7.6 Complex Eigenvalues

$$r_{1,2} = \lambda \pm \mu i$$

<u>Exp</u> solve this system of DE's

$$x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$$

- First solve $*^1$ for the eigenvalues r_1 and r_2:

$$|A - rI| = 0 \Rightarrow \begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0$$

$$(1-r)(-3-r) - -5 = 0$$

$$-3 - r + 3r + r^2 + 5 = 0 \qquad \Rightarrow r^2 + 2r + 2 = 0$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i \quad \text{complex eigenvalues}$$

- To find the eigenvector $\xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ corresponding to the eigenvalue $r_1 = -1 - i$ we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-r_1 & -1 \\ 5 & -3-r_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(-1-i) & -1 \\ 5 & -3-(-1-i) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2+i & -1 \\ 5 & -2+i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2+i)y_1 - y_2 = 0 \Rightarrow y_2 = (2+i)y_1$$

Take $y_1 = 1 \Rightarrow y_2 = 2+i \Rightarrow \xi_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

- It can be shown that the 2nd eigenvector $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$ which is conjugate of ξ_1. To see that we solve $*^2 \Rightarrow$

$$(A - r_2 I)\xi_2 = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1-r_2 & -1\\ 5 & -3-r_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Note $r_2 = -1+i$

$$\begin{pmatrix}2-i & -1\\ 5 & -2-i\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(2-i)z_1 - z_2 = 0 \quad \Rightarrow z_2 = (2-i)z_1$$

Take $z_1 = 1 \quad \Rightarrow z_2 = (2-i) \quad \Rightarrow \xi_2 = \begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}1\\2-i\end{pmatrix}$

- Hence, 1^{st} complex solution is $x_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix}1\\2+i\end{pmatrix} e^{(-1-i)t}$

  2^{nd} complex solution is $x_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix}1\\2-i\end{pmatrix} e^{(-1+i)t}$

- But we need to find real valued solutions:
  So we use Euler Formula to find the
  first real solution ($u(t)$ - real part) and the
  second real solution ($v(t)$ - imaginary part)

- we apply Euler Formula either on $x_1(t)$ or $x_2(t)$:

$$x_1(t) = \begin{pmatrix}1\\2+i\end{pmatrix} e^{(-1-i)t} = \begin{pmatrix}1\\2+i\end{pmatrix} e^{-t} e^{-it}$$

Euler Formular
$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \begin{pmatrix}1\\2+i\end{pmatrix} e^{-t}\left(\cos(-t) + i\sin(-t)\right)$$

$$= e^{-t}\begin{pmatrix}1\\2+i\end{pmatrix}(\cos t - i\sin t)$$

$$= e^{-t}\begin{pmatrix}\cos t - i\sin t\\ 2\cos t + \sin t + i\cos t - 2i\sin t\end{pmatrix}$$

$$x_1(t) = \underbrace{e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}}_{u(t)\text{ - real part}} + i\, \underbrace{e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}}_{v(t)\text{ - imaginary part}}$$

Hence, the gen. sol. is

$$x(t) = c_1\, u(t) + c_2\, v(t)$$

$$= c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

Remark If we take $x_2(t)$ we get the same $u(t)$ and $v(t)$ $\Rightarrow$

$$x_2(t) = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t} = e^{-t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{it} = e^{-t}\begin{pmatrix} 1 \\ 2-i \end{pmatrix}(\cos t + i\sin t)$$

$$= e^{-t}\begin{pmatrix} \cos t + i\sin t \\ 2\cos t + \sin t - i\cos t + 2i\sin t \end{pmatrix}$$

$$= \underbrace{e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}}_{u(t)\text{ - real part}} - i\, \underbrace{e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}}_{v(t)\text{ - imaginary part}}$$

the negative sign goes in c_2

Remark
- If $\lambda < 0$, then the origin is asymptotically stable spiral Eq. point
- If $\lambda > 0$, " " " " unstable spiral Eq. point
- If $\lambda = 0$, " " " " stable Eq. point