6.6 Convolution Integral (*)

Exp Show that $L\{f(t)\,g(t)\} \neq L\{f(t)\}\,L\{g(t)\}$

$$f(t) = 3 \Rightarrow F(s) = L\{f(t)\} = L\{3\} = \frac{3}{s}$$

$$g(t) = \sin t \Rightarrow G(s) = L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

$$f(t)\,g(t) = 3\sin t \Rightarrow L\{f(t)g(t)\} = L\{3\sin t\} = \frac{3}{s^2+1}$$

Hence, $L\{f(t)g(t)\} = \frac{3}{s^2+1} \neq \left(\frac{3}{s}\right)\left(\frac{1}{s^2+1}\right) = F(s)G(s)$

Def $f(t)$ convolution $g(t)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau \quad \text{--- (1)}$$

Remark The integral in (1) is also called the convolution integral of f and g

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Exp Show that $f*g = g*f$

$$f(t) * g(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$u = t - \tau$
$du = -d\tau$

$$= -\int_t^0 f(u)\,g(t-u)\,du$$

$\tau = 0 \Rightarrow u = t$
$\tau = t \Rightarrow u = 0$

$\tau = t - u$

$$= \int_0^t g(t-u)\,f(u)\,du = g(t) * f(t)$$

Th If $h(t) = f(t) * g(t)$ then

$$H(s) = L\{h(t)\} = L\{f(t) * g(t)\}$$
$$= L\{f(t)\}\, L\{g(t)\}$$
$$= F(s)\, G(s)$$

Hence, $L^{-1}\left(F(s)\,G(s)\right) = L^{-1}\left(L\{f(t)\}\, L\{g(t)\}\right)$

$$= f(t) * g(t)$$

Exp Find Laplace Transform of

① $h(t) = e^{-t} * t^3$

$$H(s) = L\{e^{-t} * t^3\} = L\{e^{-t}\}\, L\{t^3\} = \left(\frac{1}{s+1}\right)\left(\frac{3!}{s^4}\right)$$
$$= \frac{6}{s^4(s+1)}$$

② $r(t) = 2 * u_2(t)$

$$R(s) = L\{2 * u_2(t)\} = L\{2\}\, L\{u_2(t)\}$$
$$= \left(\frac{2}{s}\right)\left(\frac{e^{-2s}}{s}\right) = \frac{2e^{-2s}}{s^2}$$

③ $m(t) = \int_0^t (t-\tau)\sin 2\tau \, d\tau$

$m(t) = t * \sin 2t \Rightarrow M(s) = L\{t * \sin 2t\}$

$$M(s) = L\{t\}\, L\{\sin 2t\} = \left(\frac{1}{s^2}\right)\left(\frac{2}{s^2+4}\right) = \frac{2}{s^2(s^2+4)}$$

④ $s(t) = \cos t * t^2$

$$S(s) = L\{\cos t * t^2\} = L\{\cos t\}\, L\{t^2\} = \left(\frac{s}{s^2+1}\right)\left(\frac{2}{s^3}\right)$$

$$= \frac{2s}{s^3(s^2+1)}$$

⑤ $f(t) = \int_0^t t e^{\tau} d\tau$

(S1)

$$f(t) = \int_0^t (t - \tau + \tau) e^{\tau} d\tau = \int_0^t (t-\tau) e^{\tau} d\tau + \int_0^t \tau e^{\tau} d\tau$$

$$= t * e^t + \int_0^t \tau e^{\tau} d\tau$$

$$= t * e^t + \left(\tau e^{\tau} - e^{\tau}\right)\Big|_0^t$$

$$\begin{array}{ccc} \tau & \xrightarrow{+} & e^{\tau} \\ 1 & \xrightarrow{-} & e^{\tau} \\ 0 & & e^{\tau} \end{array}$$

$$= t * e^t + t e^t - e^t - (0 - 1)$$

$$F(s) = L\{t * e^t\} + L\{t e^t\} - L\{e^t\} + L\{1\}$$

$$= L\{t\}\, L\{e^t\} + G(s-1) - \frac{1}{s-1} + \frac{1}{s}$$

$$G(s) = L\{t\} = \frac{1}{s^2}$$

$$= \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right) + \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s} = \frac{1}{(s-1)^2} - \frac{1}{s^2}$$

(S2) $f(t) = \int_0^t t e^{\tau} d\tau = t\left(e^{\tau}\Big|_0^t\right) = t\left(e^t - 1\right)$

$$= t e^t - t$$

$$F(s) = L\{t e^t\} - L\{t\} = G(s-1) - \frac{1}{s^2}$$

$$= \frac{1}{(s-1)^2} - \frac{1}{s^2}$$

Exp Find Inverse transform of $H(s) = \dfrac{2}{s^2(s-2)}$

(S1)
$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{2}{s^2(s-2)}\right)$$

$$= L^{-1}\left(\frac{A}{s-2} + \frac{B}{s} + \frac{C}{s^2}\right)$$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

$C = -1$

$$= \frac{1}{2} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^2}\right)$$

$$= \frac{1}{2} e^{2t} - \frac{1}{2} - t$$

(S2)
$$H(s) = 2\left(\frac{1}{s^2}\right)\left(\frac{1}{s-2}\right) = 2\, L\{t\}\, L\{e^{2t}\}$$

$$h(t) = 2 L^{-1}\left(L\{t\} L\{e^{2t}\}\right) = 2t * e^{2t}$$

$$= 2\int_0^t (t-\tau)\, e^{2\tau}\, d\tau$$

$$\begin{array}{cc} t-\tau & e^{2\tau} \\ & {\scriptstyle +} \\ -1 & \frac{1}{2} e^{2\tau} \\ & {\scriptstyle -} \\ 0 & \frac{1}{4} e^{2\tau} \end{array}$$

$$= 2\left((t-\tau)\left(\tfrac{1}{2}\right) e^{2\tau} + \frac{1}{4} e^{2\tau} \Big|_0^t \right)$$

$$= 2\left(0 + \frac{1}{4} e^{2t} - \left(\frac{t}{2} + \frac{1}{4}\right)\right)$$

$$= \frac{1}{2} e^{2t} - t - \frac{1}{2}$$

Exp Solve the integral equation

$$\phi'(t) + 2\phi(t) + \int_0^t 2\,\phi(u)\,du = 0 \quad , \phi(0) = 1$$

$$\phi'(t) + 2\phi(t) + 2 * \phi(t) = 0$$

$$L\{\phi'(t)\} + 2\,L\{\phi(t)\} + L\{2 * \phi(t)\} = L\{0\}$$

$$s\,\Phi(s) - \phi(0) + 2\,\Phi(s) + L\{2\}\,L\{\phi(t)\} = \frac{0}{s}$$

$$(s+2)\,\Phi(s) - 1 + \frac{2}{s}\,\Phi(s) = 0$$

$$\left(s + 2 + \frac{2}{s}\right)\Phi(s) = 1$$

$$\left(\frac{s^2 + 2s + 2}{s}\right)\Phi(s) = 1$$

$$\Phi(s) = \frac{s}{s^2 + 2s + 2}$$

$$\phi(t) = L^{-1}\left(\frac{s}{s^2 + 2s + 2}\right)$$

$$= L^{-1}\left(\frac{s+1-1}{(s+1)^2 + 1}\right)$$

$$= L^{-1}\left(\frac{s+1}{(s+1)^2 + 1}\right) - L^{-1}\left(\frac{1}{(s+1)^2 + 1}\right)$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Exp show that $1 * \delta(t-2) = u_2(t)$

$$L\{1 * \delta(t-2)\} = L\{1\}\, L\{\delta(t-2)\}$$
$$= \left(\frac{1}{s}\right) e^{-2s}$$

Hence,

$$1 * \delta(t-2) = L^{-1}\left(\frac{e^{-2s}}{s}\right) = u_2(t)$$

Exp Solve the IVP : $y'' = \delta(t-\pi) - \delta(t-e)$, $y(0)=0$, $y'(0)=0$

$$L\{y''\} = L\{\delta(t-\pi)\} - L\{\delta(t-e)\}$$

$$s^2 Y(s) - sy(0) - y'(0) = e^{-\pi s} - e^{-es}$$

$$s^2 Y(s) = e^{-\pi s} - e^{-es}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2} - \frac{e^{-es}}{s^2}$$

$$y(t) = L^{-1}\left(\frac{e^{-\pi s}}{s^2}\right) - L^{-1}\left(\frac{e^{-es}}{s^2}\right)$$

$$= u_\pi(t)\,(t-\pi) - u_e(t)\,(t-e)$$

Exp Let $f(t) = u_1(t)$ and $g(t) = t$. Find $(f*g)(t)$

$$L\{f*g\} = L\{u_1(t) * t\} = L\{u_1(t)\}\, L\{t\} = \frac{e^{-s}}{s}\left(\frac{1}{s^2}\right)$$

Hence,

$$f*g = L^{-1}\left(\frac{e^{-s}}{s^3}\right) = \frac{1}{2} L^{-1}\left(\frac{2e^{-s}}{s^3}\right) = \frac{1}{2} u_1(t)\,(t-1)^2$$

Exp Solve the integral equation:

$$h(t) + \int_0^t (t-\xi)\, h(\xi)\, d\xi = t + u_2(t)$$

$$h(t) + t * h(t) = t + u_2(t)$$

$$H(s) + \frac{1}{s^2} H(s) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$H(s)\left(1+\frac{1}{s^2}\right) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$\left(\frac{s^2+1}{s^2}\right) H(s) = \frac{1}{s^2} + \frac{e^{-2s}}{s}$$

$$(s^2+1)\, H(s) = 1 + s\, e^{-2s}$$

$$H(s) = \frac{1}{s^2+1} + \frac{s\, e^{-2s}}{s^2+1}$$

$$h(t) = L^{-1}\left(\frac{1}{s^2+1}\right) + L^{-1}\left(\frac{s\, e^{-2s}}{s^2+1}\right)$$

$$= \sin t + u_2(t) \cos(t-2)$$

Exp Show that $L\{f(ct)\} = \frac{1}{c} F\left(\frac{s}{c}\right)$ where $F(s) = L\{f(t)\}$

$$L\{f(ct)\} = \int_0^\infty e^{-st} f(ct)\, dt = \int_0^\infty e^{-s\left(\frac{u}{c}\right)} f(u)\, \frac{du}{c}$$

$u = ct$

$du = c\, dt$

$\frac{du}{c} = dt$

$$= \frac{1}{c} \int_0^\infty e^{-\left(\frac{s}{c}\right) u} f(u)\, du$$

$t = \frac{u}{c}$

$t = 0 \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$= \frac{1}{c} F\left(\frac{s}{c}\right)$$