \pi)\}$$

$$= L\left\{u_{\pi}(t) \left(\sin(t-\pi)\cos\pi + \sin(\pi)\cos(t-\pi)\right)\right\}$$

$$= L\{u_{\pi}(t)\left(-\sin(t-\pi) + 0\right)\}$$

$$= -L\{u_{\pi}(t)\sin(t-\pi)\}$$

$$= -e^{-\pi s} L\{\sin t\} = -e^{-\pi s} \frac{1}{s^2+1}$$

Exp Find $L\{u_3(t)\, e^{4t}\}$

(S1) $L\{u_3(t)\, e^{4(t-3+3)}\}$

$$= L\{u_3(t)\, e^{4(t-3)} e^{12}\}$$

$$= e^{12} L\{u_3(t)\, e^{4(t-3)}\}$$

$$= e^{12} e^{-3s} L\{e^{4t}\}$$

$$= e^{-3(s-4)} \frac{1}{s-4}$$

$$= \frac{e^{-3(s-4)}}{s-4}$$

(S2) $L\{u_3(t)\, e^{4t}\}$ Apply (A) Remark

$$= F(s-4)$$

where $F(s) = L\{u_3(t)\}$

$$= \frac{e^{-3s}}{s}$$

Hence, $F(s-4) = \dfrac{e^{-3(s-4)}}{s-4}$

Exp Find Laplace Inverse of $\frac{2(s-1)e^{-2s}}{s^2-2s+2}$

$$L^{-1}\left(\frac{2(s-1)e^{-2s}}{s^2-2s+2}\right)$$

$$L^{-1}\left(\frac{2(s-1)e^{-2s}}{(s-1)^2+1}\right) = 2L^{-1}\left(\frac{(s-1)e^{-2s}}{(s-1)^2+1}\right)$$

$$= 2\, u_2(t)\, e^{t-2} \cos(t-2)$$

Exp Find $F(s)$ if $f(t)=\begin{cases} 3, & 0 \le t < 2 \\ t^2-4t+4, & 2 \le t \end{cases}$

$$f(t) = 3 + (t^2-4t+4-3)\, u_2(t)$$

$$= 3 + (t^2-4t+1)\, u_2(t)$$

$$= 3 + \left((t-2)^2-3\right) u_2(t)$$

$$= 3 + u_2(t)(t-2)^2 - 3u_2(t)$$

$$F(s) = L\{3\} + L\{u_2(t)(t-2)^2\} - 3L\{u_2(t)\}$$

$$= \frac{3}{s} + e^{-2s} L\{t^2\} - 3\frac{e^{-2s}}{s}$$

$$= \frac{3}{s} + e^{-2s}\frac{2}{s^3} - 3\frac{e^{-2s}}{s}$$

$$= \frac{3(1-e^{-2s})}{s} + \frac{2e^{-2s}}{s^3}$$