Ch 7

Solving linear Systems of DE's

- In this chapter, we will learn how to solve 2x2 linear system of ODE's of the form:

$$x_1' = a_{11} x_1 + a_{12} x_2 \quad , \quad x_1(t_0) = x_1^0$$
$$x_2' = a_{21} x_1 + a_{22} x_2 \quad , \quad x_2(t_0) = x_2^0$$

Homo. System with constant coefficients

where $x_1' = \dfrac{dx_1}{dt}$ and $x_2' = \dfrac{dx_2}{dt}$

- We can write this linear system using matrix form:

$$x' = Ax \quad , \quad \overset{\circ}{x} = x(0) = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} \quad \text{where} \qquad (1)$$

$$x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Note that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ is vector of unknowns

A is called the coefficient matrix

$x_0 = x(0) = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ is vector of initial conditions

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Question How to solve the linear system of two ODE's ?

Answer Assume exponensial solution of the form

$$x(t) = \xi\, e^{rt}$$

where $r \in \mathbb{R}$ is an eigenvalue and

$\xi \in \mathbb{R}^2$ is the corresponding nonzero eigenvector given by

$$\xi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y_1, y_2 \in \mathbb{R}$$

- Hence, $\frac{dx}{dt} = x' = r\,\xi\, e^{rt}$
- Now substitute x and x' in (1) $\Rightarrow$

$$x' = Ax \quad \Rightarrow \quad r\,\xi\, e^{rt} = A\,\xi\, e^{rt}$$

$$A\,\xi\, e^{rt} - r\,\xi\, e^{rt} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A\,\xi - r\,\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{(A - rI)\,\xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \;\text{---}\; *^2 \quad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- since ξ is nonzero vector $\Rightarrow$ we must have

$$\boxed{|A - rI| = 0} \;\text{---}\; *^1 \quad \text{where } |\;| \text{ means determinante}$$

- This means the square matrix $A - rI$ is singular

- To solve the linear system of two ODE's (1):

  → First solve $*^1$ for the eigenvalues $r_1 \neq r_2$

  → Second solve $*^2$ for the corresponding eigenvectors ξ_1 and ξ_2

  → 1^{st} solution $X_1(t) = \xi_1 e^{r_1 t}$

  → 2^{nd} solution $X_2(t) = \xi_2 e^{r_2 t}$

  → General solution $X(t) = c_1 X_1(t) + c_2 X_2(t)$

  $$= c_1 \xi_1 e^{r_1 t} + c_2 \xi_2 e^{r_2 t}$$

  → To find c_1 and c_2 we use the initial vector x^0

Remark There are three possible cases for the values of the eigenvalues r_1 and r_2:

$\boxed{1}$ If $r_1 \neq r_2 \in \mathbb{R}$, then we will study the solution in section 7.5

$\boxed{2}$ If $r_{1,2} = \lambda \pm \mu i$, then we will study the solution in section 7.6

$\boxed{3}$ If $r_1 = r_2 = r \in \mathbb{R}$, then we will study the solution in section 7.8

$\boxed{4}$ $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is trivial solution for (1) or Eq. solution.

But we look for nontrivial solution for (1).