6.4 Solving IVP's with Step Functions

In this section we will find cont. and diff. solution for a given IVP with discont. step functions.

Exp Solve the IVP:

$$y'' + 4y = 6 + t\,u_3(t) \quad , \quad y(0)=0 \ , \ y'(0)=0$$

$$L\{y''\} + L\{4y\} = L\{6\} + L\{u_3(t)\,(t-3+3)\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + 4\,L\{y\} = \frac{6}{s} + L\{u_3(t)(t-3)\} + 3\,L\{u_3(t)\}$$

$$(s^2+4)\,L\{y\} = \frac{6}{s} + e^{-3s}\,L\{t\} + 3\,\frac{e^{-3s}}{s}$$

$$(s^2+4)\,L\{y\} = \frac{6}{s} + e^{-3s}\left(\frac{1}{s^2}\right) + \frac{3e^{-3s}}{s}$$

$$L\{y\} = \frac{6}{s(s^2+4)} + \frac{e^{-3s}}{s^2(s^2+4)} + \frac{3e^{-3s}}{s(s^2+4)}$$

$$y(t) = 6L^{-1}\left(\frac{1}{s(s^2+4)}\right) + L^{-1}\left(\frac{e^{-3s}}{s^2(s^2+4)}\right) + 3L^{-1}\left(\frac{e^{-3s}}{s(s^2+4)}\right)$$

$$= 6\,L^{-1}\left(\frac{A}{s} + \frac{Bs+C}{s^2+4}\right) + L^{-1}\left[e^{-3s}\left(\frac{D}{s} + \frac{E}{s^2} + \frac{Fs+G}{s^2+4}\right)\right] +$$

$$3\,L^{-1}\left[e^{-3s}\left(\frac{A}{s} + \frac{Bs+C}{s^2+4}\right)\right]$$

Using Partial Fraction: $A = \frac{1}{4}$, $B = -\frac{1}{4}$, $C = 0$ | $D = 0$, $E = \frac{1}{4}$, $F = 0$, $G = -\frac{1}{4}$

$$y(t) = 6\,L^{-1}\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right) + L^{-1}\left[e^{-3s}\left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)\right] + 3\,L^{-1}\left[e^{-3s}\left(\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4}\right)\right]$$

$$y(t) = \frac{6}{4} L^{-1}\left(\frac{1}{s}\right) - \frac{6}{4} L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{4} L^{-1}\left(\frac{e^{-3s}}{s^2}\right) - \frac{1}{8} L^{-1}\left(\frac{2e^{-3s}}{s^2+4}\right)$$

$$+ \frac{3}{4} L^{-1}\left(\frac{e^{-3s}}{s}\right) - \frac{3}{4} L^{-1}\left(\frac{se^{-3s}}{s^2+4}\right)$$

$$= \frac{3}{2}(1) - \frac{3}{2}\cos 2t + \frac{1}{4} u_3(t)(t-3) - \frac{1}{8} u_3(t) \sin 2(t-3)$$

$$+ \frac{3}{4} u_3(t) - \frac{3}{4} u_3(t) \cos 2(t-3)$$

$$y(t) = \frac{3}{2}(1-\cos 2t) + \frac{1}{4} u_3(t)\left[t \cancel{-3} - \frac{1}{2}\sin 2(t-3) \cancel{+3} - 3\cos 2(t-3)\right]$$

$$= \frac{3}{2}(1-\cos 2t) + \frac{1}{4} u_3(t)\left[t - \frac{1}{2}\sin 2(t-3) - 3\cos 2(t-3)\right]$$

Note that y and y' are cont. at $t=3$

Exp Solve the IVP:

$$y'' + y = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t \end{cases} \qquad \begin{array}{l} y(0)=0 \\ y'(0)=0 \end{array}$$

$$y'' + y = 1 + (t-1)u_1(t)$$

$$L\{y''\} + L\{y\} = L\{1\} + L\{u_1(t)(t-1)\}$$

$$s^2 L\{y\} - sy(0) - y'(0) + L\{y\} = \frac{1}{s} + e^{-s} L\{t\}$$

$$(s^2+1) L\{y\} = \frac{1}{s} + e^{-s}\frac{1}{s^2}$$

$$L\{y\} = \frac{1}{s(s^2+1)} + \frac{e^{-s}}{s^2(s^2+1)}$$

$$y(t) = L^{-1}\left(\frac{1}{s(s^2+1)}\right) + L^{-1}\left(\frac{e^{-s}}{s^2(s^2+1)}\right)$$

$$y(t) = L^{-1}\left(\frac{A}{s} + \frac{Bs+C}{s^2+1}\right) + L^{-1}\left[e^{-s}\left(\frac{D}{s} + \frac{E}{s^2} + \frac{Fs+G}{s^2+1}\right)\right]$$

Using Partial Fraction $\Rightarrow A = 1,\ B = -1,\ C = 0$

$\Rightarrow D = 0,\ E = 1,\ F = 0,\ G = -1$

$$y(t) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{e^{-s}}{s^2}\right) + L^{-1}\left(\frac{e^{-s}}{s^2+1}\right)$$

$$= 1 - \cos t + u_1(t)(t-1) + u_1(t)\sin(t-1)$$

Exp Solve the IVP: $y'' = e^t u_1(t)$, $y(0) = 1$, $y'(0) = 0$

$$L\{y''\} = L\{e^t u_1(t)\} = L\{u_1(t)e^{t-1+1}\} = e\,L\{u_1(t)e^{t-1}\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) = e\,e^{-s} L\{e^t\} = e\,e^{-s}\frac{1}{s-1}$$

$$s^2 L\{y\} - s = e\,\frac{e^{-s}}{s-1}$$

$$L\{y\} = \frac{1}{s} + e\,\frac{e^{-s}}{s^2(s-1)}$$

Using Partial Fraction
$A = -1$
$B = -1$
$C = 1$

$$y(t) = L^{-1}\left(\frac{1}{s}\right) + e\,L^{-1}\left(\frac{e^{-s}}{s^2(s-1)}\right)$$

$$= 1 + e\,L^{-1}\left[e^{-s}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\right)\right]$$

$$= 1 + e\,L^{-1}\left[e^{-s}\left(\frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\right)\right]$$

$$= 1 - e\left[u_1(t) + u_1(t)(t-1) - u_1(t)e^{t-1}\right]$$

$$= 1 - e\,u_1(t)\left[t - e^{t-1}\right]$$

y and y' are cont. at $t=1$

Exp Solve the IVP: $2y'' + y' + 2y = g(t)$, $y(0) = y'(0) = 0$

where $g(t) = \begin{cases} 0 , & 0 \le t < 5 \\ 1 , & 5 \le t < 20 \\ 0 , & 20 \le t \end{cases}$

$$g(t) = 0 + (1-0)\,u_5(t) + (0-1)\,u_{20}(t) = u_5(t) - u_{20}(t)$$

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$

$$2\left(s^2 L\{y\} - s\,y(0) - y'(0)\right) + \left(sL\{y\} - s\,y(0)\right) + 2L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2)\,L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$L\{y\} = \frac{e^{-5s}}{s(2s^2+s+2)} - \frac{e^{-20s}}{s(2s^2+s+2)}$$

$$y(t) = L^{-1}\left(e^{-5s} H(s)\right) - L^{-1}\left(e^{-20s} H(s)\right)$$

$$= u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$$

$$= u_5(t)\left[\frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}(t-5)} \cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}(t-5)} \sin\frac{\sqrt{15}}{4}(t-5)\right]$$

$$- u_{20}(t)\left[\frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}(t-20)} \cos\frac{\sqrt{15}}{4}(t-20) - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}(t-20)} \sin\frac{\sqrt{15}}{4}(t-20)\right]$$

Note that y and y' are cont. at $t=5$ and $t=20$

$$H(s) = \frac{1}{s(2s^2+s+2)}$$

$$= \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$= \frac{\frac{1}{2}}{s} + \frac{-s-\frac{1}{2}}{2s^2+s+2}$$

Using Partial Fraction

$$h(t) = L^{-1}(H(s))$$

$$= L^{-1}\left(\frac{\frac{1}{2}}{s}\right) - L^{-1}\left(\frac{s+\frac{1}{2}}{2(s^2+\frac{s}{2}+1)}\right)$$

$$= \frac{1}{2} - \frac{1}{2} L^{-1}\left(\frac{s+\frac{1}{4}+\frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2} - \frac{1}{2} L^{-1}\left(\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right) - \frac{1}{2\sqrt{15}} L^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

$$= \frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}t} \cos\frac{\sqrt{15}}{4}t - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}t} \sin\frac{\sqrt{15}}{4}t$$