7.8 Repeated Eigenvalues

$r_1 = r_2 = r$

- If we solve $*^1$ $|A - rI| = 0$ and we get $r_1 = r_2 = r$, then we solve $*^2$ $(A - rI)\xi = \begin{pmatrix}0\\0\end{pmatrix}$ to find $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$
- Now to find the second eigenvector $\xi_2 = \begin{pmatrix}z_1\\z_2\end{pmatrix}$ we solve $*^3$

$$\boxed{(A - rI)\xi_2 = \xi_1} \quad \cdots \; *^3$$

- Hence, the 1st solution is $x_1(t) = \xi_1 e^{rt}$

  the 2nd solution is $x_2(t) = \xi_2 e^{rt} + \xi_1 t e^{rt}$

- The gen. sol. is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

$$= c_1 \xi_1 e^{rt} + c_2\left(\xi_1 t e^{t} + \xi_2 e^{rt}\right)$$

Exp Solve this linear system: $x_1' = x_1 - 4x_2$, $x_1(0) = 3$

$x_2' = 4x_1 - 7x_2$, $x_2(0) = 2$

- First find the eigenvalues by solving $*^1 \Rightarrow |A - rI| = 0$

$$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0 \quad \Rightarrow (1-r)(-7-r) - -16 = 0$$

$$-7 - r + 7r + r^2 + 16 = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)(r+3) = 0$$

$$r_1 = r_2 = r = -3 \quad \text{repeated eigenvalues}$$

- Now we find the eigenvector $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ corresponding to the eigenvalue $r = -3$ by solving $*^2 \Rightarrow$

$$(A-rI)\xi_1=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1-r & -4\\ 4 & -7-r\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}4 & -4\\ 4 & -4\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow 4y_1-4y_2=0 \Rightarrow y_1=y_2$$

Take $y_1=1 \Rightarrow y_2=1 \Rightarrow \xi_1=\begin{pmatrix}y_1\\y_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

- Hence, 1^{st} solution is $x_1(t)=\xi_1 e^{rt}=\begin{pmatrix}1\\1\end{pmatrix}e^{-3t}$
- To find the 2^{nd} eigenvecter $\xi_2=\begin{pmatrix}z_1\\z_2\end{pmatrix}$ we solve $*^3 \Rightarrow$

$$(A-rI)\xi_2=\xi_1 \Rightarrow \begin{pmatrix}1-r & -4\\ 4 & -7-r\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}4 & -4\\ 4 & -4\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow 4z_1-4z_2=1 \Rightarrow z_1=\frac{1}{4}+z_2$$

Take $z_2=K \Rightarrow z_1=\frac{1}{4}+K \Rightarrow \xi_2=\begin{pmatrix}z_1\\z_2\end{pmatrix}=\begin{pmatrix}\frac{1}{4}+K\\K\end{pmatrix}=\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix}+K\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}$

To be cancelled since it is multiple of ξ_1

- Hence, 2^{nd} solution is $x_2(t)=\xi_2 e^{rt}+\xi_1 t e^{rt}$

$$x_2(t)=\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix}e^{-3t}+\begin{pmatrix}1\\1\end{pmatrix}te^{-3t}$$

- Thus, the gen. sol. becomes:

$$x(t)=c_1x_1(t)+c_2x_2(t)$$

$$=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-3t}+c_2\left[\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix}e^{-3t}+\begin{pmatrix}1\\1\end{pmatrix}te^{-3t}\right]$$

To find c_1 and c_2 $\Rightarrow$ we use IC's:

$$X(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

- origin is called improper node in repeated roots $r_1 = r_2 = r$
- If $r<0$, then origin is Asy. stable Eq. point.
- If $r>0$, then origin is unstable Eq. point.

$$c_1 + \tfrac{1}{4} c_2 = 3$$
$$c_1 + 0\, c_2 = 2 \quad \Rightarrow \boxed{c_1 = 2} \quad \Rightarrow \boxed{c_2 = 4}$$

Hence, $X(t) = \begin{pmatrix}2\\2\end{pmatrix}e^{-3t} + \begin{pmatrix}1\\0\end{pmatrix}e^{-3t} + \begin{pmatrix}4\\4\end{pmatrix}t\, e^{-3t}$

Exp Solve $f'(t) = f(t) + g(t)$, $f(0) = 1$
$g'(t) = 2f(t) - g(t)$, $g(0) = -1$

$x_1(t) = f(t)$
$x_2(t) = g(t)$

(S1) Find eigenvalues using $*^1 \Rightarrow |A - rI| = 0$

$$\begin{vmatrix}1-r & 1\\ 2 & -1-r\end{vmatrix} = 0 \Rightarrow (1-r)(-1-r) - 2 = 0$$
$$-1 - r + r + r^2 - 2 = 0$$
$$r^2 - 3 = 0$$
$$(r-\sqrt{3})(r+\sqrt{3}) = 0$$
$$r_1 = \sqrt{3},\ r_2 = -\sqrt{3}$$ real different eigenvalues

- To find $\xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix}$ for the corresponding $r_1 = \sqrt{3}$, we solve $*^2$:

$$(A - r_1 I)\xi_1 = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}1-\sqrt{3} & 1\\ 2 & -1-\sqrt{3}\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(1-\sqrt{3})y_1 + y_2 = 0 \Rightarrow y_2 = (\sqrt{3}-1)y_1$$

Take $y_1 = 1 \Rightarrow y_2 = \sqrt{3} - 1 \Rightarrow \xi_1 = \begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}1\\ \sqrt{3}-1\end{pmatrix}$

Hence, 1^{st} solution is $X_1(t) = \xi_1 e^{r_1 t} = \begin{pmatrix}1\\ \sqrt{3}-1\end{pmatrix}e^{\sqrt{3}t}$

• To find $\xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ for the corresponding $r_2 = -\sqrt{3}$, we solve $*^2$:

$$(A - r_2 I)\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1+\sqrt{3} & 1 \\ 2 & -1+\sqrt{3} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (1+\sqrt{3}) z_1 + z_2 = 0$$
$$z_2 = -(1+\sqrt{3}) z_1$$

Take $z_1 = 1 \Rightarrow z_2 = -(1+\sqrt{3}) \Rightarrow \xi_2 = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix}$

Hence, 2^{nd} solution is $X_2(t) = \xi_2 e^{r_2 t} = \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$

• Thus, gen. sol. is $X(t) = c_1 X_1(t) + c_2 X_2(t)$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} e^{\sqrt{3}t} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$$

• To find c_1 and $c_2 \Rightarrow$ we use IC's :

$$X(0) = c_1 \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\boxed{c_1 + c_2 = 1}$ — ①

$(\sqrt{3}-1)c_1 - (1+\sqrt{3})c_2 = -1 \quad \Rightarrow \sqrt{3}c_1 - \sqrt{3}c_2 - (c_1 + c_2) = -1$

$$\sqrt{3}c_1 - \sqrt{3}c_2 - 1 = -1$$
$$\sqrt{3}c_1 - \sqrt{3}c_2 = 0$$
$$\boxed{c_1 = c_2} — ②$$

$c_1 = c_2 = \frac{1}{2}$

Hence, the gen. sol. becomes:

$$X(t) = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix} e^{\sqrt{3}t} + \frac{1}{2}\begin{pmatrix} 1 \\ -1-\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}$$

$$= \begin{pmatrix} \dfrac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2} \\ \left(\dfrac{e^{\sqrt{3}t} - e^{-\sqrt{3}t}}{2}\right)\sqrt{3} - \left(\dfrac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2}\right) \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{3}t \\ \sqrt{3}\sinh\sqrt{3}t - \cosh\sqrt{3}t \end{pmatrix}$$

52 $f'(t) = f(t) + g(t)$, $f(0) = 1$

$g'(t) = 2f(t) - g(t)$, $g(0) = -1$

$L\{f'(t)\} = L\{f(t)\} + L\{g(t)\}$

$L\{g'(t)\} = 2L\{f(t)\} - L\{g(t)\}$

$sF(s) - f(0) = F(s) + G(s)$

$sG(s) - g(0) = 2F(s) - G(s)$

$F(s)(s-1) = 1 + G(s) \quad \Rightarrow \boxed{G(s) = (s-1)F(s) - 1}$ — 1

$G(s)(s+1) = -1 + 2F(s)$ ... 2

substitute 1 in 2 $\Rightarrow \big((s-1)F(s) - 1\big)(s+1) = -1 + 2F(s)$

$(s-1)(s+1)F(s) - (s+1) + 1 - 2F(s) = 0$

$(s^2-1)F(s) - s \cancel{-1} \cancel{+1} - 2F(s) = 0$

$(s^2-3)F(s) = s \quad \Rightarrow \quad F(s) = \dfrac{s}{s^2-3}$

$f(t) = L^{-1}\left(\dfrac{s}{s^2-3}\right) = \cosh\sqrt{3}\,t$

$f'(t) = \sqrt{3}\sinh\sqrt{3}\,t \quad \Rightarrow$ But $f'(t) = f(t) + g(t)$

$\sqrt{3}\sinh\sqrt{3}\,t = \cosh\sqrt{3}\,t + g(t)$

Hence, $g(t) = \sqrt{3}\sinh\sqrt{3}\,t - \cosh\sqrt{3}\,t$

Note that from 51 we have the gen. sol. is

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} \cosh\sqrt{3}\,t \\ \sqrt{3}\sinh\sqrt{3}\,t - \cosh\sqrt{3}\,t \end{pmatrix}$$