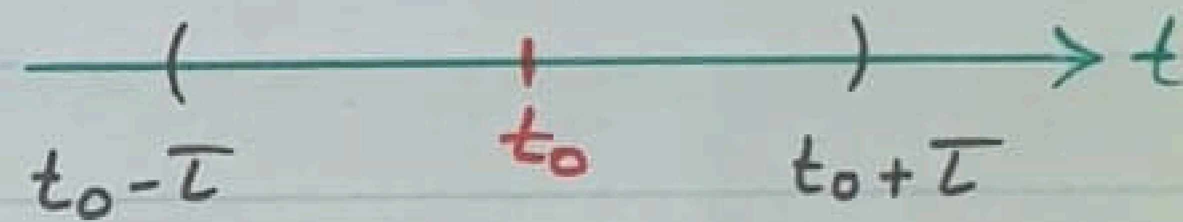
Impulse Functions

In some applications, it is necessary to deal with impulsive (اندفاع) nature. For example, voltages or forces with large magnitude that act over short time intervals. Such applications lead to DE's of the form:

$$a y'' + b y' + c y = d_\tau(t - t_0)$$

where the forcing function $d_\tau(t - t_0) = \begin{cases} \frac{1}{2\tau}, & t_0 - \tau < t < \tau + t_0 \\ 0, & \text{Otherwise} \end{cases}$

where $\tau > 0$ and t_0 is the center.



$|t - t_0| < \tau$

- When the center $t_0 = 0 \Rightarrow$ the forcing function becomes

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$

- To measure the strength of the forcing function $d_\tau(t)$ we use the Integral

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\, dt = \int_{-\tau}^{\tau} \frac{1}{2\tau}\, dt = \frac{1}{2\tau}(\tau - -\tau) = 1$$

Exp* Note that $\lim_{\tau \to 0} d_\tau(t) = 0$

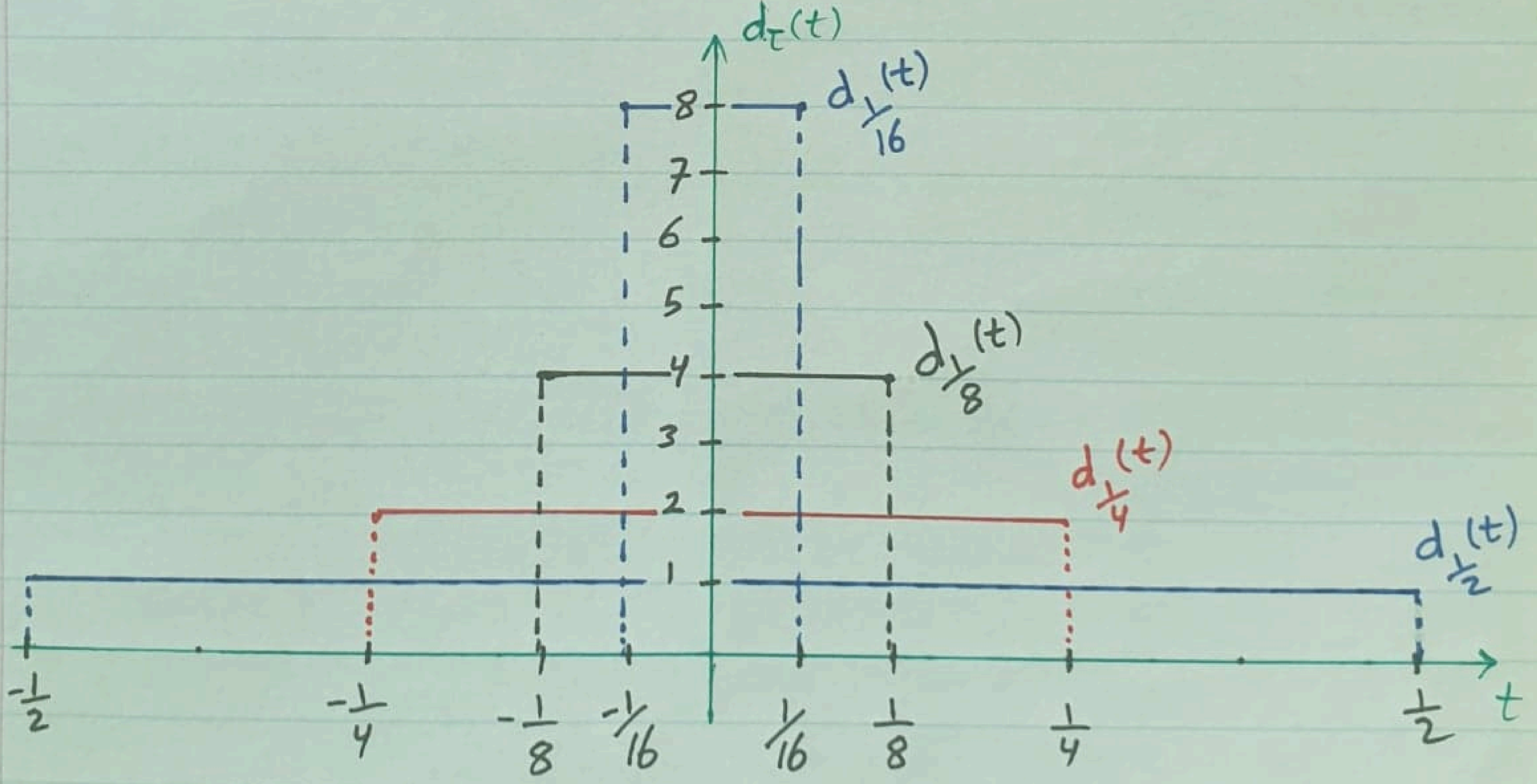
when $\tau = \frac{1}{2} \Rightarrow d_{\frac{1}{2}}(t) = \begin{cases} 1 & , -\frac{1}{2} < t < \frac{1}{2} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{4} \Rightarrow d_{\frac{1}{4}}(t) = \begin{cases} 2 & , -\frac{1}{4} < t < \frac{1}{4} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{8} \Rightarrow d_{\frac{1}{8}}(t) = \begin{cases} 4 & , -\frac{1}{8} < t < \frac{1}{8} \\ 0 & , \text{otherwise} \end{cases}$

$\tau = \frac{1}{16} \Rightarrow d_{\frac{1}{16}}(t) = \begin{cases} 8 & , -\frac{1}{16} < t < \frac{1}{16} \\ 0 & , \text{otherwise} \end{cases}$

$\vdots$



Def The Unit Impulse Function δ at point t_0 is defined by

$$\delta(t-t_0) = \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

Special Case when $t_0 = 0 \Rightarrow$ The unit impulse function becomes

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

Remark

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) \quad \text{when } t \neq 0$$

Hence, $\delta(t-t_0) = \lim_{\tau \to 0} d_\tau(t-t_0)$ when $t \neq t_0$

Exp show that $L\{\delta(t-t_0)\} = e^{-t_0 s}$

Proof $L\{\delta(t-t_0)\} = L\{\lim_{\tau \to 0} d_\tau(t-t_0)\}$ by Remark above

$$= \lim_{\tau \to 0} \int_0^{\infty} e^{-st} d_\tau(t-t_0)\,dt = \lim_{\tau \to 0} \int_{t_0-\tau}^{t_0+\tau} e^{-st}\left(\frac{1}{2\tau}\right) dt$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \cdot \frac{-1}{s} \left[e^{-st} \Big|_{t_0-\tau}^{t_0+\tau} \right]$$

$$= \lim_{\tau \to 0} \frac{-1}{2s\tau} \left[e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)} \right]$$

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} \frac{1}{s\tau}\left[\frac{e^{s\tau} - e^{-s\tau}}{2}\right] e^{-t_0 s}$$

$$= e^{-t_0 s} \lim_{\tau\to 0} \frac{\sinh s\tau}{s\tau}$$

$$= e^{-t_0 s} \lim_{\tau\to 0} \frac{s\cosh s\tau}{s}$$

$$= e^{-t_0 s}$$

Exp Find

① $L\{\delta(t-\pi)\} = e^{-\pi s}$

② $L\{\delta(t-1)\} = e^{-s}$

③ $L\{\delta(t)\} = L\{\delta(t-0)\} = e^{-0s} = e^{0} = 1$

④ $L^{-1}(e^{-3s}) = \delta(t-3)$

⑤ $L^{-1}(4) = 4L^{-1}(1) = 4\,\delta(t)$

⑥ $L\{4\} = \frac{4}{s}$

Exp Find

$$L^{-1}\left(L^{-1}\left(\frac{7}{s}\right)\right) = L^{-1}(7) = 7L^{-1}(1) = 7\,\delta(t)$$

Exp Solve this IVP: $2y'' + y' + 2y = \delta(t-5)$, $y(0) = y'(0) = 0$

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-5)\}$$

$$2\left(s^2 Y(s) - sy(0) - y'(0)\right) + \left(sY(s) - sy(0)\right) + 2Y(s) = e^{-5s}$$

$$(2s^2 + s + 2)\, Y(s) = e^{-5s}$$

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2}$$

$$y(t) = L^{-1}\left(e^{-5s} H(s)\right)$$

$$= u_5(t)\, h(t-5)$$

$$= \frac{2}{\sqrt{15}} u_5(t)\, e^{-\frac{(t-5)}{4}} \sin \frac{\sqrt{15}}{4}(t-5)$$

$$H(s) = \frac{1}{2s^2 + s + 2}$$

$$h(t) = L^{-1}\left(\frac{1}{2(s^2 + \frac{s}{2} + 1)}\right)$$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{1}{2} \frac{4}{\sqrt{15}} L^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \frac{2}{\sqrt{15}} e^{-\frac{t}{4}} \sin \frac{\sqrt{15}}{4} t$$

Remark (D) To find Laplace Transform for product of two functions: one is the unit impulse function $\delta(t - t_0)$
$\Rightarrow$ we use the following result

Exp show that $L\{\delta(t-t_0)f(t)\} = e^{-t_0 s} f(t_0)$

where f is cont.

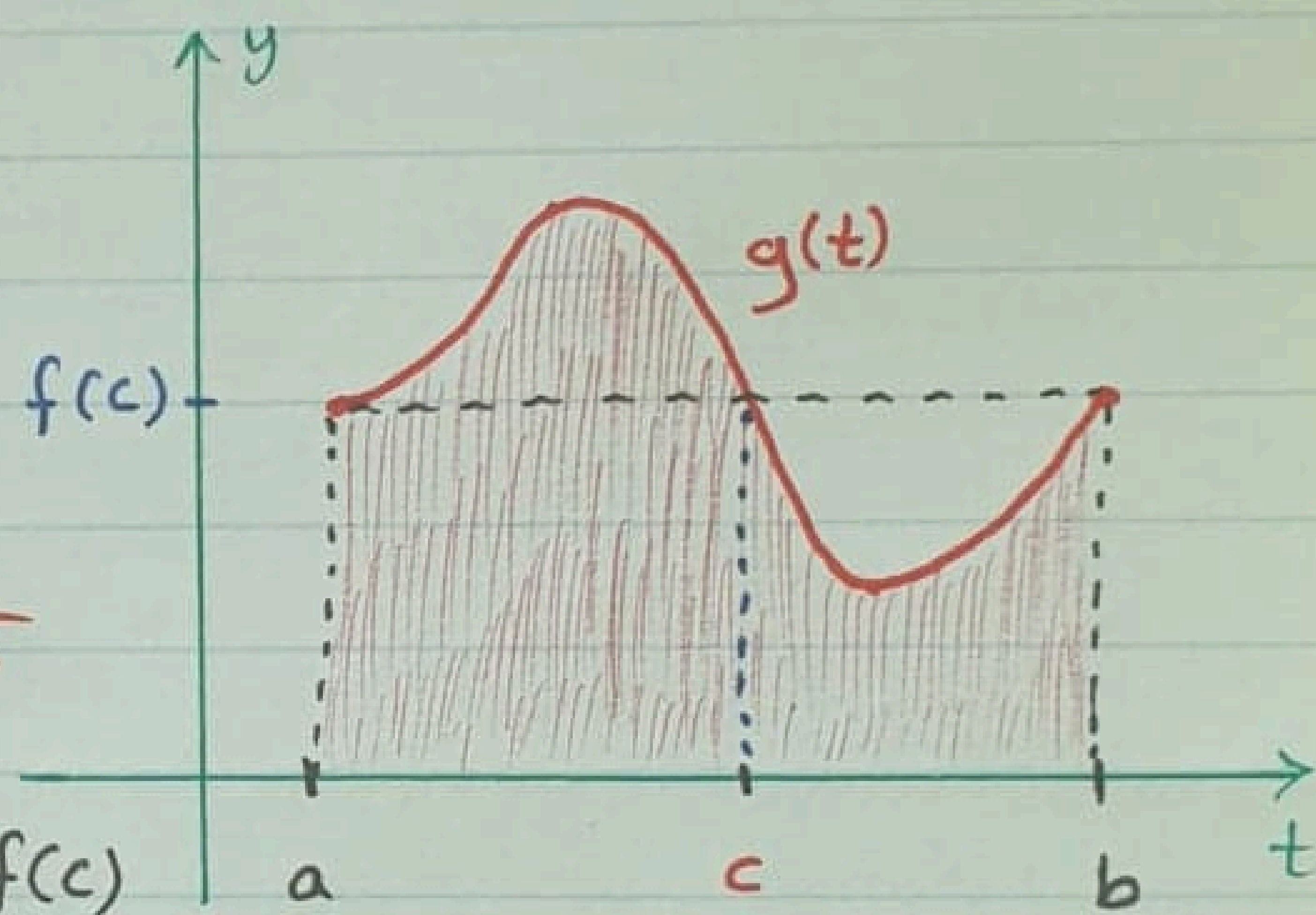
$$L\{\delta(t-t_0)f(t)\} = \int_0^\infty e^{-st}\,\delta(t-t_0)f(t)\,dt$$

$$= \lim_{\tau\to 0}\int_0^\infty e^{-st}\,d_\tau(t-t_0)\,f(t)\,dt$$

where the forcing function

$$d_\tau(t-t_0) = \begin{cases} \frac{1}{2\tau}, & t_0-\tau < t < \tau+t_0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \lim_{\tau\to 0}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,\frac{1}{2\tau}\,f(t)\,dt$$



$$= \lim_{\tau\to 0}\frac{1}{2\tau}\,g(c)\,(b-a)$$

$t_0-\tau < c < t_0+\tau$

$$= \lim_{\tau\to 0}\frac{1}{2\tau}\,e^{-cs}\big(t_0+\tau-(t_0-\tau)\big)f(c)$$

$$= \lim_{\tau\to 0}\frac{1}{2\tau}(2\tau)\,e^{-cs}f(c)$$

$$= \lim_{\tau\to 0} e^{-cs}f(c)$$

$$= e^{-t_0 s}f(t_0)$$

since as $\tau\to 0 \Rightarrow c\to t_0$

g satisfies MVT for integrals on $[a,b] \Rightarrow$

$$\int_a^b g(t)\,dt = f(c)(b-a)$$

for some $c\in(a,b)$

Hence, $g(t) = e^{-st}f(t)$ satisfies MVT on $(t_0-\tau,\ t_0+\tau)$ since g is cont. function.

Exp ① $L\{\delta(t-\pi)\cos t\} = e^{-\pi s}\cos\pi = -e^{-\pi s}$

② $L\{\delta(t-\ln 2)\, e^{t}\} = e^{-(\ln 2)s}\, e^{\ln 2} = 2e^{-(\ln 2)s}$

③ $L\{\delta(t)\sin t\} = L\{\delta(t-0)\sin t\}$

$= e^{-0s}\sin 0 = (1)(0) = 0$

Exp Solve the IVP: $y'' + y = \delta(t-\pi)\cos t + u_2(t) + \delta(t)$

$y(0) = y'(0) = 0$

$L\{y''\} + L\{y\} = L\{\delta(t-\pi)\cos t\} + L\{u_2(t)\} + L\{\delta(t)\}$

$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = e^{-\pi s}\cos\pi + \dfrac{e^{-2s}}{s} + 1$

$(s^2+1)Y(s) = -e^{-\pi s} + \dfrac{e^{-2s}}{s} + 1$

$Y(s) = -\dfrac{e^{-\pi s}}{s^2+1} + \dfrac{e^{-2s}}{s(s^2+1)} + \dfrac{1}{s^2+1}$

$y(t) = -L^{-1}\left(\dfrac{e^{-\pi s}}{s^2+1}\right) + L^{-1}\left(e^{-2s}H(s)\right) + L^{-1}\left(\dfrac{1}{s^2+1}\right)$

$= -u_\pi(t)\sin(t-\pi) + u_2(t)\,h(t-2) + \sin t$

$= -u_\pi(t)\sin(t-\pi) + u_2(t)\left(1-\cos(t-2)\right) + \sin t$

$H(s) = \dfrac{1}{s(s^2+1)}$

$= \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

$= \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$h(t) = L^{-1}\left(\dfrac{1}{s}\right) - L^{-1}\left(\dfrac{s}{s^2+1}\right)$

$= 1 - \cos t$

$y(t) = -u_\pi(t)\sin(t-\pi) - u_2(t)\cos(t-2) + u_2(t) + \sin t$