7.5 Real Different Eigenvalues

$r_1 \neq r_2 \in \mathbb{R}$

Exp Find the general solution for the linear system:

$$x_1' = x_1 + x_2 \quad , \quad x_1(0) = 3$$
$$x_2' = 4x_1 + x_2 \quad , \quad x_2(0) = -2$$

- Note that $x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\overset{\circ}{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
- First we solve $*^1$ for r_1 and r_2 $\Rightarrow$

$$|A - rI| = 0 \quad \Rightarrow \left| \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0$$

$$(1-r)^2 - 4 = 0 \quad \Rightarrow (1-r)^2 = 4 \quad \Rightarrow |1-r| = 2$$

either $1-r = 2 \Rightarrow r_1 = -1$
or $1-r = -2 \Rightarrow r_2 = 3$

$\left.\right\}$ eigenvalues are real different

- To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = -1$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} 1-r_1 & 1 \\ 4 & 1-r_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1--1 & 1 \\ 4 & 1--1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- It's enough to take $2y_1 + y_2 = 0 \quad \Rightarrow y_2 = -2y_1$

Take $y_1 = 1 \Rightarrow y_2 = -2 \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Hence, the 1^{st} solution is $x_1(t) = \xi_1 e^{r_1 t}$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

. To find the eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ corresponding to the eigenvalue $r_2 = 3$ we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_2 & 1 \\ 4 & 1-r_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

. It's enough to set $-2z_1 + z_2 = 0 \Rightarrow z_2 = 2z_1$

Take $z_1 = 1 \Rightarrow z_2 = 2 \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence, the 2^{nd} solution is $x_2(t) = \xi_2 e^{r_2 t}$

$$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

. Hence, the gen. sol. is :

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

$$= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

. To find the constants c_1 and c_2 we use the IC :

$$x(0) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\left.\begin{array}{l} 3 = c_1 + c_2 \\ -2 = -2c_1 + 2c_2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 2 \\ c_2 = 1 \end{array}$$

So the gen. sol. becomes:

$$x(t) = 2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$$

$$= \begin{pmatrix} 2 \\ -4 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{3t}$$

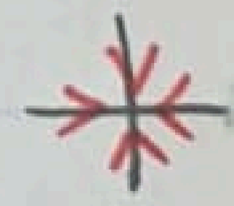
Remark ① $x_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$ and $x_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

are two independent solution since

$$W(x_1(t), x_2(t))(t) = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = 2e^{2t} - -2e^{2t} = 4e^{2t} \neq 0$$

Hence, $\{x_1(t), x_2(t)\}$ form fundemental set of solution

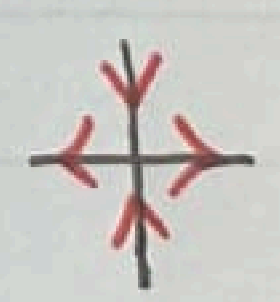
② • If r_1 and r_2 are negative, then origin is asymptotically stable Eq. point



• If r_1 and r_2 are positive, then origin is unstable Eq. point

• If $r_1 r_2 < 0$, then origin is saddle point which is unstable Eq. point

or



Exp Find two independent solutions for the system

$$x' = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x$$

• First solve $*^1$ for r_1 and r_2 $\Rightarrow |A - rI| = 0$

$$\begin{vmatrix} 1-r & 2 \\ 2 & 4-r \end{vmatrix} = 0 \quad \Rightarrow \quad (1-r)(4-r) - 4 = 0$$

$$4 - r - 4r + r^2 - 4 = 0$$

$$r^2 - 5r = 0$$

$$r(r-5) = 0$$

$$r_1 = 0 \,,\; r_2 = 5$$

eigenvalues are real different

• To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = 0$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_1 & 2 \\ 2 & 4-r_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Enough to set $y_1 + 2y_2 = 0$

$$y_1 = -2y_2$$

Take $y_2 = 1 \Rightarrow y_1 = -2 \quad \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Hence, the 1^{st} solution is $x_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{0t}$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

• To find the eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ corresponding to the eigenvalue $r_2 = 5$ we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_2 & 2 \\ 2 & 4-r_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ $\Rightarrow$ Enough to set $2z_1 - z_2 = 0$

$\Rightarrow \quad z_2 = 2z_1$

Take $z_1 = 1 \Rightarrow z_2 = 2 \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hence, the 2^{nd} solution is $x_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$

- To show $x_1(t)$ and $x_2(t)$ are independent $\Rightarrow$

$$W(x_1(t), x_2(t))(t) = \begin{vmatrix} -2 & e^{5t} \\ 1 & 2e^{5t} \end{vmatrix} = -4e^{5t} - e^{5t} = -5e^{5t} \neq 0$$

Hence, $x_1(t)$ and $x_2(t)$ are independent solutions and so $\{x_1(t), x_2(t)\}$ forms fundemental set of solutions

- The gen. sol. is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

$$= c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$$

Exp Find the eigenvalues of the system:

$$x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$$

Solve $*' \Rightarrow |A - rI| = 0 \Rightarrow \begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = 0$

$(-3-r)(-2-r) - 2 = 0$

$6 + 3r + 2r + r^2 - 2 = 0$

$r^2 + 5r + 4 = 0$

$(r+1)(r+4) = 0$

$r_1 = -1, \quad r_2 = -4$

eigenvalues

One can find

$\xi_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \Rightarrow x_1(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$

$\xi_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \Rightarrow x_2(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$