Student Solutions Manual

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Elementary Differential Equations and Boundary Value Problems

Ninth Edition

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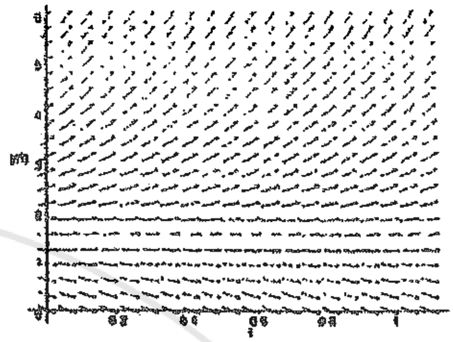
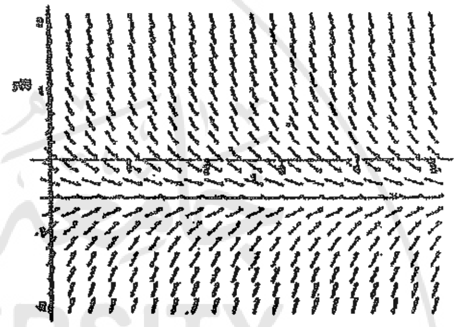
CHAPTER 1

Section 1.1, Page 7



2. For $y > 3/2$ we see that $y' > 0$ and thus $y(t)$ is increasing there. For $y < 3/2$ we have $y' < 0$ and thus $y(t)$ is decreasing there. Hence $y(t)$ diverges from $3/2$ as $t \to \infty$.



4. Observing the direction field, we see that for $y > -1/2$ we have $y' < 0$, so the solution is decreasing. Likewise, for $y < -1/2$ we have $y' > 0$ and thus $y(t)$ is increasing. These results are consistent with the given D.E. Since the slopes get closer to zero as y gets closer to $-1/2$, we conclude that $y \to -1/2$ as $t \to \infty$.

7. If all solutions approach 3, then 3 is the equilibrium solution and we want $\frac{dy}{dt} < 0$ for $y > 3$ and $\frac{dy}{dt} > 0$ for $y < 3$. Thus $\frac{dy}{dt} = 3 - y$, which is not unique as there are other possible answers, such as $\frac{dy}{dt} = 6 - 2y$.

9. If solutions diverge from 2 then we want $\frac{dy}{dt} > 0$ for $y > 2$ and $\frac{dy}{dt} < 0$ for $y < 2$. Thus $\frac{dy}{dt} = y - 2$ is a possible D.E.

11. For $y = 0$ and $y = 4$ we have $y' = 0$ and thus $y = 0$ and $y = 4$ are equilibrium solutions. For $y > 4$, $y' < 0$ so if $y(0) > 4$ the solution approaches $y = 4$ from above. If $0 < y(0) < 4$, then $y' > 0$ and the solutions "grow" to $y = 4$ as $t \to \infty$. For $y(0) < 0$ we see that $y' < 0$ and the solutions diverge from 0.

13. Since $y' = y^2$, $y = 0$ is the equilibrium solution and $y' > 0$ for all y. Thus if $y(0) > 0$, solutions will diverge from 0 and if $y(0) < 0$, solutions will aproach $y = 0$ as $t \to \infty$.

16. From Fig. 1.1.6 we see that $y = 2$ is an equilibrium solution and thus (c) and (g) are the only posible D.E. to

consider. Since $\frac{dy}{dt} > 0$ for $y > 2$, and $\frac{dy}{dt} < 0$ for $y < 2$ we conclude that (c) is the correct answer.

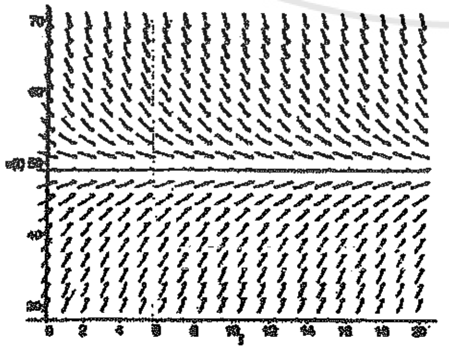
19. From Fig. 1.1.9 we see that $y = 0$ and $y = 3$ are equilibrium solutions, so (e) and (h) are the only possible D.E. Furthermore, in Fig. 1.1.9 we have $\frac{dy}{dt} < 0$ for $y > 3$ and for $y < 0$, and $\frac{dy}{dt} > 0$ for $0 < y < 3$. This tells us that (h) is the desired D.E.

21a. Let $q(t)$ be the number of grams of the chemical in the water at any time. Then $\frac{q(t)}{1,000,000}$ represents the concentration of the chemical in the pond at any time and hence $\frac{300q(t)}{1,000,000}$ is the rate at which the chemical leaves the pond per hour and $300(.01)$ represents the amount of the chemical coming into the pond per hour. Thus $\frac{dq}{dt} = 300(.01) - \frac{300q}{1,000,000} = 300(10^{-2} - 10^{-6}q)$.

21b. The equilibrium solution occurs when $q' = 0$, or $q = 10^4$gm. Since $q' > 0$ for $q < 10^4$ gm and $q' < 0$ for $q > 10^4$ gm all solutions approach the equilibriium solution indpendent of the amount present at $t = 0$.

22. Let V be the volume, S the surface area and a the constant of proportionality. Then the D.E. expressing the evaporation is $\frac{dV}{dt} = -aS$, $a > 0$. Now $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$, so $S = 4\pi\left(\frac{3}{4\pi}\right)^{2/3} V^{2/3}$. Thus $\frac{dV}{dt} = -skV^{2/3}$, for $k>0$.

25d.



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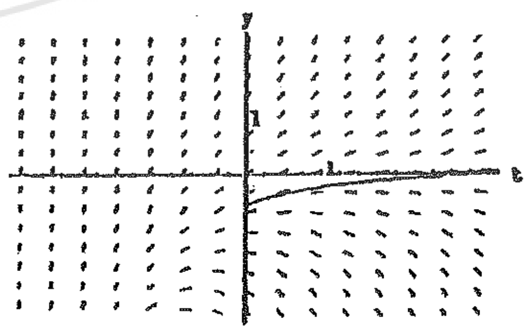

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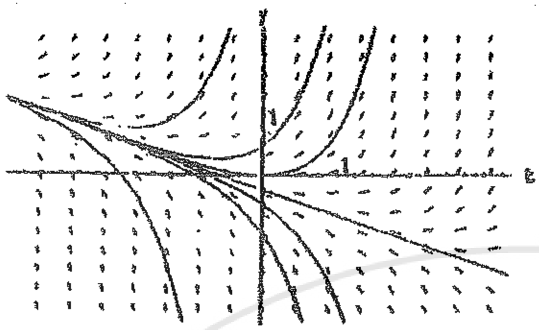

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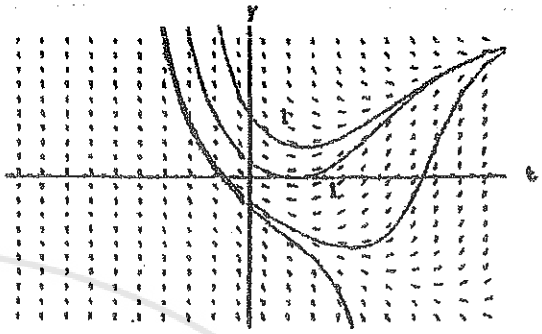

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1b. $dy/dt = -2y+5$ can be rewritten as $\frac{dy}{y-5/2} = -2dt$. Thus $\ln|y-5/2| = -2t+c_1$, or $y-5/2 = ce^{-2t}$. $y(0) = y_0$ yields $c = y_0 - 5/2$, so $y = 5/2 + (y_0-5/2)e^{-2t}$. If $y_0 > 5/2$, the solution starts above the equilibrium solution and decreases exponentially and approaches 5/2 as $t\to\infty$. Conversely, if $y_0 < 5/2$, the solution starts below 5/2 and grows exponentially and approaches 5/2 from below as $t\to\infty$.

2b. As in 1b., the D.E. can be rewritten as $\frac{dy}{y-5/2} = 2dt$. Integrating (as above) and solving yields $y = 5/2 + ce^{2t}$, so if $y(0) = y_0$ we find $y_0 = 5/2 + c$. Hence $y(t) = 5/2 + (y_0 - 5/2)e^{2t}$. Again $y = 5/2$ is the equilbrium solution, but all other solutions diverge from this due to the positive exponential.

3a. Note that $\frac{dy}{dt} = -a(y - b/a)$ and thus $\frac{dy}{y-b/a} = -a$. Integration, as in the text, yields $\ln|y-b/a| = -at + c$, or $y = \frac{b}{a} + Ce^{-at}$.

3c. (i) If a increases then b/a is smaller. Thus the equilibrium solution is lower and it is reached sooner, since e^{-at} decays faster for larger a. (ii) If b increases then b/a (the equilibrium solution) is larger, but a is constant, so there is no change in the rate of approach. Similar analysis will yield the result for iii).

5a. Rewrite Eq.(ii) as $\frac{dy/dt}{y} = a$ and thus $\ln|y| = at+C$; or $y_1 = ce^{at}$.

5b. If $y = y_1(t) + k$, then $\frac{dy}{dt} = \frac{dy_1}{dt}$. Substituting both these into Eq.(i) we get $\frac{dy_1}{dt} = a(y_1+k) - b$. Since $\frac{dy_1}{dt} = ay_1$, this leaves $ak - b = 0$ and thus $k = b/a$. Hence $y = y_1(t) + b/a$ is the solution to Eq(i).

5c. Substitution of $y_1 = ce^{at}$ shows this is the same as that given in Eq.(17).

7b. From Eq.(11) we have $p = 900 + ce^{t/2}$. If $p(0) = p_0$, then $c = p_0 - 900$ and thus $p = 900 + (p_0 - 900)e^{t/2}$. If $p_0 < 900$, this decreases, so if we set $p = 0$ and solve for T (the time of extinction) we obtain $e^{T/2} = 900/(900-p_0)$, or $T = 2\ln[900/(900-p_0)]$ months.

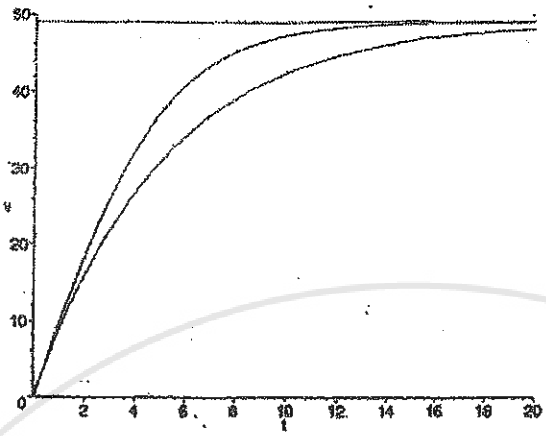
9a. The solution to this problem is given by Eq.(26), which has a limiting velocity of 49 m/sec. Substituting $v = 48.02$ (which is 98% of 49) into Eq.(26) yields $e^{-t/5} = .02$. Solving for t we have $t = -5\ln(.02) = 19.56$ sec.

9b. Use Eq.(29) with $t = 19.56$.

11a. If the drag force is proportional to v^2 then $F = 98 - kv^2$ is the net force acting on the falling mass (m = 10kg.). Thus $10\frac{dv}{dt} = 98 - kv^2$, which has a limiting velocity of $v^2 = 98/k$. Setting $v^2 = 49^2$ gives $k = 98/49^2$ and hence $\frac{dv}{dt} = \frac{49^2 - v^2}{10/k} = (49^2-v^2)/245$.

11b. From part a we have $\frac{dv}{49^2-v^2} = \frac{dt}{245}$, which yields $\frac{1}{49}\tanh^{-1}\left(\frac{v}{49}\right) = \frac{t}{245} + C_0$. Setting t=0 and v=0, the initial point, we then have $0 = 0 + C_0$, or $C_0 = 0$. Thus $\tanh^{-1}\left(\frac{v}{49}\right) = t/5$, or $v(t) = 49\tanh(t/5)$ m/sec.

11c.



11d. In the graph of 11c, the solution for the linear drag force lies below the solution for the quadratic drag force. This latter solution approaches equilibrium faster since, as the velociy increases, there is a larger drag force.

11e. Note that $\int \tanh(x)dx = \ln(\cosh(x)) + C$.

12a. $\frac{dQ}{dt} = -rQ$ yields $\frac{dQ/dt}{Q} = -r$, or $\ln|Q| = -rt + c_1$. Thus $Q = ce^{-rt}$ and $Q(0) = 100$ yields $c = 100$. Hence $Q = 100e^{-rt}$. Setting $t = 1$, we have $82.04 = 100e^{-r}$, which yields $r = .1980$/wk or $r = .02828$/day.

15a. Rewrite the D.E. as $\frac{du}{u-T} = -kdt$ and then integrate to find $\ln|u-T| = -kt + c$. Thus $u - T = \pm Ce^{-kt}$. For $t = 0$ we have $u_0 - T = \pm C$ and thus $u(t) = T + (u_0 - T)e^{-kt}$.

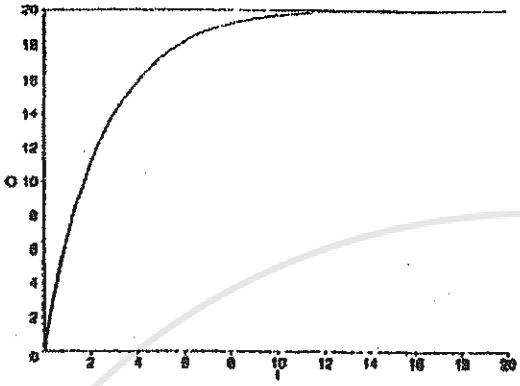
15b. Set $u(\tau) - T = \frac{u_0 - T}{2}$ when $t = \tau$ in the solution of part (a).

17a. Rewrite the D.E. as $\frac{dQ/dt}{Q-CV} = \frac{-1}{CR}$, thus, upon integrating and simplifying, we get $Q = De^{-t/CR} + CV$. $Q(0) = 0 \Rightarrow D = -CV$ and thus $Q(t) = CV(1 - e^{-t/CR})$.

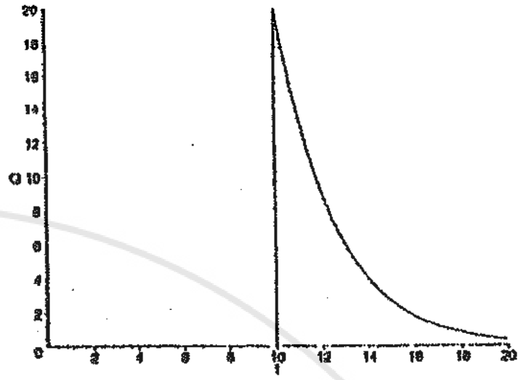
17b. $\lim_{t\to\infty} Q(t) = CV$ since $\lim_{t\to\infty} e^{-t/CR} = 0$.

17c. In this case $R\frac{dQ}{dt} + \frac{Q}{C} = 0$, $Q(t_1) = CV$. The solution of this D.E. is $Q(t) = Ee^{-t/CR}$, so $Q(t_1) = Ee^{-t_1/CR} = CV$, or $E = CVe^{t_1/CR}$. Thus $Q(t) = CVe^{t_1/CR}e^{-t/CR} = CVe^{-(t-t_1)/CR}$ for $t \ge t_1$.

17a. CV = 20, CR = 2.5



17c. CV = 20, CR = 2.5, $t_1 = 10$



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2. The D.E. is second order since there is a second derivative of y appearing in the equation. The equation is nonlinear due to the y^2 term (as well as due to the y^2 term multiplying the y'' term).

6. This is a third order D.E. since the highest derivative is y''' and it is linear since y and all its derivatives appear to the first power only. The terms t^3 and $\cos^2 t$ do not affect the linearity of the D.E.

8. For $y_1(t) = e^{-3t}$ we have $y_1'(t) = -3e^{-3t}$ and $y_1''(t) = 9e^{-3t}$. Substitution of these into the D.E. yields $9e^{-3t} + 2(-3e^{-3t}) - 3(e^{-3t}) = (9-6-3)e^{-3t} = 0$.

11. Substituting $y_1 = t^{1/2}$ into the D.E. we get

$$2t^2\left(\frac{-1}{4}t^{-3/2}\right) + 3t\left(\frac{1}{2}t^{-1/2}\right) - t^{1/2} = \left(\frac{-1}{2}\right)t^{1/2} + \left(\frac{3}{2}\right)t^{1/2} - t^{1/2} = 0.$$

14. Recall that if $u(t) = \int_0^t f(s)ds$, then $u'(t) = f(t)$.

16. Differentiating e^{rt} twice and substituting into the D.E. yields $r^2e^{rt} - e^{rt} = (r^2-1)e^{rt}$. If $y = e^{rt}$ is to be a solution of the D.E. then the last quantity must be zero for all t. Thus $r^2-1 = 0$ since e^{rt} is never zero.

19. Differentiating t^r twice and substituting into the D.E. yields $t^2[r(r-1)t^{r-2}] + 4t[rt^{r-1}] + 2t^r = [r^2+3r+2]t^r$. If $y = t^r$ is to be a solution of the D.E. the last term must be zero for all t and thus $r^2 + 3r + 2 = 0$.

22. The D.E. is second order since there are second partial derivatives of u(x,y). The D.E. is nonlinear due to the product of u(x,y) times u_x (or u_y).

26. Since $\frac{\partial u_1}{\partial t} = -\alpha^2 e^{-\alpha^2 t}\sin x$ and $\frac{\partial^2 u_1}{\partial x^2} = -e^{-\alpha^2 t}\sin x$ we have

$\alpha^2[-e^{-\alpha^2 t}\sin x] = -\alpha^2 e^{-\alpha^2 t}\sin x$, which is true for all t and x.

29a. The free-body diagram is essentially shown in Fig. 1.3.1. The gravitational force is shown. The only other force is the tension, T, which acts towards the hinge along L.

29b. The component of the gravitational force (mg) along the tangent to the circular arc is given by $-mg\sin\theta$. The minus sign arises since θ is positive in the counterclockwise direction and the gravitational force is clockwise. Since T acts perpendicular to the tangent, there is no component of the tension in the tangential direction. Newton's Second Law states that F = ma. In this problem, since the motion is circular, it is appropriate to use polar coordinates and thus $a \neq \frac{dv}{dt}$ as we have used earlier. Since r = L is constant, the linear acceleration tangent to the circular motion is given by $L\frac{d^2\theta}{dt^2}$ and thus Newton's Second Law gives $-mg\sin\theta = mL\frac{d^2\theta}{dt^2}$.

29c. Dividing by mL and rearranging terms gives $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$.

CHAPTER 2

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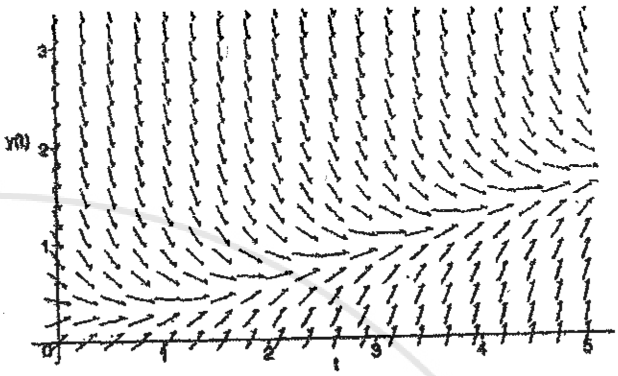
1a.



1b. All solutions seem to approach a line in the region where the negative and positive slopes meet each other.

1c. $\mu(t) = \exp(\int 3dt) = e^{3t}$. Thus $e^{3t}(y'+3y) = e^{3t}(t+e^{-2t})$ or $\frac{d}{dt}(ye^{3t}) = te^{3t} + e^t$. Integration of both sides yields $ye^{3t} = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + e^t + c$, where integration by parts is used on the right side, with $u = t$ and $dv = e^{3t}dt$. Division by e^{3t} gives $y(t) = ce^{-3t} + t/3 - 1/9$, so y approaches $t/3 - 1/9$ as $t \to \infty$. This is the line identified in part b.

2c. $\mu(t) = e^{-2t}$.

3c. $\mu(t) = e^t$.

4c. $\mu(t) = \exp(\int \frac{dt}{t}) = e^{\ln t} = t$, so $(ty)' = 3t\cos 2t$, and integration by parts yields the general solution.

6c. The equation must be divided by t so that it is in the form of Eq.(3): $y' + (2/t)y = (\sin t)/t$. Thus $\mu(t) = \exp(\int \frac{2dt}{t} = t^2$, and $(t^2y)' = t\sin t$. Integration then yields $t^2y = -t\cos t + \sin t + c$.

7c. $\mu(t) = e^{t^2}$.

8c. $\mu(t) = \exp(\int \frac{4tdt}{1+t^2}) = (1+t^2)^2$.

11c. $\mu(t) = e^t$ so $(e^ty)' = 5e^t\sin 2t$. To integrate the right side you can integrate by parts (twice), use an integral table, or use a symbolic computational software program to find $e^ty = e^t(\sin 2t - 2\cos 2t) + c$.

13. $\mu(t) = e^{-t}$ so that $(e^{-t}y)' = 2te^t$ and thus $e^{-t}y = 2\int te^tdt + c = 2(te^t - \int e^tdt) + c = 2(te^t - e^t) + c$. Thus $y(t) = 2(t-1)e^{2t} + ce^t$, so setting $t = 0$ we have $1 = -2 + c$, or $c = 3$. Hence $y(t) = 2(t-1)e^{2t} + 3e^t$.

15. $\mu(t) = \exp(\int \frac{2dt}{t}) = t^2$ so that $(t^2y)' = t^3 - t^2 + t$.
Integrating and dividing by t^2 gives
$y = t^2/4 - t/3 + 1/2 + c/t^2$. Setting $t = 1$ and $y = 1/2$ we have $c = 1/12$.

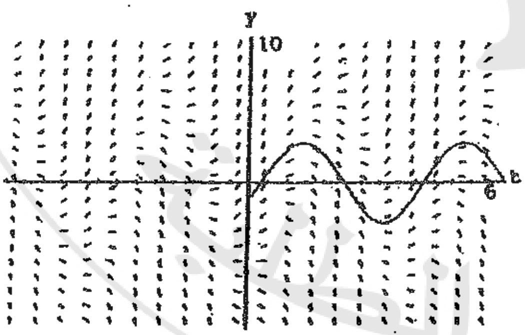
18. $\mu(t) = t^2$. Thus $(t^2y)' = t\sin t$ and
$t^2y = -t\cos t + \sin t + c$. Setting $t = \pi/2$ and $y = 1$ yields $c = \pi^2/4 - 1$.

20. $\mu(t) = \exp\int \frac{t+1}{t}dt = \exp(t + \ln(t)) = te^t$.

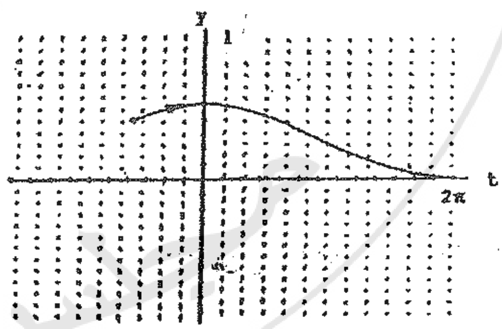
21b. $\mu(t) = e^{-t/2}$ so $(e^{-t/2}y)' = 2e^{-t/2}\cos t$. Integrating (see comments in Prob.11) and dividing by $e^{-t/2}$ yields
$y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + ce^{t/2}$. Thus $y(0) = -\frac{4}{5} + c = a$,
or $c = a + \frac{4}{5}$ and $y(t) = -\frac{4}{5}\cos t + \frac{8}{5}\sin t + (a + \frac{4}{5})e^{t/2}$.

21c. If $(a + \frac{4}{5}) = 0$, then the solution is oscillatory for all t, while if $(a + \frac{4}{5}) \neq 0$, the solution is unbounded as $t \to \infty$. Thus $a_0 = -\frac{4}{5}$.

21a.



25a.



25b. $\mu(t) = \exp\int \frac{2dt}{t} = t^2$, so $(t^2y)' = \sin t$ and
$y(t) = \frac{-\cos t}{t^2} + \frac{c}{t^2}$. Setting $t = -\frac{\pi}{2}$ yields
$\frac{4c}{\pi^2} = a$ or $c = \frac{a\pi^2}{4}$ and hence $y(t) = \frac{a\pi^2/4 - \cos t}{t^2}$, which is unbounded as $t \to 0$ unless $a\pi^2/4 = 1$ or $a_0 = 4/\pi^2$.

25c. For $a = 4/\pi^2$ $y(t) = \dfrac{1 - \cos t}{t^2}$. To find the limit as $t \to 0$ L'Hospital's Rule must be used:

$$\lim_{t\to 0} y(t) = \lim_{t\to 0}\frac{\sin t}{2t} = \lim_{t\to 0}\frac{\cos t}{2} = \frac{1}{2}.$$

26b. $\mu(t) = \exp\int\frac{\cos(t)}{\sin(t)}dt = \exp(\ln(\sin t)) = \sin(t)$ and thus $(\sin(t)y)' = e^t$. Hence $\sin(t)y = e^t + c$ or $y(t) = \dfrac{e^t + c}{\sin(t)}$. Setting $t = 1$ and $y = a$ we get $c = a\sin 1 - e$ so $y(t) = (e^t - e + a\sin 1)/\sin(t)$. If $y(t)$ is to remain finite as $t \to 0$ the numerator, $e^t - e + a\sin 1$, must approach 0 as $t \to 0$ and hence $a_0 = (e-1)/\sin 1$.

26c. Using a_0 we have $y(t) = (e^t - 1)/\sin(t)$, which approaches 1 as $t \to 0$, using l'Hospital's Rule.

30. $(e^{-t}y)' = e^{-t} + 3e^{-t}\sin t$ so $e^{-t}y = -e^{-t} - 3e^{-t}\left(\dfrac{\sin t + \cos t}{2}\right) + c$ or

$y(t) = -1 - \left(\dfrac{3}{2}\right)e^{-t}(\sin t + \cos t) + ce^t$. Thus

$y(0) = -1 - \dfrac{3}{2} + c = y_0$ or $c = y_0 + \dfrac{5}{2}$. Now, if $y(t)$ is to remain bounded as $t \to \infty$, we must have $c = 0$ so that $y_0 = -5/2$.

32. Write the first term of Eq.(47) as $\dfrac{\int_0^t e^{s^2/4}ds}{e^{t^2/4}}$. In applying L'Hospital's Rule, the derivative of the numerator term is $e^{t^2/4}$ by the Fundamental Theorem of Calculus. The derivative of the denominator is $(t/2)e^{t^2/4}$ and thus the limit of both terms in Eq.(47) is 0 as $t \to \infty$.

33. $\mu(t) = e^{at}$ so the D.E. can be written as $(e^{at}y)' = be^{at}e^{-\lambda t} = be^{(a-\lambda)t}$. If $a \neq \lambda$, then integration and solution for y yields $y = [b/(a-\lambda)]e^{-\lambda t} + ce^{-at}$. Then $\lim_{t\to\infty} y$ is zero since both λ and a are positive numbers.

If $a = \lambda$, then the D.E. becomes $(e^{at}y)' = b$, which yields $y = (bt+c)/e^{\lambda t}$ as the solution. L'Hospital's Rule gives

$$\lim_{t\to\infty} y = \lim_{t\to\infty}\frac{(bt+c)}{e^{\lambda t}} = \lim_{t\to\infty}\frac{b}{\lambda e^{\lambda t}} = 0.$$

35. There is no unique answer for this situation. One possible answer is to assume $y(t) = ce^{-2t} + 3 - t$ (which satisfies the given condition), then $y'(t) = -2ce^{-2t} - 1$. Eliminating ce^{-2t} between the two equations yields $y' + 2y = 5 - 2t$.

39. By Eq.(iii), Prob.38, $y(t) = A(t)\exp(-\int(-2)dt) = A(t)e^{2t}$. Differentiating $y(t)$ and substituting into the D.E. yields $A'(t) = t^2$ since the terms involving $A(t)$ add to zero. Thus $A(t) = t^3/3 + c$, which substituted into $y(t)$ yields the solution.

42. Since $p(t) = \frac{1}{2}$, $y(t) = A(t)\exp(-\int\frac{dt}{2}) = A(t)e^{-t/2}$ and $A'(t) = (3/2)t^2e^{t/2}$. Integration of A' and substituting in $y(t)$ yields the desired solution.

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Problems 1 through 20 follow the pattern of the examples worked in this section. The first eight problems, however, do not have I.C. so the integration constant, c, cannot be found.

1. Write the equation in the form $ydy = x^2dx$. Integrating the left side with respect to y and the right side with respect to x yields
$\frac{y^2}{2} = \frac{x^3}{3} + C$, or $3y^2 - 2x^3 = c$.

4. For $y \neq -3/2$ multiply both sides of the equation by $3 + 2y$ to get the separated equation
$(3+2y)dy = (3x^2-1)dx$. Integration then yields
$3y + y^2 = x^3 - x + c$.

6. We need $x \neq 0$ and $|y| \leq 1$ for this problem to be defined. Separating the variables we get $(1-y^2)^{-1/2}dy = x^{-1}dx$. Integrating each side yields $\arcsin y = \ln|x| + c$, so $y = \sin[\ln|x| + c]$, $x \neq 0$ (note that $|y| < 1$). Also, $y = \pm 1$ satisfy the D.E., since both sides are zero.

10a. Separating the variables we get $ydy = (1-2x)dx$, so

$\frac{y^2}{2} = x - x^2 + c$. Setting $x = 1$ and $y = -2$ we have $2 = c$ and thus $y^2 = 2x - 2x^2 + 4$ or $y = -\sqrt{2x - 2x^2 + 4}$. The negative square root must be used since $y(1) = -2$.

10b. The graph is the bottom half of the ellipse $2x^2 - 2x + y^2 = 4$.

10c. Rewriting $y(x)$ as $-\sqrt{2(2-x)(x+1)}$, we see that y is defined for $-1 \le x \le 2$, However, since y' does not exist for $x = -1$ or $x = 2$, the solution is valid only for the open interval $-1 < x < 2$.

13. Separate variables by factoring the denominator of the right side to get $ydy = \frac{2x}{1+x^2}dx$. Integration yields $y^2/2 = \ln(1+x^2)+c$ and use of the I.C. gives $c = 2$. Thus $y = \pm [2\ln(1+x^2)+4]^{1/2}$, but we must discard the plus square root because of the I.C. Since $1 + x^2 > 0$, the solution is valid for all x.

15. Separating variables and integrating yields $y + y^2 = x^2 + c$. Setting $y = 0$ when $x = 2$ yields $c = -4$ or $y^2 + y = x^2-4$. To solve for y complete the square on the left side by adding 1/4 to both sides. This yields $y^2 + y + \frac{1}{4} = x^2 - 4 + \frac{1}{4}$ or $(y + \frac{1}{2})^2 = x^2 - 15/4$. Taking the square root of both sides yields $y + \frac{1}{2} = \pm\sqrt{x^2 - 15/4}$, where the positive square root must be taken in order to satisfy the I.C. Thus $y = -\frac{1}{2} + \sqrt{x^2 - 15/4}$, which is defined for $x^2 \ge 15/4$ or $x \ge \sqrt{15}/2$.

17a. Separating variables gives $(2y-5)dy = (3x^2-e^x)dx$ and integration then gives $y^2 - 5y = x^3 - e^x + c$. Setting $x = 0$ and $y = 1$ we have $1 - 5 = 0 - 1 + c$, or $c = -3$ and thus $y^2 - 5y - (x^3-e^x-3) = 0$. Using the quadratic formula then gives

$$y(x) = \frac{5 \pm \sqrt{25+4(x^3-e^x-3)}}{2} = \frac{5}{2} - \sqrt{\frac{13}{4} + x^3 - e^x},$$

where the negative square root is chosen so that $y(0) = 1$.

17c. The interval of definition for y must be found numerically. Approximate values can be found by plotting $y_1(x) = \frac{13}{4} + x^3$ and $y_2(x) = e^x$ and noting the values of x where the two curves cross.

19a. We start with cos3ydy = -sin2xdx and integrate to get $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + c$. Setting $y = \pi/3$ when $x = \pi/2$ (from the I.C.) we find that $0 = -\frac{1}{2} + c$ or $c = \frac{1}{2}$, so that $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x$ (using the appropriate trigonometric identity). To solve for y we must choose the branch that passes through the point $(\pi/2,\pi/3)$ and thus $3y = \pi - \arcsin(3\cos^2 x)$, or $y = \frac{\pi}{3} - \frac{1}{3}\arcsin(3\cos^2 x)$.

19c. The solution in part a is defined only for $0 \le 3\cos^2 x \le 1$, or $-\sqrt{1/3} \le \cos x \le \sqrt{1/3}$. Taking the indicated square roots and then finding the inverse cosine of each side yields $.9553 \le x \le 2.1863$, or $|x-\pi/2| \le 0.6155$, as the approximate interval.

21. We have $(3y^2-6y)dy = (1+3x^2)dx$ so that $y^3-3y^2 = x + x^3 - 2$, once the I.C. are used. From the D.E., the integral curve will have a vertical tangent when $3y^2 - 6y = 0$, or $y = 0,2$. For $y = 0$ we have $x^3 + x - 2 = 0$, which is satisfied for $x = 1$, which is the only zero of the function $w = x^3 + x - 2$. Likewise, for $y = 2$, $x = -1$. Thus the solution is valid on $|x| < 1$.

23. Separating variables gives $y^{-2}dy = (2+x)dx$, so $-y^{-1} = 2x + \frac{x^2}{2} + c$. $y(0) = 1$ yields $c = -1$ and thus
$$y = \frac{-1}{\frac{x^2}{2} + 2x - 1} = \frac{2}{2 - 4x - x^2}.$$ This gives
$$\frac{dy}{dx} = \frac{8 + 4x}{(2-4x-x^2)^2},$$ so the minimum value is attained at $x = -2$. Note that the solution is defined for $-2 - \sqrt{6} < x < -2 + \sqrt{6}$ (by finding the zeros of the

denominator) and has vertical asymptotes at the end points of the interval.

25. Separating variables and integrating yields $3y + y^2 = \sin 2x + c$. $y(0) = -1$ gives $c = -2$ so that $y^2 + 3y + (2-\sin 2x) = 0$. The quadratic formula, along with the I.C., then gives $y = -\frac{3}{2} + \sqrt{\sin 2x + 1/4}$, which is defined for $-.126 < x < 1.697$ (found by solving $\sin 2x = -.25$ for x and noting $x = 0$ is the initial point). Thus we have $\frac{dy}{dx} = \frac{\cos 2x}{(\sin 2x+\frac{1}{4})^{1/2}}$, which yields $x = \pi/4$ as the only critical point in the above interval. Using the second derivative test or graphing the solution indicates the critical point is a maximum.

27a. By sketching the direction field or by using the D.E. we note that $y' < 0$ for $y > 4$ and y' approaches zero as y approaches 4. For $0 < y < 4$, $y' > 0$ and again approaches zero as y approaches 4. Thus $\lim_{t\to\infty} y = 4$ if $y_0 > 0$. For $y_0 < 0$, $y' < 0$ for all y and hence y becomes negatively unbounded ($-\infty$) as t increases. If $y_0 = 0$, then $y' = 0$ for all t, so $y = 0$ for all t.

27b. Separating variables and using a partial fraction expansion we have $(\frac{1}{y} - \frac{1}{y-4})dy = \frac{4}{3}tdt$. Hence $\ln\left|\frac{y}{y-4}\right| = \frac{2}{3}t^2 + c_1$ and thus $\left|\frac{y}{y-4}\right| = e^{c_1}e^{2t^2/3} = ce^{2t^2/3}$, where c is positive. For $y(0) = y_0 = .5$ this becomes $\left|\frac{.5}{.5 - 4}\right| = c$ and thus $c = \frac{1}{7}$. Using this value for c and solving for y yields $y(t) = \frac{4}{1 + 7e^{-2t^2/3}}$. Setting this equal to 3.98 and solving for t yields $t = 3.29527$.

29. Separating variables yields $\frac{cy+d}{ay+b}\,dy = dx$. If $a \neq 0$ and $ay+b \neq 0$ then $dx = (\frac{c}{a}+\frac{ad-bc}{a(ay+b)})dy$. Integration then yields the desired answer.

30a. Divide numerator and denomintor by $x \neq 0$.

30c. If $v = y/x$ then $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and thus the D.E. becomes $v + x\frac{dv}{dx} = \frac{v-4}{1-v}$. Subtracting v from both sides yields $x\frac{dv}{dx} = \frac{v^2-4}{1-v}$.

30d. The last equation in (c) separates into $\frac{1-v}{v^2-4}dv = \frac{1}{x}dx$. To integrate the left side use partial fractions to write $\frac{1-v}{v-4} = \frac{A}{v-2} + \frac{B}{v+2}$, which yields $A = -1/4$ and $B = -3/4$. Integration then gives $-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| - k$, or $\ln|x^4||v-2||v+2|^3 = 4k$ after manipulations using properties of the ln function. Thus $x^4|v-2||v+2|^3 = C$.

30e. Recalling that $v = y/x$ gives the desired solution.

31a. Simplifying the right side of the D.E. gives $dy/dx = 1 + (y/x) + (y/x)^2$ so the equation is homogeneous.

31b. $y = vx$ gives $\frac{dy}{dx} = v + x\frac{dv}{dx}$, so substitution leads to $v + x\frac{dv}{dx} = 1 + v + v^2$ or $\frac{dv}{1 + v^2} = \frac{dx}{x}$. Integrating, we get $\arctan v = \ln|x| + c$ and substituting for v we obtain $\arctan(y/x) - \ln|x| = c$.

33b. Dividing the numerator and denominator of the right side by x and substituting $y = vx$ we get $v + x\frac{dv}{dx} = \frac{4v - 3}{2-v}$ which can be rewritten as $x\frac{dv}{dx} = \frac{v^2 + 2v - 3}{2 - v}$. Note that $v = -3$ and $v = 1$ are solutions of this equation. For $v \neq 1, -3$ separating variables gives $\frac{2 - v}{(v+3)(v-1)}\,dv = \frac{1}{x}dx$. Applying a partial fraction decomposition to the left side we obtain $[\frac{1}{4}\frac{1}{v-1} - \frac{5}{4}\frac{1}{v+3}]dv = \frac{dx}{x}$, and upon integrating both sides we find that $\frac{1}{4}\ln|v-1| - \frac{5}{4}\ln|v+3| = \ln|x| + c$.

Substituting for v and performing some algebraic manipulations we get the solution in the implicit form $|y-x| = c|y+3x|^5$. $v = 1$ and $v = -3$ yield $y = x$ and $y = -3x$, respectively, as solutions also.

35b. As in Prob.33, substituting $y = vx$ into the D.E. we get $v + x\frac{dv}{dx} = \frac{1+3v}{1-v}$, or $x\frac{dv}{dx} = \frac{(v+1)^2}{1-v}$. Note that $v = -1$ (or $y = -x$) satisfies this D.E. Separating variables yields $\frac{1-v}{(v+1)^2}dv = \frac{dx}{x}$. Integrating the left side by parts (let $u = 1-v$ and $dw = \frac{dv}{(v+1)^2}$) we obtain $\frac{v-1}{v+1} - \ln|v+1| = \ln|x| + c$. Letting $v = \frac{y}{x}$ then yields $\frac{y-x}{y+x} - \ln|\frac{y+x}{x}| = \ln|x| + c$, or $\frac{y-x}{y+x} - \ln|y+x| = c$. The answer in the text can be obtained by integrating the left side, above, using partial fractions. By differentiating both answers, it can be verified that indeed both forms satisfy the D.E.

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2. Let S(t) be the amount of salt that is present at any time t, then $S(0) = 0$ is the original amount of salt in the tank, 2γ is the amount of salt entering per minute, and $2(S/120)$ is the amount of salt leaving per minute (all amounts measured in grams). Thus $dS/dt = 2\gamma - 2S/120$, $S(0) = 0$. This is a linear equation, which has $e^{t/60}$ as its integrating factor. Thus the general solution is $S(t) = 120\gamma + ce^{-t/60}$. $S(0) = 0$ gives $c = -120\gamma$, so $S(t) = 120\gamma(1 - e^{-t/60})$ and hence $S(t) \to 120\gamma$ grams as $t \to \infty$.

3. We must first find the amount of salt that is present after 10 minutes. For the first 10 minutes (if we let Q(t) be the amount of salt in the tank): $\frac{dQ}{dt} = \frac{1}{2}(2) - 2\frac{Q(t)}{100}$, $Q(0) = 0$. This is a linear equation which has the solution $Q(t) = 50(1 - e^{-t/50})$, as in Prob. 2, and thus $Q(10) = 50(1-e^{-.2}) = 9.063$ lbs. of salt in the tank after the first 10 minutes. At this point no more salt is allowed to enter, so the new I.V.P. (letting P(t) be the amount of

salt in the tank after the first 10 minutes) is:
$\frac{dP}{dt} = (0)(2) - 2\frac{P(t)}{100}$, $P(0) = Q(10) = 9.063$. The solution of this problem is $P(t) = 9.063e^{-.02t}$, which yields $P(10) = 7.42$ lbs.

4. Salt flows out of the tank at the rate of $\frac{Q(t)}{200+t}(2)$ lb/min. since the volume of water in the tank at any time t is $200 + (1)(t)$ gallons (due to the fact that water flows into the tank faster than it flows out). Thus the I.V.P. is $dQ/dt = (3)(1) - \frac{2}{200+t}Q(t)$, $Q(0) = 100$, which is a linear equation with $(200+t)^2$ as its integrating factor.

8a. Set $S_0 = 0$ in Eq.(16) (or solve Eq.(15) with $S(0) = 0$).

8b. Set $r = .075$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to part (a) and then solve for k.

8c. Set $k = \$2,000$, $t = 40$ and $S(t) = \$1,000,000$ in the answer to (a) and then solve numerically for r.

9. Let S(t) be the amount of the loan remaining at time t, then $dS/dt = .1S - k$, $S(0) = \$8,000$. Solving this for S(t) yields $S(t) = 8000e^{.1t} - 10k(e^{.1t}-1)$. Setting $S = 0$ and substitution of $t = 3$ gives $k = \$3,086.64$ per year. For 3 years this totals \$9,259.92, so \$1,259.92 has been paid in interest.

10. Since we are assuming continuity, either convert the monthly payment into an annual payment or convert the yearly interest rate into a monthly interest rate for 240 months. Then proceed as in Prob. 9.

11a. Using Eq. (15) we have $\frac{dS}{dt} = \frac{.09}{12}S - 800(1+\frac{t}{120})$ or
$\frac{dS}{dt} - \frac{3}{400}S = -(800+\frac{20}{3}t)$, $S(0) = 100,000$. Using an integrating factor and integration by parts (or using a D.E. solver) we get $S(t) = \frac{6,080,000}{27} + \frac{8000}{9}t + ce^{3t/400}$. Using the I.C. yields $c = \frac{-3,380,000}{27}$. Substituting this value into S, setting $S(t) = 0$, and solving numerically for t yields $t \cong 135.36$ months.

14a. We have $\frac{dy}{y} = (.1+.2\sin t)dt$, by separating variables, and thus $y(t) = c\exp(.1t-.2\cos t)$. $y(0) = 1$ gives $c = e^{.2}$, so $y(t) = \exp(.2+.1t-.2\cos t)$. Setting $y = 2$ yields $\ln 2 = .2 + .1\tau - .2\cos\tau$, which can be solved numerically to give $\tau = 2.9632$. If $y(0) = y_0$, then as above, $y(t) = y_0\exp(.2+.1t-.2\cos t)$. Thus if we set $y = 2y_0$ we get the same numerical equation for τ and hence the doubling time has not changed.

16. If T is the temperature of the coffee at any time t, then $\frac{dT}{dt} = -k(T - 70^\circ)$; $T(0) = 200^\circ$, $T(1) = 190^\circ$. The solution of this linear equation will involve k (the cooling rate) and the integration constant c. Use $T(0) = 200$ to find c and then use $T(1) = 190$ to evaluate k.

18a. Eq.(i) is a linear equation with the integrating factor e^{kt}. Thus $(e^{kt}u)' = k(T_0 + T_1\cos\omega t)e^{kt}$ and hence $e^{kt}u = T_0e^{kt} + kT_1\int\cos\omega t e^{kt}dt + c$. Evaluating the integral (by parts or by a symbolic software package) and dividing by e^{kt} yields $u(t) = T_0 + kT_1\frac{k\cos\omega t + \omega\sin\omega t}{k^2 + \omega^2} + ce^{-kt}$. Note that the last term approaches zero as $t \to \infty$ for any I.C., and that the rest of the solution oscillates about $u(t) = T_0$

18c. Recall that $R\cos[\omega(t-\tau)] = R\cos\omega t\cos\omega\tau + R\sin\omega t\sin\omega\tau$. Comparing this with the oscillatory portion of the above solution we have $R\cos\omega\tau = \frac{k^2T_1}{k^2+\omega^2}$ and $R\sin\omega\tau = \frac{k\omega T_1}{k^2+\omega^2}$ since these are the coefficients of $\cos\omega t$ and $\sin\omega t$ respectively. By squaring and adding we find $R^2 = \frac{k^2T_1^2}{k^2+\omega^2}$ and by dividing we find $\tan\omega\tau = \omega/k$.

19a. The required D.E. is $dQ/dt = kr + P - \frac{Q(t)}{V}r$, since kr is the rate of water pollutant entering the lake, P is the rate of pollutant entering directly and $Q(t)r/V$ is the rate at which the pollutant leaves the lake. The I.C. is $Q(0) = Vc_0$. Since $c = Q(t)/V$, the I.V.P. may be rewritten $Vc'(t) = kr + P - rc$, $c(0) = c_0$, which has the solution $c(t) = k + \frac{P}{r} + (c_0 - k - \frac{P}{r})e^{-rt/V}$.

19b. Set $k = 0$, $P = 0$, $t = T$ and $c(T) = .5c_0$ in the solution found in (a).

20a. If we measure x positively upward from the ground, then Eq.(4) of Section 1.1 becomes $m\frac{dv}{dt} = -mg$, since there is no air resistance. Thus the I.V.P. for v(t) is $dv/dt = -g$, $v(0) = 20$, which gives $v(t) = 20 - gt$. Since $\frac{dx}{dt} = v(t)$ we get $x(t) = 20t - (g/2)t^2 + c$. Then $x(0) = 30$ gives $c = 30$ and thus $x(t) = 20t - (g/2)t^2 + 30$. At the maximum height $v(t_m) = 0$ and thus $t_m = 20/9.8 = 2.04$ sec., which when substituted in the equation for x(t) yields the maximum height.

21. If v is positive in the upward direction then the drag force $-\frac{1}{30}v$ is downward when v is positive and upward when v is negative. The I.V.P. in this case is $m\frac{dv}{dt} = -\frac{1}{30}v - mg$, $v(0) = 20$.

23a. The I.V.P. is $m\frac{dv}{dt} = mg - .75v$, $v(0) = 0$ and v is measured positively downward. Since $m = 180/32$, the D.E. becomes $\frac{dv}{dt} = 32 - \frac{2}{15}v$ and thus $v(t) = 240(1-e^{-2t/15})$ so that $v(10) = 176.7$ ft/sec.

23b. Integration of v(t) as found in (a) yields $x(t) = 240t + 1800(e^{-2t/15}-1)$, x is measured positively down from the altitude of 5000 feet. Set $t = 10$ to find the distance traveled when the parachute opens.

23c. After the parachute opens the I.V.P. is $m\frac{dv}{dt} = mg-12v$, $v(0) = 176.7$, which has the solution $v(t) = 161.7e^{-32t/15} + 15$ and where $t = 0$ now represents the time the parachute opens. Letting $t\to\infty$ yields the limiting velocity of 15 ft/sec.

23d. Integrate v(t) as found in (c) to find $x(t) = 15t - 75.8e^{-32t/15} + C_2$. $C_2 = 75.8$ since $x(0) = 0$, x now being measured from the point where the parachute opens. Setting $x = 3925.5$ will then yield the length of time the skydiver is in the air after the parachute opens.

26a. As in Prob.21, $m\frac{dv}{dt} = -mg - kv$, $v(0) = v_0$.

26b. From part (a) $v(t) = -\frac{mg}{k} + [v_0 + \frac{mg}{k}]e^{-kt/m}$. As $k \to 0$ this has the indeterminant form of $-\infty + \infty$. Thus rewrite $v(t)$ as $v(t) = [-mg + (v_0k + mg)e^{-kt/m}]/k$ which has the indeterminant form of 0/0, as $k \to 0$ and hence L'Hospital's Rule may be applied with k as the variable.

27a. The equation of motion is $m(dv/dt) = w-R-B$ which, in this problem, is $\frac{4}{3}\pi a^3\rho(dv/dt) = \frac{4}{3}\pi a^3\rho g - 6\pi\mu av - \frac{4}{3}\pi a^3\rho' g$. The limiting velocity occurs when $dv/dt = 0$.

27b. Since the droplet is motionless, $v = dv/dt = 0$, we have the equation of motion $0 = (\frac{4}{3})\pi a^3\rho g - Ee - (\frac{4}{3})\pi a^3\rho' g$, where ρ is the density of the oil and ρ' is the density of air. Solving for e yields the answer.

28. All three parts can be answered from one solution if k represents the resistance and if the method of solution of Example 4 is used. Thus we have $m\frac{dv}{dt} = mv\frac{dv}{dx} = mg - kv$, $v(0) = 0$, where we have assumed the velocity is a function of x. The solution of this I.V.P. involves a logarithmic term, and thus the answers to parts (a) and (c) must be found using a numerical procedure.

29a. Use Eq.(30)

29b. Note that 32 ft/sec^2 = 78,545 m/hr^2.

30b. From part a) $\frac{dx}{dt} = v = u\cos A$ and hence $x(t) = (u\cos A)t + d_1$. Since $x(0) = 0$, we have $d_1 = 0$ and $x(t) = (u\cos A)t$. Likewise $\frac{dy}{dt} = -gt + u\sin A$ and therefore $y(t) = -gt^2/2 + (u\sin A)t + d_2$. Since $y(0) = h$ we have $d_2 = h$ and $y(t) = -gt^2/2 + (u\sin A)t + h$.

30d. Let t_w be the time the ball reaches the wall. Then

$x(t_w) = L = (u\cos A)t_w$ and thus $t_w = \dfrac{L}{u\cos A}$. For the ball to clear the wall $y(t_w) \geq H$ and thus (setting $t_w = \dfrac{L}{u\cos A}$, $g = 32$ and $h = 3$ in y) we get

$$\frac{-16L^2}{u^2\cos^2 A} + L\tan A + 3 \geq H.$$

30e. Setting $L = 350$ and $H = 10$ we get $\dfrac{-161.98}{\cos^2 A} + 350\dfrac{\sin A}{\cos A} \geq 7$ or $7\cos^2 A - 350\cos A\sin A + 161.98 \leq 0$. This can be solved numerically or by plotting the left side as a function of A and finding where the zero crossings are.

30f. Setting $L = 350$, and $H = 10$ in the answer to part d yields $\dfrac{-16(350)^2}{u^2\cos^2 A} + 350\tan A = 7$, where we have chosen the equality sign since we want to just clear the wall. Solving for u^2 we get $u^2 = \dfrac{1{,}960{,}000}{175\sin 2A - 7\cos^2 A}$. Now u will have a minimum when the denominator has a maximum. Thus $350\cos 2A + 7\sin 2A = 0$, or $\tan 2A = -50$, which yields $A = .7954$ rad. and $u = 106.89$ ft./sec.

<u>Section 2.4, Page 75</u>

1. If the equation is written in the form of Eq.(1), then $p(t) = (\ln t)/(t-3)$ and $g(t) = 2t/(t-3)$. These are defined and continuous on the intervals $(0,3)$ and $(3,\infty)$, but since the initial point is $t = 1$, the solution will be continuous on $0 < t < 3$.

4. $p(t) = 2t/(2-t)(2+t)$ and $g(t) = 3t^2/(2-t)(2+t)$, which have discontinuities at $t = \pm 2$. Since $y_0 = -3$, the solution will be continuous on $-\infty < t < -2$.

8. Theorem 2.4.2 guarantees a unique solution to the D.E. through any point (t_0,y_0) such that $t_0^2 + y_0^2 < 1$ since $\dfrac{\partial f}{\partial y} = -y/(1-t^2-y^2)^{1/2}$ is defined and continuous only for $1-t^2-y^2 > 0$. Note also that $f = (1-t^2-y^2)^{1/2}$ is defined

and continuous in this region as well as on the boundary $t^2+y^2 = 1$. The boundary can't be included in the final region due to the discontinuity of $\frac{\partial f}{\partial y}$ there.

11. In this case $f = \dfrac{1+t^2}{y(3-y)}$ and $\dfrac{\partial f}{\partial y} = \dfrac{1+t^2}{y(3-y)^2} - \dfrac{1+t^2}{y^2(3-y)}$, which are both continuous everywhere except for $y = 0$ and $y = 3$.

13. The D.E. may be written as $ydy = -4tdt$ so that $\dfrac{y^2}{2} = -2t^2+c$, or $y^2 = C-4t^2$. The I.C. then yields $y_0^2 = C$, so that $y^2 = y_0^2 - 4t^2$ or $y = \pm\sqrt{y_0^2-4t^2}$, which is defined for $4t^2 < y_0^2$ or $|t| < |y_0|/2$. Note that $y_0 \neq 0$ since Theorem 2.4.2 does not hold there.

17. From the direction field (or the given D.E.) it is noted that for $t > 0$ and $y < 0$ that $y' < 0$, so $y \to -\infty$ for $y_0 < 0$. Likewise, for $0 < y_0 < 3$, $y' > 0$ and $y' \to 0$ as $y \to 3$, so $y \to 3$ for $0 < y_0 < 3$ and for $y_0 > 3$, $y' < 0$ and again $y' \to 0$ as $y \to 3$, so $y \to 3$ for $y_0 > 3$. For $y_0 = 3$, $y' = 0$ and $y = 3$ for all t and for $y_0 = 0$, $y' = 0$ and $y = 0$ for all t.

22a. For $y_1 = 1-t$, $y_1' = -1$, so substitution into the D.E. gives

$$-1 = \frac{-t+[t^2+4(1-t)]^{1/2}}{2} = \frac{-t+[(t-2)^2]^{1/2}}{2} = \frac{-t+|t-2|}{2}.$$

By the definition of the absolute value, the right side is -1 if $(t-2) \geq 0$. Setting $t = 2$ in y_1 we get $y_1(2) = -1$, as required by the I.C.

22b. By Theorem 2.4.2 we are guaranteed a unique solution only where $f(t,y) = \dfrac{-t+(t^2+4y)^{1/2}}{2}$ and $f_y(t,y) = (t^2+4y)^{-1/2}$ are continuous. In this case the initial point $(2,-1)$ lies in the region $t^2 + 4y \leq 0$, so $\dfrac{\partial f}{\partial y}$ is not continuous and hence the theorem is not applicable and there is no contradiction.

22c. For $y = ct + c^2$ follow the steps of Prob. 22a. If $y = y_2(t)$ then we must have $ct + c^2 = -t^2/4$ for all t, which is not possible since c is a constant.

23b. $\phi(t) = t^{-1}$ gives $\phi'(t) = -t^{-2}$ so $\phi' + \phi^2 = 0$. $\phi(t) = ct^{-1}$ gives $\phi'(t) = -ct^{-2}$, so $\phi' + \phi^2 \neq 0$ unless $c = 0$ or $c = 1$.

25. $[y_1(t) + y_2(t)]' + p(t)[y_1(t) + y_2(t)] =$
$y_1'(t) + p(t)y_1(t) + y_2'(t) + p(t)y_2(t) = 0 + g(t)$.

27a. For $n = 1$, we have $y' + [p(t)-q(t)]y = 0$, which is linear. Thus Eq.(3) gives
$y(t) = c\mu^{-1}(t) = ce^{-\int[p(t)-q(t)]dt}$, since $g(t) = 0$.

27b. Let $v = y^{1-n}$ then $\frac{dv}{dt} = (1-n)y^{-n}\frac{dy}{dt}$ so $\frac{dy}{dt} = \frac{1}{1-n}y^n\frac{dv}{dt}$, for $n \neq 1$. Substituting into the D.E. yields
$\frac{y^n}{1-n}\frac{dv}{dt} + p(t)y = q(t)y^n$ or
$v' + (1-n)p(t)y^{1-n} = (1-n)q(t)$, or
$v' + (1-n)p(t)v = (1-n)q(t)$, which is a linear D.E. for v.

28. $n = 3$ so $v = y^{-2}$ and $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$ or $\frac{dy}{dt} = -\frac{1}{2}y^3\frac{dv}{dt}$. Substituting this into the D.E. gives
$-\frac{1}{2}y^3\frac{dv}{dt} + \frac{2}{t}y = \frac{1}{t^2}y^3$. Simplifying and using
$y^{-2} = v$ then gives the linear D.E. $v' - \frac{4}{t}v = -\frac{2}{t^2}$. Thus
$\mu(t) = \frac{1}{t^4}$ and $v(t) = ct^4 + \frac{2}{5t} = \frac{2+5ct^5}{5t}$. Solving for y
gives $y = \pm[5t/(2+5ct^5)]^{1/2}$.

29. $n = 2$ so $v = y^{-1}$ and $\frac{dv}{dt} = -y^{-2}\frac{dy}{dt}$. Thus the D.E.
becomes $-y^2\frac{dv}{dt} - ry = -ky^2$ or $\frac{dv}{dt} + rv = k$. Hence
$\mu(t) = e^{rt}$ and $v = k/r + ce^{-rt}$. Setting $v = 1/y$ then yields the solution.

32. Since $g(t)$ is continuous on the interval $0 \le t \le 1$ and hence we solve the I.V.P.
$y_1' + 2y_1 = 1$, $y_1(0) = 0$ on that interval to obtain

$y_1 = 1/2 - (1/2)e^{-2t}$, $0 \le t \le 1$. For $1<t$, $g(t) = 0$; and hence we solve $y_2' + 2y_2 = 0$ to obtain $y_2 = ce^{-2t}$, $1<t$. The solution y of the original I.V.P. must be continuous at t = 1 (since its derivative must exist) and hence we need c in y_2 so that y_2 at 1 has the same value as y_1 at 1. Thus

$ce^{-2} = 1/2 - e^{-2}/2$ or $c = (1/2)(e^2-1)$ and we obtain

$$y = \begin{cases} 1/2 - (1/2)e^{-2t} & 0 \le t \le 1 \\ 1/2(e^2-1)e^{-2t} & 1 \le t \end{cases} \quad \text{and}$$

$$y' = \begin{cases} e^{-2t} & 0 \le t \le 1 \\ (1-e^2)e^{-2t} & 1 < t. \end{cases}$$

Evaluating the two parts of y' at $t_0 = 1$ we see that they are different, and hence y' is not continuous at $t_0 = 1$.

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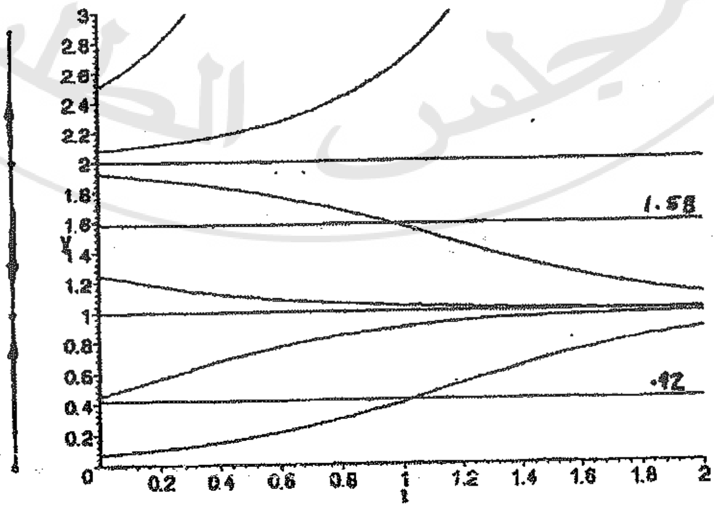
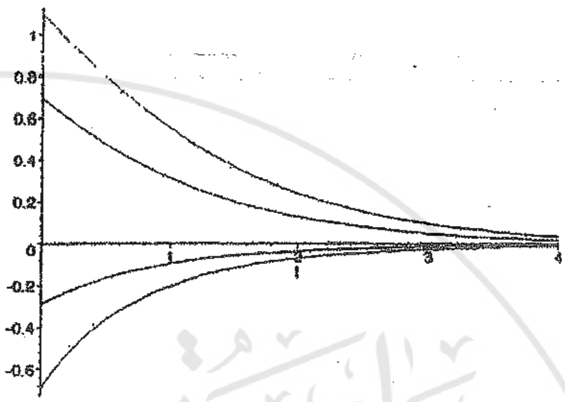
3. From the graph, or by setting $\frac{dy}{dt} = y(y-1)(y-2) = 0$, we find that y = 0,1,2 are the critical points. The graph of y(y-1)(y-2) is positive for $0 < y < 1$ and $2 < y$ and negative for $1 < y < 2$. Thus y(t) is increasing ($\frac{dy}{dt} > 0$) for $0 < y < 1$ and $2 < y$ and decreasing ($\frac{dy}{dt} < 0$) for $1 < y < 2$. Therefore 0 and 2 are unstable critical points while 1 is an asymptotically stable critical point.



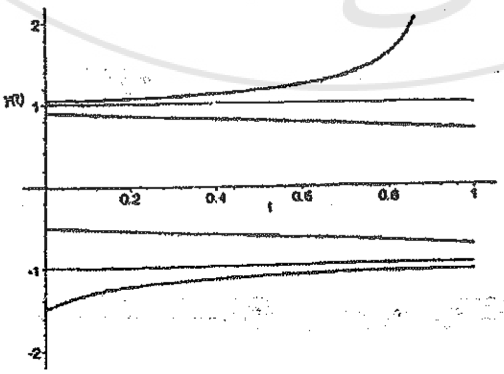
The phase line and several solutions are shown. The inflection points of the solutions (1.58 and .42) are found by determining where $y(y-1)(y-2)$ has its relative max and min points. These lines are are also shown.

5. $\frac{dy}{dt}$ is zero only when $e^{-y}-1=0$, or $y = 0$. Since $\frac{dy}{dt}>0$ for $y<0$ and $\frac{dy}{dt}<0$ for $y>0$ we conclude that $y=0$ is an asymptotically stable critical point.



7c. Separate variables to get $\frac{dy}{(1-y)^2} = kdt$. Integration yields $\frac{1}{1-y} = kt + c$, or $y = 1 - \frac{1}{kt + c} = \frac{kt + c - 1}{kt + c}$. Setting $t = 0$ and $y(0) = y_0$ yields $y_0 = \frac{c-1}{c}$ or $c = \frac{1}{1-y_0}$. Hence $y(t) = \frac{(1-y_0)kt + y_0}{(1-y_0)kt + 1}$. For $y_0 < 1$ we have $y \rightarrow (1-y_0)k/(1-y_0)k = 1$ as $t \rightarrow \infty$. For $y_0 > 1$ the denominator will have a zero for some value of t, depending on the values chosen for y_0 and k. Thus the solution has a discontinuity at that point.

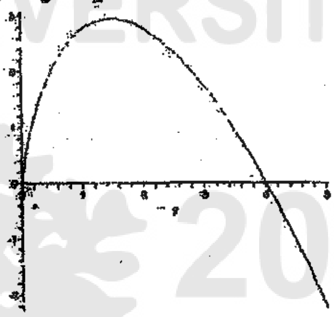
9. Setting $\frac{dy}{dt} = 0$ we find $y = 0, \pm 1$ are the critical points. We have $\frac{dy}{dt} > 0$ for $|y| > 1$ while $\frac{dy}{dt} < 0$ for $|y| < 1$ we conclude that $y = -1$ is asymptotically stable, $y = 0$ is semistable, and $y = 1$ is unstable.



11. $y = b^2/a^2$ and $y = 0$ are the only critical points. For $0 < y < b^2/a^2$, $\frac{dy}{dt} < 0$ and thus $y = 0$ is asymptotically stable. For $y > b^2/a^2$, $dy/dt > 0$ and thus $y = b^2/a^2$ is unstable. All solutions that start above $y = b^2/a^2$ will continue to increase and all solutions that start for y between 0 and b^2/a^2 will decay to zero. If $y(0) = b^2/a^2$ then $y(t) = b^2/a^2$ for all t, since this is an equilibrium point.

14. If $f'(y_1) < 0$ then the slope of f is negative at y_1 and thus $f(y) > 0$ for $y < y_1$ and $f(y) < 0$ for $y > y_1$ since $f(y_1) = 0$. Hence y_1 is an asymptotically stable critical point. A similar argument will yield the result for $f'(y_1) > 0$.

16a. Setting $\frac{dy}{dt} = 0$ we have $ry\ln(K/y) = 0$, so $y = 0$ and $y = K$ are the critical points. The graph for $r = 2$ and $K = 4$ is shown, and we see that $\frac{dy}{dt} > 0$ for $0 < t < K = 4$ and $\frac{dy}{dt} < 0$ for $t > K = 4$. Thus $y = 0$ is unstable and $y = K$ is asymptotically stable.



16b. The derivative of $y\ln(K/y)$ is $\ln(K/y) - 1$, so the graph of $\frac{dy}{dt}$ vs y has a maximum point at $y = K/e$. Thus $\frac{dy}{dt}$ is positive and increasing for $0 < y < K/e$ and hence $y(t)$ is concave up for that interval. Similarly $\frac{dy}{dt}$ is positive and decreasing for $K/e < y < K$ and thus $y(t)$ is concave down for that interval.

16c. $\ln(K/y)$ is very large for small values of y and thus $(ry)\ln(K/y) > ry(1 - y/K)$ for small y. Since $\ln(K/y)$ and $(1 - y/K)$ are both strictly decreasing functions of y and since $\ln(K/y) = (1 - y/K)$ only for $y = K$, we may conclude that $\frac{dy}{dt} = (ry)\ln(K/y)$ is never less than $\frac{dy}{dt} = ry(1-y/K)$.

17a. If $u = \ln(y/K)$ then $y = Ke^u$ and $\frac{dy}{dt} = Ke^u\frac{du}{dt}$ so that the D.E. becomes $du/dt = -ru$.

18a. The D.E. is $dV/dt = k - \alpha\pi r^2$. The volume of a cone of height L and radius r is given by $V = \pi r^2 L/3$ where $L = hr/a$ from symmetry. Solving for r yields the desired solution.

18b. Equilibrium is given by $k - \alpha\pi r^2 = 0$.

18c. The equilibrium height must be less than h.

20b. Use the results of Problem 14.

20c. Y is defined to be Ey_2, where y_2 was found in part a.

20d. Differentiate Y with respect to E.

21a. Set $\frac{dy}{dt} = 0$ and solve for y using the quadratic formula.

21b. Use the results of Prob. 14.

21d. If $h > rK/4$ there are no critical points (see part a) and $\frac{dy}{dt} < 0$ for all t.

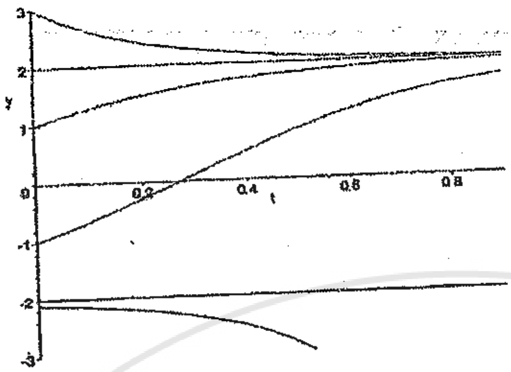
24a. If $z = x/n$ then $dz/dt = \frac{1}{n}\frac{dx}{dt} - \frac{x}{n^2}\frac{dn}{dt}$. Use of Equations (i) and (ii) then gives the D.E. (iii) and the I.C. is $z(0) = 1$ since $n(0) = x(0)$.

24b. Separate variables to get $\frac{dz}{z(1-vz)} = -\beta dt$. Using partial fractions this becomes $\frac{dz}{z} + \frac{vdz}{1-vz} = -\beta dt$. Integration and solving for z yields the answer.

24c. From part b, find $z(20)$ when $\beta = v = 1/8$.

25b. For $a = 0$, $\frac{dy}{dt}$ is always negative, so $y = 0$ is semistable. For $a > 0$ we have $\frac{dy}{dt} < 0$ for $|y| > \sqrt{a}$ and $\frac{dy}{dt} > 0$ for $|y| < -\sqrt{a}$, so $y = \sqrt{a}$ is asymptotically stable and $y = \sqrt{a}$ is unstable.

25c. The graphs are shown (on the next page) for $a = 4$. Note that $\frac{d}{dy}(a-y^2) = 0$ for $y = 0$ and thus $y = 0$ is an inflection point for the solution that crosses the t-axis.



28a. Observe that $x = p$ and $x = q$ are critical points. Also note that $dx/dt > 0$ for $x < \min(p,q)$ and $x > \max(p,q)$ while $dx/dt < 0$ for x between $\min(p,q)$ and $\max(p,q)$. Thus $x = \min(p,q)$ is an asymptotically stable point while $x = \max(p,q)$ is unstable. To solve the D.E., separate variables and use partial fractions to obtain $\frac{1}{q-p}\left[\frac{dx}{q-x} - \frac{dx}{p-x}\right] = \alpha dt$. Integration and solving for x yields the solution.

28b. $x = p$ is a semistable critical point and since $\frac{dx}{dt} > 0$, $x(t)$ is an increasing function. Thus for $x(0) = 0$, $x(t)$ approaches p as $t \to \infty$. To solve the D.E., separate variables and integrate.

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3. $M(x,y) = 3x^2-2xy+2$ and $N(x,y) = 6y^2-x^2+3$, so $M_y = -2x = N_x$ and thus the D.E. is exact. Integrating $M(x,y)$ with respect to x we get $\psi(x,y) = x^3 - x^2y + 2x + h(y)$. Taking the partial derivative of this with respect to y and setting it equal to $N(x,y)$ yields $-x^2+h'(y) = 6y^2-x^2+3$, so that $h'(y) = 6y^2 + 3$ and $h(y) = 2y^3 + 3y$. Substitute this $h(y)$ into $\psi(x,y)$ and recall that the equation which defines $y(x)$ implicitly is $\psi(x,y) = c$. Thus $x^3 - x^2y + 2x + 2y^3 + 3y = c$ is the equation that implicitly defines the solution.

5. Writing the equation in the form $M(x,y)dx + N(x,y)dy = 0$ gives $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Thus $M_y = b = N_x$ and the equation is exact. Integrating $M(x,y)$ with respect to x yields $\psi(x,y) = (a/2)x^2 + bxy + h(y)$. Differentiating ψ with respect to y (x constant) and setting $\psi_y(x,y) = N(x,y)$ we find that $h'(y) = cy$ and thus

$h(y) = (c/2)y^2$. Hence the solution is given by $(a/2)x^2 + bxy + (c/2)y^2 = C$ or $ax^2 + 2bx + cy^2 = k$

7. $M_y(x,y) = e^x\cos y - 2\sin x = N_x(x,y)$ and thus the D.E. is exact. Integrating $M(x,y)$ with respect to x gives $\psi(x,y) = e^x\sin y + 2y\cos x + h(y)$. Finding $\psi_y(x,y)$ from this and setting that equal to $N(x,y)$ yields $h'(y) = 0$ and thus $h(y)$ is a constant. Hence an implicit solution of the D.E. is $e^x\sin y + 2y\cos x = c$. The solution $y = 0$ is also valid since it satisfies the D.E. for all x.

9. $M_y = N_x$ so the D.E. is exact. If you try to find $\psi(x,y)$ by integrating $M(x,y)$ with respect to x you must integrate by parts. Instead find $\psi(x,y)$ by integrating $N(x,y)$ with respect to y to obtain $\psi(x,y) = e^{xy}\cos 2x - 3y + g(x)$. Now find $g(x)$ by differentiating $\psi(x,y)$ with respect to x and set that equal to $M(x,y)$, which yields $g'(x) = 2x$ or $g(x) = x^2$. As before the implicit solution is $\psi(x,y) = c$

12. As long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2 + y^2)^{3/2}$. This gives the exact equation $xdx + ydy = 0$. The solution to this equation is given implicitly by $x^2 + y^2 = c$. If you apply Theorem 2.6.1 and its construction without the simplification, you get $(x^2 + y^2)^{-1/2} = C$ which can be written as $x^2 + y^2 = c$ under the same assumption required for the simplification.

14. $M_y = 1$ and $N_x = 1$, so the D.E. is exact. Integrating $M(x,y)$ with respect to x yields $\psi(x,y) = 3x^3 + xy - x + h(y)$. Differentiating this with respect to y and setting $\psi_y(x,y) = N(x,y)$ yields $h'(y) = -4y$ or $h(y) = -2y^2$. Thus the implicit solution is $3x^3 + xy - x - 2y^2 = c$. Setting $x = 1$ and $y = 0$ gives $c = 2$ so that $2y^2 - xy + (2+x-3x^3) = 0$ is the implicit solution satisfying the given I.C. Use the quadratic formula to find $y(x)$, where the negative square root is used in order to satisfy the I.C. The solution will be valid for $24x^3 + x^2 - 8x - 16 > 0$. Using a numerical procedure (or graphically) this cubic equation has one zero for $x \cong .9846$.

15. We want $M_y(x,y) = 2xy + bx^2$ to be equal to $N_x(x,y) = 3x^2 + 2xy$. Thus we must have $b = 3$. This gives $\psi(x,y) = \frac{1}{2}x^2y^2 + x^3y + h(y)$ and consequently $h'(y) = 0$. After multiplying through by 2, the solution is given implicitly by $x^2y^2 + 2x^3y = c$.

19. $M_y(x,y) = 3x^2y^2$ and $N_x(x,y) = 1 + y^2$ so the equation is not exact by Theorem 2.6.1. Multiplying by the integrating factor $\mu(x,y) = 1/xy^3$ we get $x + \frac{(1+y^2)}{y^3}y' = 0$, which is an exact equation since $M_y = N_x = 0$ (it is also separable). In this case $\psi = \frac{1}{2}x^2 + h(y)$ and $h'(y) = y^{-3} + y^{-1}$ so that $x^2 - y^{-2} + 2\ln|y| = c$ gives the solution implicitly. Note that $y(x) = 0$ also satisfies the given D.E.

22. Multiplication of the given D.E. (which is not exact) by $\mu(x,y) = xe^x$ yields $(x^2 + 2x)e^x\sin y\,dx + x^2e^x\cos y\,dy$, which is exact since $M_y(x,y) = N_x(x,y) = (x^2+2x)e^x\cos y$. To solve this exact equation it's easiest to integrate $N(x,y) = x^2e^x\cos y$ with respect to y to get $\psi(x,y) = x^2e^x\sin y + g(x)$. Finding ψ_x and setting that equal to $(x^2+2x)e^x\sin y$ yields $g'(x) = 0$.

23. This problem is similar to the derivation leading up to Eq.(26). Assuming that μ depends only on y, we find from Eq.(25) that $\mu' = Q\mu$, where $Q = (N_x - M_y)/M$ must depend on y alone. Solving this last D.E. yields $\mu(y)$ as given. This method provides an alternative approach to Problems 25 through 31.

25. The equation is not exact so we must attempt to find an integrating factor. Since $\frac{1}{N}(M_y - N_x) = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$ is a function of x alone there is an integrating factor depending only on x, as shown in Eq.(26). Then $d\mu/dx = 3\mu$, and the integrating factor is $\mu(x) = e^{3x}$. Multiplying all terms in the given D.E. by e^{3x} will then yield an exact D.E.

26. An integrating factor can be found which is a function of x only, yielding $\mu(x) = e^{-x}$. Alternatively, you might recognize that $y' - y = e^{2x} - 1$ is a linear first order equation which can be solved as in Section 2.1.

27. Using the results of Prob. 23, it can be shown that $\mu(y) = y$ is an integrating factor. Thus multiplying the D.E. by y gives $ydx + (x - y\sin y)dy = 0$, which can be identified as an exact equation. Alternatively, one can rewrite the last equation as $(ydx + xdy) - y\sin y\, dy = 0$. The first term is $d(xy)$ and the last can be integrated by parts. Thus we have $xy + y\cos y - \sin y = c$.

29. Simplify the D.E. by multiplying by siny (which is really an integrating factor) to obtain
$e^x\sin y dx + e^x\cos y dy + 2ydy = 0$, which is exact. The first two terms are just $d(e^x\sin y)$ and thus, $e^x\sin y + y^2 = c$.

31. Using the results of Prob. 24, it can be shown that $\mu(xy) = xy$ is an integrating factor. Thus, multiplying by xy we have $(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0$, which can be identified as an exact equation. Alternatively, we can observe that the above equation can be written as $d(x^3y) + d(3x^2) + d(y^3) = 0$, so that $x^3y + 3x^2 + y^3 = c$.

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1d. The exact solution to this I.V.P. is $y = \phi(t) = t + 2 - e^{-t}$.

3a. The Euler formula is $y_{n+1} = y_n + h(2y_n - t_n + 1/2)$ for $n = 0,1,2,3$ and with $t_0 = 0$ and $y_0 = 1$. Thus, for $h = .1$,
$y_1 = y_0 + .1(2y_0 - t_0 + 1/2) = 1.25$,
$y_2 = 1.25 + .1[2(1.25) - (.1) + 1/2] = 1.54$,
$y_3 = 1.54 + .1[2(1.54) - (.2) + 1/2] = 1.878$, and
$y_4 = 1.878 + .1[2(1.878) - (.3) + 1/2] = 2.2736$.

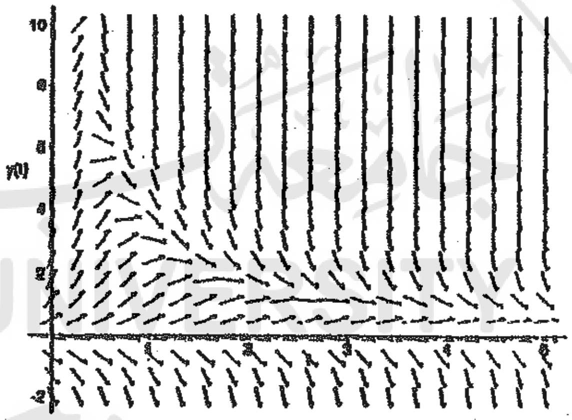
3b. Use the same formula as in Prob. 3a, except now $h = .05$ and $n = 0,1...7$. Notice that only results for $n = 1,3,5$ and 7 are needed to compare with part a.

3c. Again, use the same formula as above with $h = .025$ and $n = 0,1...15$. Notice that only results for $n = 3,7,11$ and 15 are needed to compare with parts a and b.

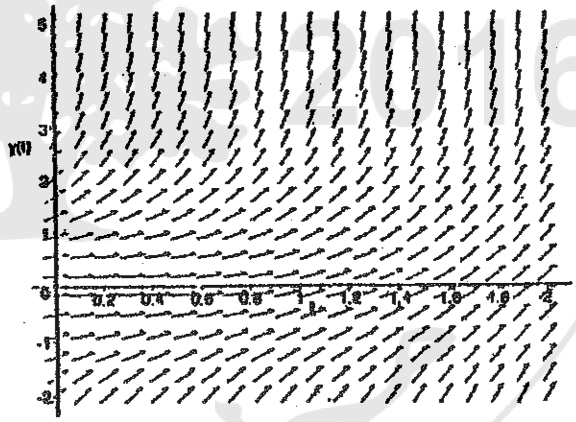
3d. $y' = 1/2 - t + 2y$ is a first order linear D.E. Rewrite the equation in the form $y' - 2y = 1/2 - t$ and multiply both sides by the integrating factor e^{-2t} to obtain $(e^{-2t}y)' = (1/2 - t)e^{-2t}$. Integrating the right side by parts and multiplying by e^{2t} we obtain $y = ce^{2t} + t/2$. The I.C. $y(0) = 1 \rightarrow c = 1$ and hence the solution of the I.V.P. is $y = \phi(t) = e^{2t} + t/2$. Thus $\phi(0.1) = 1.2714$, $\phi(0.2) = 1.59182$, $\phi(0.3) = 1.97212$, and $\phi(0.4) = 2.42554$.

4d. The exact solution to this I.V.P. is $y = \phi(t) = (6\cos t + 3\sin t - 6e^{-2t})/5$.

6. For $y(0) > 0$ the solutions appear to converge to a number between 0 and 2. Note that $y=0$ is an equilibrium solution For $y(0)<0$ the solutions diverge.



9. All solutions seem to diverge.



13a. The Euler formula is

$$y_{n+1} = y_n + h\left(\frac{4-t_n y_n}{1 + y_n^2}\right), \text{ where } t_0 = 0 \text{ and}$$

$y_0 = y(0) = -2$. Thus, for $h = .1$, we get

$$y_1 = -2 + .1(4/5) = -1.92$$

$$y_2 = -1.92 + .1\left(\frac{4-.1(-1.92)}{1 + (1.92)^2}\right) = -1.83055$$

$$y_3 = -1.83055 + .1\left(\frac{4-.2(-1.83055)}{1+(1.83055)^2}\right) = -1.7302$$

$$y_4 = -1.7302 + .1(\frac{4-.3(-1.7302)}{1+(1.7302)^2}) = -1.617043$$

$$y_5 = -1.617043 + .1(\frac{4-.4(-1.617043)}{1+(1.617043)^2}) = -1.488494.$$

Thus, $y(.5) \cong -1.488494$.

15a. The Euler formula is

$y_{n+1} = y_n + .1\ (\frac{3t_n^2}{3y_n^2-4})$, where $t_0 = 1$ and $y_0 = 0$. Thus

$y_1 = 0 + .1\ (\frac{3}{-4}) = -.075$ and

$$y_2 = -.075 + .1\ (\frac{3(1.1)^2}{3(-.075)^2-4}) = -.166134.$$

15c. There are two factors that explain the large differences. From the D.E., the slope of y, y', becomes very "large" for values of y near -1.155. Also, the slope changes sign at $y = -1.155$. Thus for part a, $y(1.7) \cong y_7 = -1.178$, which is close to -1.155 and the slope y' here is large and positive, creating the large change in $y_8 \cong y(1.8)$. For part b, $y(1.65) \cong -1.125$, resulting in a large negative slope, which yields $y(1.70) \cong -3.133$. The slope at this point is now positive and the remainder of the solutions "grow" to -3.098 for the approximation to $y(1.8)$.

16. For the four step sizes given, the approximte values for $y(.8)$ are 3.5078, 4.2013, 4.8004 and 5.3428. Thus, since these changes are still rather "large", it is hard to give an estimate other than $y(.8)$ is at least 5.3428. By using $h = .005$, $.0025$ and $.001$, we find further approximate values of $y(.8)$ to be 5.576, 5.707 and 5.790. Thus a better estimate now is for $y(.8)$ to be between 5.8 and 6. No reliable estimate is obtainable for $y(1)$, which is consistent with the direction field of Prob.9.

18. It is helpful, in understanding this problem, to also calculate $y'(t_n) = y_n(.1y_n^2 - t_n)$. For $\alpha = 2.38$ this term remains positive and grows very large for $t_n > 2$. On the other hand, for $\alpha = 2.37$ this term decreases and eventually becomes negative for $t_n \cong 1.6$ (for $h = .01$). For $\alpha = 2.37$ and $h = .1$, $.05$ and $.01$, $y(2.00)$ has the approximations of 4.48, 4.01 and 3.50 respectively. A small step size must be used, due to the sensitivety of the slope field, given by $y_n(.1y_n^2 - t_n)$.

22. Using Eq.(8) we have $y_{n+1} = y_n + h(2y_n - 1) = (1+2h)y_n - h$. Setting $n + 1 = k$ (and hence $n = k-1$) this becomes $y_k = (1 + 2h)y_{k-1} - h$, for $k = 1,2,\ldots$. Since $y_0 = 1$, we have $y_1 = 1 + 2h - h = 1 + h = (1 + 2h)/2 + 1/2$, and hence $y_2 = (1 + 2h)y_1 - h = (1 + 2h)^2/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^2/2 + 1/2$;
$y_3 = (1 + 2h)y_2 - h = (1 + 2h)^3/2 + (1 + 2h)/2 - h$
$= (1 + 2h)^3/2 + 1/2$. Continuing in this fashion (or using induction) we obtain $y_k = (1 + 2h)^k/2 + 1/2$. For fixed $t > 0$ choose $h = t/k$. Then substitute for h in the last formula to obtain $y_k = (1 + 2t/k)^k/2 + 1/2$. Letting $k \to \infty$ we find (See hint for Prob 20d.)
$y(t) = \lim_{k\to\infty} y_k = e^{2t}/2 + 1/2$, which is the exact solution.

Section 2.8, Page 118

1. Let $s = t-1$ and $w(s) = y(t(s)) - 2$, then when $t = 1$ and $y = 2$ we have $s = 0$ and $w(0) = 0$. Also,
$\frac{dw}{ds} = \frac{dw}{dt}\cdot\frac{dt}{ds} = \frac{d}{dt}(y-2)\frac{dt}{ds} = \frac{dy}{dt}$ (since $t = s+1$) and hence
$\frac{dw}{ds} = (s+1)^2 + (w+2)^2$, upon substitution into the given D.E.

4a. Following Ex. 1 of the text, from Eq.(7) we have
$\phi_{n+1}(t) = \int_0^t f(s,\phi_n(s))ds$, where $f(t,\phi) = -1 - \phi$. Thus if

$\phi_0(t) = 0$, then $\phi_1(t) = -\int_0^t ds = -t$;

$\phi_2(t) = -\int_0^t (1-s)ds = -t + \frac{t^2}{2}$;

$\phi_3(t) = -\int_0^t (1-s + \frac{s^2}{2})ds = -t + \frac{t^2}{2} - \frac{t^3}{2\cdot3}$;

$\phi_4(t) = -\int_0^t (1 - s + \frac{s^2}{2} - \frac{s^3}{3!})ds = -t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!}$.

Based upon these we hypothesize that $\phi_n(t) = \sum_{k=1}^{n} \frac{(-1)^k t^k}{k!}$

and use mathematical induction to verify this form for $\phi_n(t)$. First, let $n = 1$, then $\phi_1(t) = -t$, so it is certainly true for $n = 1$. Then, using Eq.(7) again we have:

Based upon these we hypothesize that:

$\phi_n(t) = \sum_{k=1}^{n} \frac{t^{2k-1}}{1\cdot3\cdot5\cdots(2k-1)}$ and use mathematical induction to verify this form for $\phi_n(t)$, which is clearly true for $n = 1$. Using Eq.(7) again we have:

$$\phi_{n+1}(t) = \int_0^t \left(\sum_{k=1}^{n} \frac{s^{2k}}{1\cdot3\cdot5\cdots(2k-1)} + 1\right)ds$$

$$= \sum_{k=1}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)} + t$$

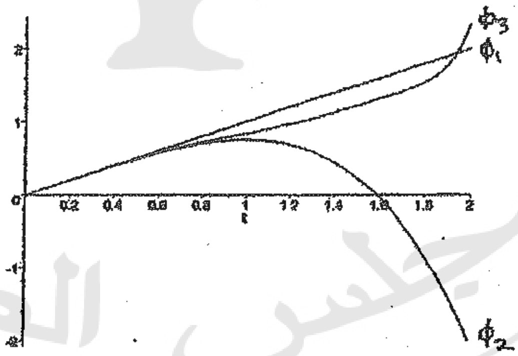
$$= \sum_{k=0}^{n} \frac{t^{2k+1}}{1\cdot3\cdot5\cdots(2k+1)}$$

$$= \sum_{i=1}^{n+1} \frac{t^{2i-1}}{1\cdot3\cdot5\cdots(2i-1)},$$ where $i = k+1$. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

7b. Your plot should show that the estimates appear to be converging.

10a. From Eq.(7) we have $\phi_{n+1} = \int_0^t [1-\phi_n^3(s)]ds$. Setting $\phi_0(t) = 0$ yields the desired iterates.

10b. The iterates appear to diverge.



11. First, recall that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7)$. Now, for this problem $\phi_1(t) \int_0^t [1-\sin\phi_0(s)]ds = t$ and hence

$$\phi_{n+1}(t) = -\int_0^t [1 + \phi_n(s)]\,ds = -t - \sum_{k=1}^{n} \frac{(-1)^k t^{k+1}}{(k+1)!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^{k+1} t^{k+1}}{(k+1)!} = \sum_{i=1}^{n+1} \frac{(-1)^i t^i}{i!},$$ where $i = k+1$. Since this is the same form for $\phi_{n+1}(t)$ as derived from $\phi_n(t)$ above, we have verified by mathematical induction that $\phi_n(t)$ is as given.

4c. From part a, let $\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=1}^{\infty} \frac{(-1)^k t^k}{k!}$

$$= -t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots .$$

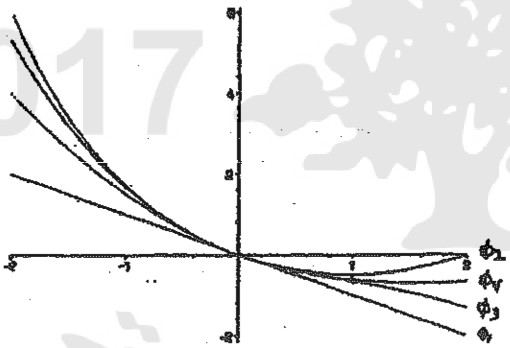
Since this is a power series, recall from calculus that:

$e^{at} = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \ldots$. If we let

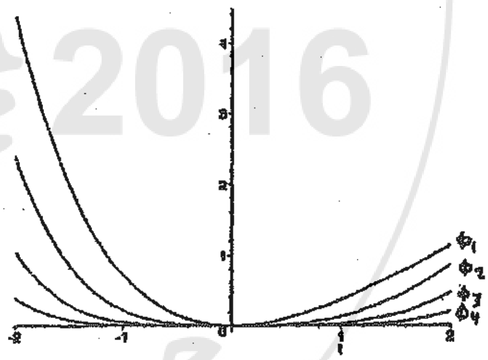
$a = -1$, then we have $e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \ldots = 1 + \phi(t)$.

Hence $\phi(t) = e^{-t} - 1$.

4b.



4d.



From the plot in 4d, it appears that ϕ_4 is a very good estimate for $|t| < 1$.

7a. As in Prob.4,

$$\phi_1(t) = \int_0^t (s\phi_0(s) + 1)ds = s\Big|_0^t = t$$

$$\phi_2(t) = \int_0^t (s^2+1)ds = (\frac{s^3}{3} + s)\Big|_0^t = t + \frac{t^3}{3}$$

$$\phi_3(t) = \int_0^t (s^2 + \frac{s^4}{3} + 1)ds = (\frac{s^3}{3} + \frac{s^5}{3\cdot 5} + s)\Big|_0^t = t + \frac{t^3}{3} + \frac{t^5}{3\cdot 5}.$$

$$\phi_2(t) = \int_0^t [1-\sin(s)]ds = \int_0^t [1 - (s - \frac{s^3}{3!} + \frac{s^5}{5!} - O(s^7)]ds$$

$$= t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + O(t^8).$$ For ϕ_3 we need to find $\sin[\phi_2(t)]$, which is given by

$$\sin[\phi_2(t)] = \phi_2(t) - \phi_2^3(t)/3! + \phi_2^5/5! + O(t^7)$$

$$= (t - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}) - \frac{(t - \frac{t^2}{2!})^3}{3!} + \frac{t^5}{5!} + O(t^7),$$

where we have retained only the terms less than $O(t^7)$. Now use this in $\phi_3(t) = \int_0^t [1-\sin(\phi_2(s))]ds$, which gives the desired answer up to $O(t^8)$.

13. If $x=0$ then $\phi_n(x) = 0$ for all n and $\lim_{n\to\infty}\phi_n(x) = 0$. If $0<x<1$ let $x = \frac{1}{r}$, so $r>1$ and $\lim_{n\to\infty}\phi_n(x) = \lim_{n\to\infty}\phi_n(\frac{1}{r^n}) = 0$. If $x=1$ then $\phi_n(x) = 1$ for all n and $\lim_{n\to\infty}\phi_n(x) = 1$, so that indeed $\phi_n(x)$ converges to a discontinuous functiom.

Section 2.9, Page 130

2. Using the given difference equation we have for $n=0$, $y_1 = y_0/2$; for $n=1$, $y_2 = 2y_1/3 = y_0/3$; and for $n=2$, $y_3 = 3y_2/4 = y_0/4$. Thus we guess that $y_n = y_0/(n+1)$, and the given equation then gives $y_{n+1} = \frac{n+1}{n+2}y_n = y_0/(n+2)$, which, by mathematical induction, verifies $y_n = y_0/(n+1)$ as the solution for all n. $\lim_{n\to\infty} y_n = 0$, as y_0 is constant.

5. From the given equation we have $y_1 = .5y_0+6$.

$$y_2 = .5y_1 + 6 = (.5)^2y_0 + 6(1 + \frac{1}{2})$$ and

$$y_3 = .5y_2 + 6 = (.5)^3y_0 + 6(1 + \frac{1}{2} + \frac{1}{4}).$$ In general, then

$$y_n = (.5)^ny_0 + 6(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}})$$

$$= (.5)^ny_0 + 6(\frac{1 - (1/2)^n}{1 - 1/2})$$

$= (.5)^n y_0 + 12 - (.5)^n 12$

$= (.5)^n(y_0-12) + 12$. Mathematical induction can now be used to prove that this is the correct solution. Note that $y_n \to 12$ as $n \to \infty$ and thus $y_n = 12$ is an equilibrium solution.

7. From Eq.(12) we have $y_{n+1} = (1+r)y_n$ since b=0 and $r = \frac{.07}{365}$ is the daily interest rate. Thus $y_1 = (1+r)y_0$, $y_2 = (1+r)^2 y_0$, ..., $y_n = (1+r)^n y_0$. Setting n = 365 we have $(1+r)^{365} = 1.0725$, so the effective annual yield is 7.25%.

10. As in Ex.(1), the governing equation is $y_{n+1} = \rho y_n - b$, which has the solution $y_n = \rho^n y_0 - \frac{1-\rho^n}{1-\rho}b$ (Eq.(14) with a negative b). Setting $y_{360} = 0$ and solving for b we obtain

$$b = \frac{(1-\rho)\rho^{360}y_0}{1-\rho^{360}},$$ where $\rho = 1.0075$ for part a.

13. You must solve Eq.(14) numerically for ρ when n = 240, $y_{240} = 0$, b = -\$900 and y_0 = \$95,000.

14. Substituting Eq.(25), $u_n = \frac{\rho-1}{\rho} + v_n$, into Eq.(21) we get

$$\frac{\rho-1}{\rho} + v_{n+1} = \rho\left(\frac{\rho-1}{\rho} + v_n\right)\left(1 - \frac{\rho-1}{\rho} - v_n\right) \text{ or}$$

$$v_{n+1} = -\frac{\rho-1}{\rho} + (\rho-1+\rho v_n)\left(\frac{1}{\rho} - v_n\right)$$

$$= \frac{1-\rho}{\rho} + \frac{\rho-1}{\rho} - (\rho-1)v_n + v_n - \rho v_n^2 = (2-\rho)v_n - \rho v_n^2$$

15a. For $u_0 = .2$ we have $u_1 = 3.2u_0(1-\to) = .512$ and $u_2 = 3.2u_1(1-u_1) = .7995392$. Likewise $u_3 = .51288406$, $u_4 = .7994688$, $u_5 = .51301899$, $u_6 = .7994576$ and $u_7 = .5130404$. Continuing in this fashion, $u_{14} = u_{16} = .79945549$ and $u_{15} = u_{17} = .51304451$.

16. To plot the stairsteps extend the graph of Fig.2.9.2(c) outside the interval [0,1]. For part b) choose any $u_0 > 1$. Then the first step goes from the x-axis down to the parabola (in the fourth quadrant) then in the

negative direction to the line $y = x$ in the third quadrant. The steps after that are similar to those in Fig. 2.9.2.

17. For both parts of this problem a computer spreadsheet was used and an initial value of $u_0 = .2$ was chosen. Different initial values or different computer programs may need a slightly different number of iterations to reach the limiting value.

17a. The limiting value of .65517 (to 5 decimal places) is reached after approximately 100 iterations for $\rho = 2.9$. The limiting value of .66102 (to 5 decimal places) is reached after approximately 200 iterations for $\rho = 2.95$. The limiting value of .66555 (to 5 decimal places) is reached after approximately 910 iterations for $\rho = 2.99$.

17b. The solution oscillates between .63285 and .69938 after approximately 400 iterations for $\rho = 3.01$. The solution oscillates between.59016 and .73770 after approximately 130 iterations for $\rho = 3.05$. The solution oscillates between .55801 and .76457 after approximately 30 iterations for $\rho = 3.1$. For each of these cases additional iterations verified the oscillations were correct to five decimal places.

18. For an initial value of .2 and $\rho = 3.448$ we have the solution oscillating between .4403086 and .8497146. After approximately 3570 iterations the eighth decimal place is still not fixed, though. For the same initial value and $\rho = 3.45$ the solution oscillates between the four values: .43399155, .84746795, .44596778 and .85242779 after 3700 iterations.. For $\rho = 3.449$, the solution is still varying in the fourth decimal place after 3570 iterations, but there appear to be four values.

Miscellaneous Problems, Page 132

Before trying to find the solution of a D.E. it is necessary to know its type. The student should first classify the D.E. before looking at this section, which indentifies the type of each D.E. in Problems 1 through 32.

1. Linear

2. Separable

3. Exact

4. Rewrite the D.E. as $\frac{dy}{dx} + (2x-1) = -3(2x-1)$, which is linear, with integrating factor $e^{(x^2-x)}$.

5. Exact

6. Linear

7. Separable

8. Linear

9. Exact

10. Separable

11. Exact

12. Linear

13. Separable

14. Exact

15. Separable

16. Exact

17. Linear

18. Linear with $\mu(x) = e^{2x}$

19. Exact

20. Separable

21. Exact

22. Separable

23. linear, $\mu(t) = te^t$

24. Exact.

25. Exact. Your calculations might be simplified if you write the D.E. as $\frac{2xdx}{y} - \frac{x^2dy}{y^2} + \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} = 0$ and show that the first two terms are exact and the last two terms are exact. Integrating each pair is then straight forward.

26. Homogeneous (See Sect 2.2, Prob.30)

28. Linear or Homogenous

27. Letting $u = x^2$ yields $\frac{dy}{dx} = 2x\frac{dy}{du}$ and thus $\frac{dy}{du} - 2yu = 2y^3$ which is linear in $u(y)$.

29. Homogeneous

30. Integrating factor, depends on x only

31. Integrating factor, depends on y only

32. From Prob.23, Sect. 2.4, this is a Bernoulli equation with n = 2. Let $v(x) = y$, then the linear equation $xv' - v = -e^{2x}$ is obtained.

34b. For $y_1(t) = 1/t$ $y_1' = -1/t^2$ and substitution into the D.E. shows that $y_1(t)$ is indeed a solution. Comparing the D.E. with the equation in Prob.33 we see that $q_1(t) = -1/t^2$, $q_2(t) = -1/t$ and $q_3(t) = 1$. Hence, using the method suggested in Prob.33, we set $y = y_1(t) + \dfrac{1}{v(t)}$ in the D.E. to obtain $\dfrac{dv}{dt} = -(-1/t + 2y_1)v - 1$, or $\dfrac{dv}{dt} + \dfrac{1}{t}v = -1$. Thus $v(t) = \dfrac{c-t^2}{2t}$ and the second solution is $y_2(t) = \dfrac{1}{t} + \dfrac{2t}{c-t^2}$.

36. Let $v = y'$, then $v' = y''$ and thus the D.E. becomes $t^2v' + 2tv - 1 = 0$ or $t^2v' + 2tv = 1$. The left side is recognized as $(t^2v)'$ and thus we may integrate to obtain $t^2v = t + c$ (otherwise, divide both sides of the D.E. by t^2 and find the integrating factor, which is just t^2 in this case). Solving for $v = \dfrac{dy}{dt}$ we find $\dfrac{dy}{dt} = 1/t + c/t^2$ so that $y = \ln t + c_1/t + c_2$.

38. If $v = y'$, the D.E. becomes $v' + tv^2 = 0$. This equation is separable and has the solution $-v^{-1} + t^2/2 = c$ or $v = y' = -2/(c_1 - t^2)$ where $c_1 = 2c$. We must consider separately the cases $c_1 = 0$, $c_1 > 0$ and $c_1 < 0$. If $c_1 = 0$, then $y' = 2/t^2$ or $y = -2/t + c_2$. If $c_1 > 0$, let $c_1 = k^2$. Then $y' = -2/(k^2-t^2) = -(1/k)[1/(k-t) + 1/(k+t)]$, so that $y = (1/k)\ln|(k-t)/(k+t)|+c_2$. If $c_1 < 0$, let $c_1 = -k^2$. Then $y' = 2/(k^2+ t^2)$ so that $y = (2/k)\tan^{-1}(t/k) + c_2$. Finally, we note that $\dfrac{dy}{dt} = 0$ satisfies the D.E. and thus y = constant is also a solution.

42. Following the procedure outlined, let $v = dy/dt$, then $y'' = \dfrac{dv}{dt} = v\dfrac{dv}{dy}$. Thus the D.E. becomes $yv\dfrac{dv}{dy} + v^2 = 0$ so that $v = 0$ and $y\dfrac{dy}{dy} + v = 0$. The D.E.is separable with

the solution $v = c/y$ (which includes $v = 0$ for $c = 0$). Since $v = \frac{dy}{dt} = \frac{c}{y}$, we conclude that $y^2 = c_1t + c_2$, by separating variables and integrating.

45. Again let $v = y'$ and $v' = vdv/dy$ to obtain $2y^2v\frac{dv}{dy} + 2yv^2 = 1$, or $2y^2vdv + 2yv^2dy = dy$. The left side can be written as $d(y^2v^2)$ and thus integration gives $y^2v^2 = y + c_1$ and thus $v = \pm y^{-1}(y + c_1)^{1/2}$. Setting $v = y'$ and separating variables gives $\pm ydy/(y+c_1)^{1/2} = dt$. On observing that the left side of the equation can be written as $\pm[(y+c_1) - c_1]dy/(y+c_1)^{1/2}$ we integrate and find $\pm(2/3)(y-2c_1)(y+c_1)^{1/2} = t + c_2$.

47. If $v = y'$, then $v' = v\frac{dv}{dy}$ and the D.E. becomes $v\frac{dv}{dy} + v^2 = 2e^{-y}$. Dividing by $v \neq 0$ we obtain $\frac{dv}{dy} + v = 2v^{-1}e^{-y}$, which is a Bernoulli equation with $n = -1$ (see Prob.27, Sect. 2.4). Let $w(y) = v^2$, then $\frac{dw}{dy} = 2v\frac{dv}{dy}$ and the D.E. then becomes $\frac{dw}{dy} + 2w = 4e^{-y}$, which is linear in w. Its solution is $w = v^2 = ce^{-2y} + 4e^{-y}$. Setting $v = y'$ and separating variables gives $\frac{dy}{\pm\sqrt{4e^{-y}+ce^{-2y}}} = \frac{e^y dy}{\pm\sqrt{4e^y+c}} = dt$. Integrating and solving for e^y yields $e^y = (t+c_2)^2 + c_1$.

48. Since both t and y are missing, either approach used above will work. In this case it's easier to use the approach of Problems 36-41, so let $v = y'$ and thus $v' = y''$ and the D.E. becomes $v\frac{dv}{dt} = 2$.

51. The variable y is missing. Let $v = y'$, then $v' = y''$ and the D.E. becomes $vv' - t = 0$. The solution of this separable equation is $v^2 = t^2 + c_1$. Substituting $v = y'$ and applying the I.C. $y'(1) = 1$, we obtain $y' = t$. The positive square root was chosen because $y' > 0$ at $t = 1$. Solving this last equation and applying the I.C. $y(1) = 2$, we obtain $y = t^2/2 + 3/2$.

CHAPTER 3

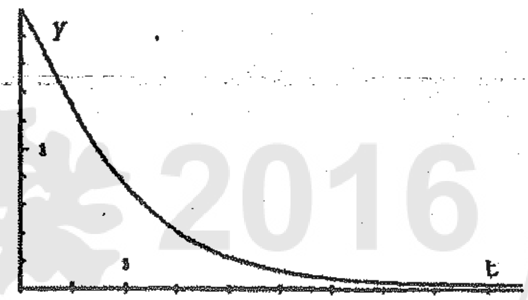
Section 3.1, Page 144

3. Assume $y = e^{rt}$, which gives $y' = re^{rt}$ and $y'' = r^2e^{rt}$. Substitution into the D.E. yields $(6r^2-r-1)e^{rt} = 0$. Since $e^{rt} \neq 0$, we have the characteristic equation $6r^2-r-1 = 0$, or $(3r+1)(2r-1) = 0$. Thus $r = -1/3, 1/2$ and $y = c_1e^{t/2} + c_2e^{-t/3}$.

5. The characteristic equation is $r^2 + 5r = 0$, so the roots are $r_1 = 0$, and $r_2 = -5$. Thus $y = c_1e^{0t} + c_2e^{-5t} = c_1 + c_2e^{-5t}$.

7. The characteristic equation is $r^2 - 9r + 9 = 0$ so the quadratic formula gives $r = (9\pm\sqrt{81-36})/2 = (9\pm3\sqrt{5})/2$. Hence $y = c_1\exp[(9+3\sqrt{5})t/2] + c_2\exp[(9-3\sqrt{5})t/2]$.

10. Substituting $y = e^{rt}$ in the D.E. we obtain the characteristic equation $r^2 + 4r + 3 = 0$, which has the roots $r_1 = -1$, $r_2 = -3$. Thus $y = c_1e^{-t} + c_2e^{-3t}$ and $y' = -c_1e^{-t} - 3c_2e^{-3t}$.



Substituting $t = 0$ we then have $c_1 + c_2 = 2$ and $-c_1 - 3c_2 = -1$, yielding $c_1 = 5/2$ and $c_2 = -1/2$. Thus $y = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$ and hence $y \to 0$ as $t \to \infty$.

15. The characteristic equation is $r^2 + 8r - 9 = 0$, so that $r_1 = 1$ and $r_2 = -9$ and the general solution is $y = c_1e^{t} + c_2e^{-9t}$. Since the I.C. are given at $t = 1$, it is convenient to write the general solution in the form $y = k_1e^{(t-1)} + k_2e^{-9(t-1)}$. Note that $c_1 = k_1e^{-1}$ and $c_2 = k_2e^{9}$. The advantage of the latter form of the general solution becomes clear when we apply the I.C. $y(1) = 1$ and $y'(1) = 0$. This latter form of y

gives $y' = k_1 e^{(t-1)} - 9k_2 e^{-9(t-1)}$ and thus setting $t = 1$ in y and y' yields the equations $k_1 + k_2 = 1$ and $k_1 - 9k_2 = 0$. Solving for k_1 and k_2 we find that $y = (9e^{(t-1)} + e^{-9(t-1)})/10$. The graph starts at $t = 1$ with zero slope and since $e^{(t-1)}$ has a positive exponent for $t > 1$, $y \to \infty$ as $t \to \infty$.

17. Comparing the given solution to Eq(17), we see that $r_1 = 2$ and $r_2 = -3$ are the two roots of the characteristic equation. Thus we have $(r-2)(r+3) = 0$, or $r^2 + r - 6 = 0$ as the characteristic equation. Hence the given solution is for the D.E. $y'' + y' - 6y = 0$.

19. The roots of the characteristic equation are $r = 1, -1$ and thus the general solution is $y(t) = c_1 e^t + c_2 e^{-t}$. $y(0) = c_1 + c_2 = \frac{5}{4}$ and $y'(0) = c_1 - c_2 = -\frac{3}{4}$, yielding $y(t) = \frac{1}{4}e^t + e^{-t}$. From this $y'(t) = \frac{1}{4}e^t - e^{-t} = 0$ or $e^{2t} = 4$ or $t = \ln 2$ and $y(\ln 2) = \frac{1}{4}(2) + \frac{1}{2} = 1$. Since $y''(t) = y(t)$ is positive at $t = \ln 2$, this is a minimum point. Note that $y(t) \to \infty$ as $t \to \infty$.

21. The general solution is $y = c_1 e^{-t} + c_2 e^{2t}$. Using the I.C. we obtain $c_1 + c_2 = \alpha$ and $-c_1 + 2c_2 = 2$, so adding the two equations we find $3c_2 = \alpha + 2$. If y is to approach zero as $t \to \infty$, c_2 must be zero. Thus $\alpha = -2$.

24. The roots of the characteristic equation are given by $r_1 = -2$, $r_2 = \alpha - 1$ and thus $y(t) = c_1 e^{-2t} + c_2 e^{(\alpha-1)t}$. Hence, for $\alpha < 1$, all solutions tend to zero as $t \to \infty$. For $\alpha > 1$, the second term becomes unbounded, but not the first, so there are no values of α for which all solutions become unbounded.

25a. The characteristic equation is $2r^2 + 3r - 2 = 0$, so $r_1 = -2$ and $r_2 = 1/2$ and $y = c_1 e^{-2t} + c_2 e^{t/2}$. The I.C. yield $c_1 + c_2 = 1$ and $-2c_1 + \frac{1}{2}c_2 = -\beta$ so that $c_1 = (1 + 2\beta)/5$ and $c_2 = (4-2\beta)/5$.

25b. Setting $\beta = 1$ and differentiating we obtain $y' = (-6e^{-2t} + e^{t/2})/5$. Setting this equal to zero and solving for t yields $t = \frac{2}{5}\ln 6$.

25c. From part (a), if $\beta = 2$ then $y(t) = e^{-2t}$ and the solution simply decays to zero. For $\beta > 2$, the solution becomes unbounded negatively, and again there is no minimum point. For $0 < \beta < 2$ there is always a minimum point, as found in part (b).

28a. The roots of the characteristic equation are given by $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. For the roots to be real and different we must have $b^2-4ac > 0$. If they are to be negative then we must have $b>0$ (since we are given $a>0$) and $c>0$. This latter condition comes from the fact that if $c \le 0$ then $\sqrt{b^2-4ac} \ge b$ and hence the numerator of r would give both positive and negative values, or a zero if $c = 0$.

<u>Section 3.2, Page 155</u>

2. $W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$.

4. $W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x$.

8. Dividing by $(t-1)$ we have $p(t) = -3t/(t-1)$, $q(t) = 4/(t-1)$ and $g(t) = \sin t/(t-1)$, so the only point of discontinuity is $t = 1$. By Theorem 3.2.1, the largest interval is $-\infty < t < 1$, since the initial point is $t_0 = -2$.

12. $p(x) = 1/(x-2)$, $q(x) = \tan x$ and $g(x) = 0$, so $x = \pi/2, 2, 3\pi/2, 5\pi/2 \ldots$ are points of discontinuity. Since $t_0 = 3$, the interval specified by Theorem 3.2.1 is $2 < x < 3\pi/2$.

14. For $y = t^{1/2}$, $y' = \frac{1}{2}t^{-1/2}$ and $y'' = -\frac{1}{4}t^{-3/2}$. Thus $yy'' + (y')^2 = -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$. $y = 1$ is also a solution since $y' = y'' = 0$. If $y = c_1(1) + c_2t^{1/2}$ is substituted

in the D.E. you will get

$(c_1+c_2t^{1/2})(-\frac{c_2}{4}t^{-3/2}) + (\frac{c_2}{2}t^{-1/2})^2 = -\frac{c_1c_2}{4}t^{-3/2}$, which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus the linear combination of two solutions is not, in general, a solution. Theorem 3.2.2 is not contradicted however, since the D.E. is not linear.

15. $y = \phi(t)$ is a solution of the D.E. so $L[\phi](t) = g(t)$. Since L is a linear operator, $L[c\phi](t) = cL[\phi](t) = cg(t)$. But, since $g(t) \neq 0$, $cg(t) = g(t)$ if and only if $c = 1$. This is not a contradiction of Theorem 3.2.2 since the linear D.E. is not homogeneous.

18. $W(f,g) = W(t,g) = \begin{vmatrix} t & g \\ 1 & g' \end{vmatrix} = tg' - g = t^2e^t$, or $g' - \frac{1}{t}g = te^t$.

This has an integrating factor of $\frac{1}{t}$ and thus $\frac{1}{t}g' - \frac{1}{t^2}g = e^t$ or $(\frac{1}{t}g)' = e^t$. Integrating and multiplying by t we obtain $g(t) = te^t + ct$.

22. From Section 3.1, e^t and e^{-2t} are two solutions, and since $W(e^t,e^{-2t}) = \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = -3e^{-t} \neq 0$ they form a fundamental set of solutions. To find the fundamental set specified by Theorem 3.2.4, let $y(t) = c_1e^t + c_2e^{-2t}$, where c_1 and c_2 satisfy $c_1 + c_2 = 1$ and $c_1 - 2c_2 = 0$ for y_1. Solving, we find $y_1 = \frac{2}{3}e^t + \frac{1}{3}e^{-2t}$. Likewise, c_1 and c_2 satisfy $c_1 + c_2 = 0$ and $c_1 - 2c_2 = 1$ for y_2, so that $y_2 = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$.

26. For $y_1 = x$, we have $x^2(0) - x(x+2)(1) + (x+2)(x) = 0$ and for $y_2 = xe^x$ we have $x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)xe^x = 0$. From Prob. 4, $W(x,xe^x) = x^2e^x \neq 0$ for $x > 0$, so y_1 and y_2 form a fundamental set of solutions for $x > 0$.

28a. From Sect. 3.1 the characteristic eq. is $r^2-r-2 = (r-2)(r+1) = 0$ and thus e^{-t} and e^{2t} are solutions. Since $\begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^t \neq 0$, e^{-t} and e^{2t} are fundamental solutions.

28b. For particular choices of c_1 and c_2, all three are solutions by Theorem 3.2.2.

28c. $W(y_2,y_3) = \begin{vmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{vmatrix} = -4e^{4t} + 4e^{4t} = 0$ and thus y_2 and y_3 do not form a fundamental set of solutions. Similar calculations show that $[y_1,y_3]$ and $[y_1,y_4]$ are fundamental solutions but y_4 and y_5 do not form a fundamental set of solutions.

29. Writing the D.E. in the form of Eq.(21), we have $p(t) = -(t+2)/t$. Thus Eq.(22) yields $W(t) = c\exp[-\int \frac{-(t+2)}{t}\,dt] = ct^2e^t$.

34. From Eq.(22) we have $W(y_1,y_2) = c\exp[-\int p(t)dt]$, where $p(t) = 2/t$ from the D.E. Thus $W(y_1,y_2) = c/t^2$. Since $W(y_1,y_2)(1) = 2$ we find $c = 2$ and thus $W(y_1,y_2)(5) = 2/25$.

38. Let c be the point in I at which both y_1 and y_2 vanish. Then $W(y_1,y_2)(c) = y_1(c)y_2'(c) - y_1'(c)y_2(c) = 0$. Since the Wronskian is zero the functions y_1 and y_2 cannot form a fundamental set.

40. Suppose that y_1 and y_2 have a point of inflection at t_0 and either $p(t_0) \neq 0$ or $q(t_0) \neq 0$. Since $y_1''(t_0) = 0$ and $y_2''(t_0) = 0$ it follows from the D.E. that $p(t_0)y_1'(t_0) + q(t_0)y_1(t_0) = 0$ and $p(t_0)y_2'(t_0) + q(t_0)y_2(t_0) = 0$. If $p(t_0) = 0$ and $q(t_0) \neq 0$ then $y_1(t_0) = y_2(t_0) = 0$, and $W(y_1,y_2)(t_0) = 0$ so the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) = 0$ then $y_1'(t_0) = y_2'(t_0) = 0$ and $W(y_1,y_2)(t_0) = 0$,

so again the solutions cannot form a fundamental set. If $p(t_0) \neq 0$ and $q(t_0) \neq 0$ then $y_1'(t_0) = -q(t_0)y_1(t_0)/p(t_0)$ and $y_2'(t_0) = -q(t_0)y_2(t_0)/p(t_0)$ and thus

$$W(y_1,y_2)(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)$$
$$= y_1(t_0)[-q(t_0)y_2(t_0)/p(t_0)] - [-q(t_0)y_1(t_0)/p(t_0)]y_2(t_0)$$
$$= 0.$$

41. Suppose that $P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]'$. On expanding the right side and equating coefficients, we find $f'(x) = R(x)$ and $P'(x) + f(x) = Q(x)$. These two conditions on f can be satisfied if $R(x) = Q'(x) - P''(x)$ which gives the necessary condition $P''(x) - Q'(x) + R(x) = 0$.

44. We have $P(x) = x$, $Q(x) = -\cos x$, and $R(x) = \sin x$ and the condition for exactness from Prob. 41 is satisfied. Also, from Prob. 41, $f(x) = Q(x) - P'(x) = -\cos x -1$, so the D.E. becomes $(xy')' - [(1 + \cos x)y]' = 0$. Hence $xy' - (1 + \cos x)y = c_1$. This is a first order linear D.E. and the integrating factor (after dividing by x) is $\mu(x) = \exp[-\int x^{-1}(1 + \cos x)dx]$. The general solution is $y = [\mu(x)]^{-1}[c_1\int_{x_0}^{x} t^{-1}\mu(t)dt + c_2]$.

46. We want to choose $\mu(x)$ and $f(x)$ so that $\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = [\mu(x)P(x)y']' + [f(x)y]'$. Expand the right side and equate coefficients of y'', y' and y. This gives $\mu'(x)P(x) + \mu(x)P'(x) + f(x) = \mu(x)Q(x)$ and $f'(x) = \mu(x)R(x)$. Differentiate the first equation and then eliminate $f'(x)$ to obtain the adjoint equation $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$.

48. $P = 1-x^2$, $Q = -2x$ and $R = \alpha(\alpha+1)$. Thus $2P' - Q = -4x + 2x = -2x$ and $P'' - Q' + R = -2 + 2 + \alpha(\alpha+1) = \alpha(\alpha+1)$, and thus, from Prob. 46, $(1-x^2)\mu'' - 2x\mu' + \alpha(\alpha+1)\mu = 0$ is the adjoint of the given D.E.

50. Write the adjoint D.E. given in Prob. 46 as $\hat{P}\mu'' + \hat{Q}\mu' + \hat{R}\mu = 0$ where $\hat{P} = P$, $\hat{Q} = 2P' - Q$, and $\hat{R} = P'' - Q' + R$. The adjoint of this equation, namely the adjoint of the adjoint, is

$\hat{P}y'' + (2\hat{P}' - \hat{Q})y' + (\hat{P}'' - \hat{Q}' + \hat{R})y = 0$. After substituting for $\hat{P}$, $\hat{Q}$, and $\hat{R}$ and simplifying, we obtain $Py'' + Qy' + Ry = 0$. This is the same as the original equation.

51. From Prob. 46 the adjoint of $Py'' + Qy' + Ry = 0$ is $P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$. The two equations are the same if $2P' - Q = Q$ and $P'' - Q' + R = R$. This will be true if $P' = Q$. Hence the original D.E. is self-adjoint if $P' = Q$. For Prob. 47, $P(x) = x^2$ so $P'(x) = 2x$ and $Q(x) = x$. Hence the Bessel equation of order ν is not self-adjoint. In a similar manner we find that Problems 48 and 49 are self-adjoint.

Section 3.3, Page 163

1. $\exp(1+2i) = e^{1+2i} = ee^{2i} = e(\cos2 + i\sin2)$.

5. Recall that
$2^{1-i} = e^{\ln(2^{1-i})} = e^{(1-i)\ln2} = e^{\ln2}e^{-i\ln2} = 2(\cos\ln2 - i\sin\ln2)$.

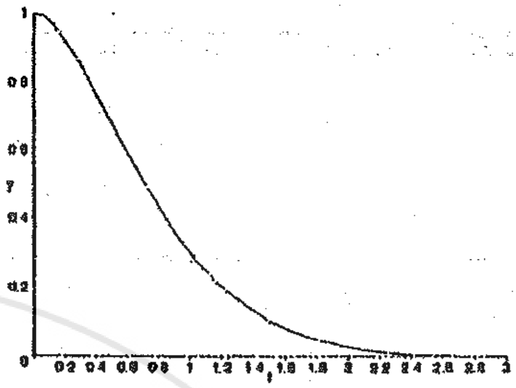
7. As in Sect. 3.1, we seek solutions of the form $y = e^{rt}$. Substituting this into the D.E. yields the characteristic equation $r^2 - 2r + 2 = 0$, which has the roots $r_1 = 1 + i$ and $r_2 = 1 - i$, using the quadratic formula. Thus $\lambda = 1$ and $\mu = 1$ and from Eq.(24) the general solution is $y = c_1e^t\cos t + c_2e^t\sin t$.

11. The characteristic equation is $r^2 + 6r + 13 = 0$, which has the roots $r = \dfrac{-6\pm\sqrt{-16}}{2} = -3 \pm 2i$. Thus $\lambda = -3$ and $\mu = 2$, so Eq.(24) becomes $y = c_1e^{-3t}\cos2t + c_2e^{-3t}\sin2t$.

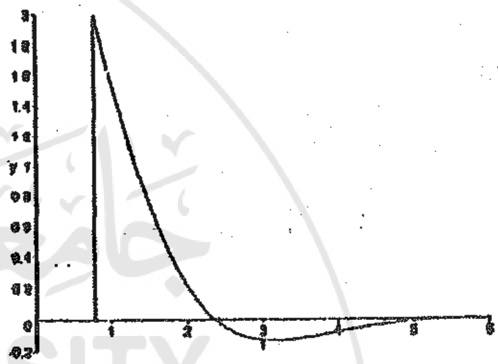
14. The characteristic equation is $9r^2 + 9r - 4$, which has the real roots $-4/3$ and $1/3$. Thus the solution has the same form as in Section 3.1, $y(t) = c_1e^{t/3} + c_2e^{-4t/3}$.

18. The characteristic equation is $r^2 + 4r + 5 = 0$, which has the roots $r_1, r_2 = -2 \pm i$. Thus $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$ and $y' = (-2c_1+c_2)e^{-2t}\cos t + (-c_1-2c_2)e^{-2t}\sin t$, so that

$y(0) = c_1 = 1$ and
$y'(0) = -2c_1 + c_2 = 0$,
or $c_2 = 2$. Hence
$y = e^{-2t}(\cos t + 2\sin t)$.
The oscillation is hard to see on this graph, but $y(t)$ does cross the t axis at $t = \tan^{-1}(-.5) = 2.68$ and periodically after that.



22. The characteristic equation is $r^2 + 2r + 2 = 0$, so $r_1, r_2 = -1 \pm i$. Since the I.C. are given at $\pi/4$ we want to alter Eq.(24) by letting $c_1 = e^{\pi/4}d_1$ and $c_2 = e^{\pi/4}d_2$. Thus, for $\lambda = -1$ and $\mu = 1$ we have $y = e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t)$; so $y' = -e^{-(t-\pi/4)}(d_1\cos t + d_2\sin t) + e^{-(t-\pi/4)}(-d_1\sin t + d_2\cos t)$.



Hence $y(\frac{\pi}{4}) = \sqrt{2}\,d_1/2 + \sqrt{2}\,d_2/2 = 2$ and

$y'(\frac{\pi}{4}) = -\sqrt{2}\,d_1 = -2$ and thus $y = \sqrt{2}\,e^{-(t-\pi/4)}(\cos t + \sin t)$.

23a. The characteristic equation is $3r^2 - r + 2 = 0$, which has the roots $r_1, r_2 = \frac{1}{6} \pm \frac{\sqrt{23}}{6}i$. Thus

$u(t) = e^{t/6}(c_1\cos\frac{\sqrt{23}}{6}t + c_2\sin\frac{\sqrt{23}}{6}t)$ and we obtain

$u(0) = c_1 = 2$ and $u'(0) = \frac{1}{6}c_1 + \frac{\sqrt{23}}{6}c_2 = 0$. Solving

for c_2 we find $u(t) = e^{t/6}(2\cos\frac{\sqrt{23}}{6}t - \frac{2}{\sqrt{23}}\sin\frac{\sqrt{23}}{6}t)$.

23b. To estimate the first time that $|u(t)| = 10$ plot the graph of $u(t)$ as found in part (a). Use this estimate in an appropriate computer software program to find $t = 10.7598$.

25a. The characteristic equation is $r^2 + 2r + 6 = 0$, so $r_1, r_2 = -1 \pm \sqrt{5}\,i$ and $y(t) = e^{-t}(c_1\cos\sqrt{5}\,t + c_2\sin\sqrt{5}\,t)$. Thus $y(0) = c_1 = 2$ and $y'(0) = -c_1 + \sqrt{5}\,c_2 = \alpha$ and hence

$$y(t) = e^{-t}\left(2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t\right).$$

25b. $y(1) = e^{-1}\left(2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\right) = 0$ and hence

$$\alpha = -2 - \frac{2\sqrt{5}}{\tan\sqrt{5}} = 1.50878.$$

25c. For $y(t) = 0$ we must have $2\cos\sqrt{5}\,t + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5}\,t = 0$ or $\tan\sqrt{5}\,t = \frac{-2\sqrt{5}}{\alpha+2}$. For $\alpha \geq 0$ (actually, for $\alpha > -2$) this yields $\sqrt{5}\,t = \pi - \arctan\frac{2\sqrt{5}}{\alpha+2}$ since $\arctan x$ is an odd function.

25d. From part (c) $\arctan\frac{2\sqrt{5}}{\alpha+2} \to 0$ as $\alpha \to \infty$, so $t \to \pi/\sqrt{5}$.

31. Let $r = \lambda+i\mu$, then $\frac{d}{dt}(e^{rt}) = \frac{d}{dt}[e^{\lambda t}(\cos\mu t + i\sin\mu t)]$

$= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + e^{\lambda t}(-\mu\sin\mu t + i\mu\cos\mu t)$

$= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + i\mu e^{\lambda t}(i\sin\mu t + \cos\mu t)$

$= e^{\lambda t}(\lambda+i\mu)(\cos\mu t + i\sin\mu t) = re^{rt}$.

33. Suppose that $t = a$ and $t = b$ $(b>a)$ are consecutive zeros of y_1. We must show that y_2 vanishes once and only once in the interval $a < t < b$. Assume that it does not vanish. Then we can form the quotient y_1/y_2 on the interval $a \leq t \leq b$. Note $y_2(a) \neq 0$ and $y_2(b) \neq 0$, otherwise y_1 and y_2 would not be a fundamental set of solutions. Next, y_1/y_2 vanishes at $t = a$ and $t = b$ and has a derivative in $a < t < b$. By Rolles theorem, the derivative must vanish at an interior point. But $\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_2'y_1}{y_2^{\,2}} = \frac{-W(y_1,y_2)}{y_2^{\,2}}$, which cannot be zero since y_1 and y_2 are fundamental solutions. Hence we have a contradiction and conclude that y_2 must vanish at a point between a and b. Finally, we show that it can vanish at only one point between a and b. Suppose that it vanishes at two points c and d between a and b. By the argument we have just given we can show that y_1 must vanish between c and d. But this contradicts the hypothesis that a and b are consecutive zeros of y_1.

34a. Using the Chain Rule we have $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dx}{dt}$ and

$$\frac{d^2y}{dt^2} = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d^2y}{dx^2}\frac{dx}{dt} = \frac{-1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2}.$$

36. Using the result of Prob.34(b) we have

$\frac{d^2y}{dx^2} + (4-1)\frac{dy}{dx} + 2y = 0$, which has the characteristic equation $r^2 + 3r + 2 = 0$. Thus $y(x) = c_1e^{-2x} + c_2e^{-x}$ so that

$$y(t) = c_1e^{-2\ln t} + c_2e^{-\ln t} = \frac{c_1}{t^2} + \frac{c_2}{t}.$$

40. Again, if $x = \ln t$, the D.E. becomes $\frac{d^2y}{dx^2} + (-1-1)\frac{dy}{dx} + 5y = 0$, so the characteristic equation is $r^2 - 2r + 5 = 0$. Finding the roots, we then have $y(x) = e^x(c_1\cos 2x + c_2\sin 2x)$ or $y(t) = t[c_1\cos(2\ln t) + c_2\sin(2\ln t)]$.

44. We use the result of Prob.43. Note that $p(t) = t$ and $q(t) = e^{-t^2} > 0$ for $-\infty < t < \infty$. Thus $(q' + 2pq)/q^{3/2} = 0$ and the D.E. can be transformed into an equation with constant coefficients by letting $x = u(t) = \int e^{-t^2/2}dt$. Substituting $x = u(t)$ in the differential equation found in part (b) of Prob.43 we obtain, after dividing by the coefficient of d^2y/dx^2, the D.E. $d^2y/dx^2 + y = 0$. Hence the general solution of the original D.E. is $y(t) = c_1\cos x(t) + c_2\sin x(t)$, $x(t) = \int e^{-t^2/2}dt$.

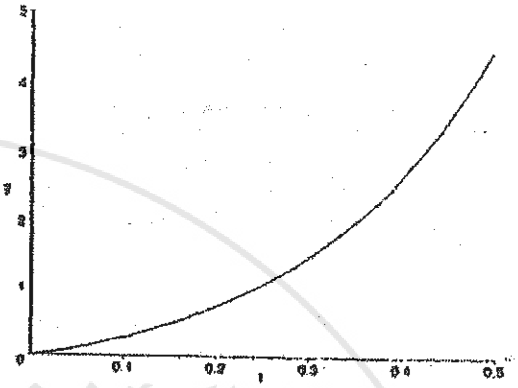
<u>Section 3.4, Page 171</u>

1. Substituting $y = e^{rt}$ into the D.E., we find that $r^2 - 2r + 1 = 0$, which gives $r_1 = 1$ and $r_2 = 1$. Since the roots are equal, the second fundamental solution is te^t, see Eq.(26), and thus the general solution is $y = c_1e^t + c_2te^t$.

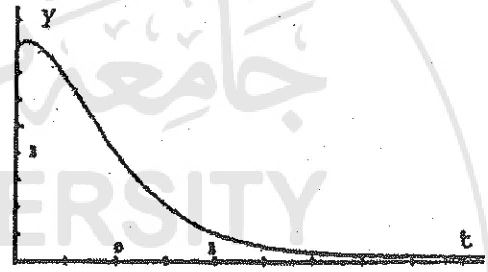
5. The characteristic equation is $r^2 - 2r + 10 = 0$, and thus $r = 1 \pm 3i$. The general solution, from Sect. 3.3, is $y(t) = e^t(c_1\cos 3t + c_2\sin 3t)$.

9. The characteristic equation is $25r^2 - 20r + 4 = 0$, which may be written as $(5r-2)^2 = 0$ and hence the roots are $r_1, r_2 = 2/5$. Thus $y = c_1e^{2t/5} + c_2te^{2t/5}$.

12. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, which has the repeated root $r = 3$. Thus $y = c_1e^{3t} + c_2te^{3t}$, which gives $y(0) = c_1 = 0$, $y'(t) = c_2(e^{3t}+3te^{3t})$ and $y'(0) = c_2 = 2$. Hence $y(t) = 2te^{3t}$, which becomes positively unbounded as $t\to\infty$.



14. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$, which has the repeated root $r = -2$. Since the I.C. are given at $t = -1$, write the general solution as $y = d_1e^{-2(t+1)} + d_2te^{-2(t+1)}$. Then $y' = -2d_1e^{-2(t+1)} + d_2e^{-2(t+1)} - 2d_2te^{-2(t+1)}$ and hence $d_1-d_2 = 2$ and $-2d_1+3d_2 = 1$ which yield $d_1 = 7$ and $d_2 = 5$. Thus $y = 7e^{-2(t+1)} + 5te^{-2(t+1)}$, a decaying exponential as shown in the graph.



17a. The characteristic equation is $4r^2 + 4r + 1 = (2r+1)^2 = 0$, so we have $y(t) = (c_1+c_2t)e^{-t/2}$. Thus $y(0) = c_1 = 1$ and $y'(0) = -c_1/2 + c_2 = 2$ and hence $c_2 = 5/2$ and $y(t) = (1 + 5t/2)e^{-t/2}$.

17b. From part(a), $y'(t) = -\frac{1}{2}(1 + 5t/2)e^{-t/2} + \frac{5}{2}e^{-t/2} = 0$, when $-\frac{1}{2} - \frac{5t}{4} + \frac{5}{2} = 0$, or $t_M = \frac{8}{5}$ and $y_M = 5e^{-4/5}$.

17c. From part(a), c_1 is the same and $y'(0) = -\frac{1}{2} + c_2 = b$ or $c_2 = b + \frac{1}{2}$ and $y(t) = [1 + (b + \frac{1}{2})t]e^{-t/2}$.

17d. From part(c), $y'(t) = -\frac{1}{2}[1 + (b+\frac{1}{2})t]e^{-t/2} + (b+\frac{1}{2})e^{-t/2} = 0$ which yields $t_M = \frac{4b}{2b+1} \to 2$ as $b \to \infty$ and

$y_M = (1 + \frac{2b+1}{2} \cdot \frac{4b}{2b+1})e^{-2b/(2b+1)} = (1 + 2b)e^{-2b/(2b+1)}$. Since $e^{-2b/(2b+1)} = e^{-2/(2+b^{-1})} \to e^{-1}$ as $b \to \infty$, $y_M \to \infty$ as $b \to \infty$.

19. If $r_1 = r_2$ then $y(t) = (c_1 + c_2t)e^{r_1t}$. Since the exponential is never zero, $y(t)$ can be zero only if $c_1 + c_2t = 0$, which yields at most one positive value of t if c_1 and c_2 differ in sign. If $r_2 > r_1$ then $y(t) = c_1e^{r_1t} + c_2e^{r_2t} = e^{r_1t}(c_1 + c_2e^{(r_2-r_1)t})$. Again, this is zero only if c_1 and c_2 differ in sign, in which case $t = \frac{\ln(-c_1/c_2)}{(r_2-r_1)}$.

21. If $r_2 \neq r_1$ then $\phi(t;r_1,r_2) = (e^{r_2t} - e^{r_1t})/(r_2 - r_1)$ is defined for all t. Note that ϕ is a linear combination of the fundamental solutions, e^{r_1t} and e^{r_2t}, of the D.E. and thus ϕ is a solution of the D.E. Think of r_1 as fixed and let $r_2 \to r_1$. The limit of ϕ as $r_2 \to r_1$ is indeterminate. If we use L'Hospital's rule (with r_2 as the variable), we find $\lim_{r_2 \to r_1} \frac{e^{r_2t} - e^{r_1t}}{r_2 - r_1} = \lim_{r_2 \to r_1} \frac{te^{r_2t}}{1} = te^{r_1t}$. Hence, the solution $\phi(t;r_1,r_2) \to te^{r_1t}$ as $r_2 \to r_1$.

25. Let $y_2 = v/t$. Then $y_2' = v'/t - v/t^2$ and $y_2'' = v''/t - 2v'/t^2 + 2v/t^3$. Substituting in the D.E. we obtain $t^2(v''/t - 2v'/t^2 + 2v/t^3) + 3t(v'/t - v/t^2) + v/t = 0$. The terms involving v add to zero since $\frac{1}{t}$ is a solution. The left side then reduces to $tv'' + v' = 0$, which is linear in v', so $v' = c_1/t$. Thus $v = c_1\ln t + c_2$ so a second solution is $y_2(t) = (c_1\ln t + c_2)/t$. However, we may set $c_2 = 0$ and $c_1 = 1$ without loss of generality and thus we have $y_2(t) = (\ln t)/t$ as a second solution. Note that in the form we actually calculated, $y_2(t)$ is a linear combination of $1/t$ and $\ln t/t$, and hence is the general solution.

27. In this case the calculations are somewhat easier if we do not use the explicit form for $y_1(x) = \sin x^2$ at the

beginning but simply set $y_2(x) = y_1v$. Substituting this form for y_2 in the D.E. gives $x(y_1v)''-(y_1v)'+4x^3(y_1v) = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $xy_1v'' + (2xy_1' - y_1)v' = 0$. Let $w = v'$ and separate variables to find $\frac{dw}{w} = (\frac{1}{x} - \frac{2y_1'}{y_1})dx$. Integration yields $\ln w = \ln x - 2\ln y_1 + C$, so $w = v' = cx/(\sin x^2)^2$. Setting $u = x^2$ allows integration of this to get $v = c_1\cot x^2 + c_2$. Setting $c_1 = 1$, $c_2 = 0$ and multiplying by $y_1 = \sin x^2$ we obtain $y_2(x) = \cos x^2$ as the second solution of the D.E.

30. Substituting $y_2(x) = y_1(x)v(x)$ in the D.E. gives $x^2(y_1v)'' + x(y_1v)' + (x^2 - \frac{1}{4})y_1v = 0$. On carrying out the differentiations and making use of the fact that y_1 is a solution, we obtain $x^2y_1v'' + (2x^2y_1' + xy_1)v' = 0$. This is a first order linear equation for $w = v'$, $w' + (2y_1'/y_1 + 1/x)w = 0$, with solution (by separating variables)

$$w = v'(x) = c\exp[-\int(2\frac{y_1'}{y_1} + \frac{1}{x})dx] = c\exp[-2\ln y_1 - \ln x]$$

$$= c\frac{1}{xy_1{}^2} = \frac{c}{x(x^{-1}\sin^2x)} = c\csc^2x,$$ where c is an arbitrary constant, which we will take to be one. Then $v(x) = \int\csc^2x\,dx = -\cot x + k$ where again k is an arbitrary constant which can be taken equal to zero. Thus $y_2(x) = y_1(x)v(x) = (x^{-1/2}\sin x)(-\cot x) = -x^{-1/2}\cos x$. The second solution is usually taken to be $x^{-1/2}\cos x$. Note that $c = -1$ would have given this solution.

31b. Let $y_2(x) = e^xv(x)$, then $y_2' = e^xv' + e^xv$, and $y_2'' = e^xv'' + 2e^xv' + e^xv$. Substituting in the D.E. we obtain $xe^xv'' + (xe^x-Ne^x)v' = 0$, or $v'' + (1-N/x)v' = 0$. This is a first order linear D.E. for v' with integrating factor $\mu(x) = \exp[\int(1-N/x)dx] = x^{-N}e^x$. Hence $(x^{-N}e^xv')' = 0$, and $v' = cx^Ne^{-x}$ which gives

$v(x) = c\int x^N e^{-x}dx + k$. On taking $k = 0$ we obtain as the second solution $y_2(x) = ce^x\int x^N e^{-x}dx$. The integral can be evaluated by using the method of integration by parts. At each stage let $u = x^N$ or x^{N-1}, or whatever the power of x that remains, and let $dv = e^{-x}$. Note that this dv is not related to the v(x) in $y_2(x)$. For $N = 2$ we have

$$y_2(x) = ce^x\int x^2e^{-x}dx = ce^x[x^2\frac{e^{-x}}{-1} - \int 2x\frac{e^{-x}}{-1}dx]$$

$$= -cx^2 + ce^x[2x\frac{e^{-x}}{-1} - \int 2\frac{e^{-x}}{-1}dx]$$

$$= c(-x^2 - 2x - 2) = -2c(1 + x + x^2/2!).$$

Choosing $c = -1/2!$ gives the desired result. For the general case $c = -1/N!$

33. $(y_2/y_1)' = (y_1y_2' - y_1'y_2)/y_1^2 = W(y_1,y_2)/y_1^2$. Abel's identity is $W(y_1,y_2) = c\exp[-\int_{t_0}^{t}p(r)dr]$. Hence $(y_2/y_1)' = cy_1^{-2}\exp[-\int_{t_0}^{t}p(r)dr]$. Integrating and setting $c = 1$ (since a solution y_2 can be multiplied by any constant) and taking the constant of integration to be zero we obtain

$$y_2(t) = y_1(t)\int_{t_0}^{t}\frac{\exp[-\int_{s_0}^{s}p(r)dr]}{[y_1(s)]^2}ds.$$

35. From Prob. 33 and Abel's formula we have

$$(\frac{y_2}{y_1})' = \frac{\exp[\int(1/t)dt]}{\sin^2(t^2)} = \frac{e^{\ln t}}{\sin^2(t^2)} = t\csc^2(t^2).$$ Thus

$y_2/y_1 = -(1/2)\cot(t^2)$ and hence we can choose $y_2 = \cos(t^2)$ since $y_1 = \sin^2(t^2)$.

38. The general solution of the D.E. is $y = c_1e^{r_1t} + c_2e^{r_2t}$ where $r_1, r_2 = (-b \pm \sqrt{b^2-4ac})/2a$ provided $b^2 - 4ac \neq 0$. In this case there are two possibilities. If $b^2 - 4ac > 0$ then $(b^2 - 4ac)^{1/2} < b$ and r_1 and r_2 are real and <u>negative</u>. Consequently $e^{r_1t} \to 0$ and $e^{r_2t} \to 0$; and hence $y \to 0$, as $t \to \infty$. If $b^2 - 4ac < 0$ then r_1 and r_2 are complex conjugates with <u>negative</u> real part. Again $e^{r_1t} \to 0$ and $e^{r_2t} \to 0$; and hence $y \to 0$, as $t \to \infty$.

Finally, if $b^2 - 4ac = 0$, then $y = c_1e^{r_1t} + c_2te^{r_1t}$ where $r_1 = -b/2a < 0$. Hence, again $y \to 0$ as $t \to \infty$. This conclusion does not hold if either $b = 0$ (since, in this case, $y(t) = c_1\cos\omega t + c_2\sin\omega t$, where $\omega^2 = \frac{c}{a}$) or $c = 0$ (since one of the solutions would be $y_1(t) = c_1$).

42. Substituting $x = \ln t$ into the D.E. gives $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 0.25y = 0$, which has the solution $y(x) = c_1e^{-x/2} + c_2xe^{-x/2}$ so that $y(t) = c_1t^{-1/2} + c_2t^{-1/2}\ln t$.

46. Again $x = \ln t$, so $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$. The roots of the characteristic are $r_{1,2} = -2 \pm 3i$ and thus $y(x) = (c_1\cos 3x + c_2\sin 3x)e^{-2x}$ which gives $y(t) = [c_1\cos(3\ln t) + c_2\sin(3\ln t)]t^{-2}$.

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1. First we find the solution of the homogeneous D.E., which has the characteristic equation $r^2-2r-3 = (r-3)(r+1) = 0$. Hence $y_c = c_1e^{3t} + c_2e^{-t}$ and we can assume $Y = Ae^{2t}$ for the particular solution. Thus $Y' = 2Ae^{2t}$ and $Y'' = 4Ae^{2t}$ and substituting into the D.E. yields $4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$. Thus $-3A = 3$ and $A = -1$, yielding $y = c_1e^{3t} + c_2e^{-t} - e^{2t}$.

4. Initially we might assume $Y = A + B\sin 2t + C\cos 2t$. However, since a constant is a solution of the related homogeneous D.E. we must modify Y by multiplying the constant A by t and thus the correct form is $Y = At + B\sin 2t + C\cos 2t$.

6. Since $y_c = c_1e^{-t} + c_2te^{-t}$ we must assume $Y = At^2e^{-t}$, so that $Y' = 2Ate^{-t} - At^2e^{-t}$ and $y'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$. Substituting in the D.E. gives $(At^2-4At+2A)e^{-t} + 2(-At^2+2At)e^{-t} + At^2e^{-t} = 2e^{-t}$. Notice that all terms on

the left involving t^2 and t add to zero and we are left with $2A = 2$, or $A = 1$. Hence $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$.

8. The assumed form is $Y = (At + B)\sin2t + (Ct + D)\cos2t$, which is appropriate for both terms appearing on the right side of the D.E. Since none of the terms appearing in Y are solutions of the homogeneous equation, we do not need to modify Y. Calculating Y' and Y'' we have $Y' = A\sin2t + C\cos2t + 2(At+B)\cos2t - 2(Ct+D)\sin2t$ and $Y'' = 4A\cos2t - 4C\sin2t - 4(At+B)\sin2t - 4(Ct+D)\cos2t$. Thus $Y''+Y = -3At\sin2t - 3Ct\cos2t - (3B+4C)\sin2t - (4A-3D)\cos2t$. Equating like coefficients yields $A = 0, -3B-4C = 3, -3C = 1$, and $4A-3D = 0$. Hence $Y(t) = -(5/9)\sin2t - (1/3)t\cos2t$.

11. First solve the homogeneous D.E. Substituting $y = e^{rt}$ gives $r^2 + r + 4 = 0$. Hence $y_c = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)]$. We replace $\sinh t$ by $(e^t - e^{-t})/2$ and then assume $Y(t) = Ae^t + Be^{-t}$. Since neither e^t nor e^{-t} are solutions of the homogeneous equation, there is no need to modify our assumption for Y. Substituting in the D.E., we obtain $6Ae^t + 4Be^{-t} = e^t - e^{-t}$. Hence, $A = 1/6$ and $B = -1/4$. The general solution is $y = e^{-t/2}[c_1\cos(\sqrt{15}\,t/2) + c_2\sin(\sqrt{15}\,t/2)] + e^t/6 - e^{-t}/4$. [For this problem we could also have found a particular solution as a linear combination of sinht and cosht: $Y(t) = A\cosh t + B\sinh t$. Substituting this in the D.E. gives $(5A + B)\cosh t + (A + 5B)\sinh t = 2\sinh t$. The solution is $A = -1/12$ and $B = 5/12$. A simple calculation shows that $-(1/12)\cosh t + (5/12)\sinh t = e^t/6 - e^{-t}/4$.]

13. $y_c = c_1e^{-2t} + c_2e^t$ so for the particular solution we assume $Y = At + B$. Since neither At or B are solutions of the homogeneous equation it is not necessary to modify the original assumption. Substituting Y in the D.E. we obtain $0 + A - 2(At+B) = 2t$ or $-2A = 2$ and $A - 2B = 0$. Solving for A and B we obtain $y = c_1e^{-2t} + c_2e^t - t - 1/2$ as the general solution. $y(0) = 0 \Rightarrow c_1 + c_2 - 1/2 = 0$ and $y'(0) = 1 \Rightarrow -2c_1 + c_2 - 1 = 1$, which yield $c_1 = -1/2$ and $c_2 = 1$. Thus $y = e^t - (1/2)e^{-2t} - t - 1/2$.

16. Since the characteristic equation is $r^2 - 2r - 3 = 0$, $y_c = c_1e^{3t} + c_2e^{-t}$. The nonhomogeneous term is the product of a linear polynomial and an exponential, so assume Y of the same form: $Y = (At+B)e^{2t}$, which we do not need to modify since these terms are not in y_c. Thus $Y' = Ae^{2t} + 2(At+B)e^{2t}$ and $Y'' = 4Ae^{2t}+4(At+B)e^{2t}$. Substituting into the D.E. we find $-3At = 3t$ and $2A - 3B = 0$, yielding $A = -1$ and $B = -2/3$. Thus, the general solution is $y = c_1e^{3t} + c_2e^{-t} - \frac{2}{3}e^{2t} - te^{2t}$.

19a. The solution of the homogeneous D.E. is $y_c = c_1e^{-3t} + c_2$. After inspection of the nonhomogeneous term, for $2t^4$ we must assume a fourth order polynominial, for t^2e^{-3t} we must assume a quadratic polynomial times the exponential, and for sin3t we must assume Csin3t + Dcos3t. Thus

$$Y(t) = (A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + (B_0t^2 + B_1t + B_2)e^{-3t} + C\sin 3t + D\cos 3t.$$

However, since e^{-3t} and a constant are solutions of the homogeneous D.E., we must multiply the coefficient of e^{-3t} and the polynomial by t. The correct form, then, is

$$Y(t) = t(A_0t^4 + A_1t^3 + A_2t^2 + A_3t + A_4) + t(B_0t^2 + B_1t + B_2)e^{-3t} + C\sin 3t + D\cos 3t.$$

22a. The solution of the homgeneous D.E. is $y_c = e^{-t}[c_1\cos t + c_2\sin t]$. After inspection of the nonhomogeneous term, we assume

$$Y(t) = Ae^{-t} + (B_0t^2 + B_1t + B_2)e^{-t}\cos t + (C_0t^2 + C_1t + C_2)e^{-t}\sin t.$$

Since $e^{-t}\cos t$ and $e^{-t}\sin t$ are solutions of the homogeneous D.E., it is necessary to multiply both the last two terms by t. Hence the correct form is

$$Y(t) = Ae^{-t} + t(B_0t^2 + B_1t + B_2)e^{-t}\cos t + t(C_0t^2 + C_1t + C_2)e^{-t}\sin t.$$

27a. Calculating Y′ and Y″ and substituting into the D.E. we get $(v''-2v'+v)e^{-t} - 3(v'-v)e^{-t} - 4ve^{-t} = 2e^{-t}$. This reduces to $v'' - 5v' = 2$, which is a 1st order D.E. for v'.

27b. The linear D.E. for w has an integrating factor of e^{-5t} and thus $(e^{-5t}w)' = 2e^{-5t}$, which gives $w = v' = -\frac{2}{5} + c_1e^{5t}$.

27c. Integration then gives $v = -\frac{2}{5}t + \frac{c_1}{5}e^{5t} + c_2$ so

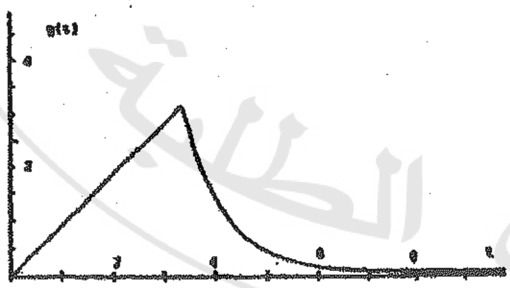
$Y(t) = ve^{-t} = -\frac{2}{5}te^{-t} + \frac{c_1}{5}e^{4t} + c_2e^{-t}$. The last two terms can be thought of as y_c and the first term as a particular solution.

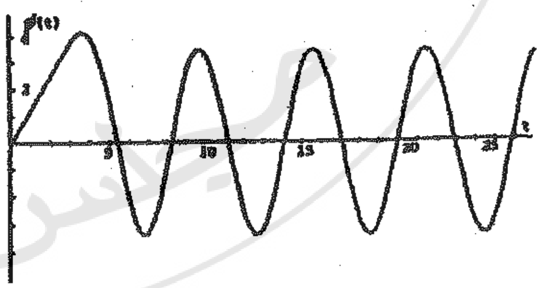
29. First solve the I.V.P. $y'' + y = t$, $y(0) = 0$, $y'(0) = 1$ for $0 \le t \le \pi$. The solution of the homogeneous D.E. is $y_c(t) = c_1\cos t + c_2\sin t$. The correct form for $Y(t)$ is $y(t) = A_0t + A_1$. Substituting in the D.E. we find $A_0 = 1$ and $A_1 = 0$. Hence, $y = c_1\cos t + c_2\sin t + t$. Applying the I.C., we obtain $y = t$, for $0 \le t \le \pi$.

For $t > \pi$ we have $y'' + y = \pi e^{\pi-t}$ so the form, now, for $Y(t)$ is $Y(t) = Ee^{\pi-t}$. Substituting $Y(t)$ in the D.E., we obtain $Ee^{\pi-t} + Ee^{\pi-t} = \pi e^{\pi-t}$ so $E = \pi/2$. Hence the general solution for $t > \pi$ is $Y = D_1\cos t + D_2\sin t + (\pi/2)e^{\pi-t}$. If y and y' are to be continuous at $t = \pi$, then the solutions and their derivatives for $t \le \pi$ and $t > \pi$ must have the same value at $t = \pi$. These conditions require $\pi = -D_1 + \pi/2$ and $1 = -D_2 - \pi/2$. Hence $D_1 = -\pi/2$, $D_2 = -(1 + \pi/2)$, and

$$y = \phi(t) = \begin{cases} t, & 0 \le t \le \pi \\ -(\pi/2)\cos t - (1 + \pi/2)\sin t + (\pi/2)e^{\pi-t}, & t > \pi. \end{cases}$$

The graphs of the nonhomogeneous term and ϕ follow.





31. According to Theorem 3.5.1, the difference of any two solutions of the linear second order nonhomogeneous D.E. is a solution of the corresponding homogeneous D.E. Hence $Y_1 - Y_2$ is a solution of $ay'' + by' + cy = 0$. In Prob. 38 of Section 3.4 we showed that if $a > 0$, $b > 0$, and $c > 0$ then every solution of this D.E. goes to zero

as $t \to \infty$. If $b = 0$, then y_c involves only sines and cosines, so $Y_1 - Y_2$ does not approach zero as $t \to \infty$.

34. From Prob. 33 we write the D.E. as $(D-4)(D+1)y = 3e^{2t}$. Thus let $(D+1)y = u$ and then $(D-4)u = 3e^{2t}$. This last equation is the same as $du/dt - 4u = 3e^{2t}$, which may be solved by multiplying both sides by e^{-4t} and integrating (see Sect. 2.1). This yields $u = (-3/2)e^{2t} + Ce^{4t}$. Substituting this form of u into $(D+1)y = u$ we obtain $dy/dt + y = (-3/2)e^{2t} + Ce^{4t}$. Again, multiplying by e^{t} and integrating gives $y = (-1/2)e^{2t} + C_1e^{4t} + C_2e^{-t}$, where $C_1 = C/5$.

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2. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{2t}$ and $y_2(t) = e^{-t}$. Assume
$Y = u_1(t)e^{2t} + u_2(t)e^{-t}$, then
$Y'(t) = [2u_1(t)e^{2t} - u_2(t)e^{-t}] + [u_1'(t)e^{2t} + u_2'(t)e^{-t}]$.
We set $u_1'(t)e^{2t} + u_2'(t)e^{-t} = 0$. Then
$Y'' = 4u_1e^{2t} + u_2e^{-t} + 2u_1'e^{2t} - u_2'e^{-t}$ and substituting in the D.E. gives $2u_1'(t)e^{2t} - u_2'(t)e^{-t} = 2e^{-t}$ (the terms involving u_1 and u_2 add to zero since e^{-t} and e^{2t} are solutions ofthe homogneous equation). Thus we have two algebraic equations for $u_1'(t)$ and $u_2'(t)$ with the solution
$u_1'(t) = 2e^{-3t}/3$ and $u_2'(t) = -2/3$. Hence
$u_1(t) = -2e^{-3t}/9$ and $u_2(t) = -2t/3$. Substituting in the expression for $Y(t)$ we obtain
$Y(t) = (-2e^{-3t}/9)e^{2t} + (-2t/3)e^{-t} = (-2e^{-t}/9) - (2te^{-t}/3)$.
Since e^{-t} is a solution of the homogeneous D.E., we can choose $Y(t) = -2te^{-t}/3$.

5. Since cost and sint are solutions of the homogeneous D.E., we assume $Y = u_1(t)\cos t + u_2(t)\sin t$. Thus
$Y' = -u_1(t)\sin t + u_2(t)\cos t$, after setting
$u_1'(t)\cos t + u_2'(t)\sin t = 0$. Finding Y'' and substituting

into the D.E. then yields $-u_1'(t)\sin t + u_2'(t)\cos t = \tan t$. The two equations for $u_1'(t)$ and $u_2'(t)$ have the solution: $u_1'(t) = -\sin^2 t/\cos t = -\sec t + \cos t$ and $u_2'(t) = \sin t$. Thus $u_1(t) = \sin t - \ln(\tan t + \sec t)$ and $u_2(t) = -\cos t$, which when substituted into the assumed form for Y, simplified, and added to the homogeneous solution yields

$y = c_1\cos t + c_2\sin t - (\cos t)\ln(\tan t + \sec t)$.

11. Two linearly independent solutions of the homogeneous D.E. are $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$. Applying Theorem 3.6.1 with $W(y_1,y_2)(t) = -e^{5t}$, we obtain

$$Y(t) = -e^{3t}\int \frac{e^{2s}g(s)}{-e^{5s}}ds + e^{2t}\int \frac{e^{3s}g(s)}{-e^{5s}}ds$$
$$= \int [e^{3(t-s)} - e^{2(t-s)}]g(s)ds.$$

The complete solution is then obtained by adding $c_1e^{3t} + c_2e^{2t}$ to $Y(t)$. Note: since we are taking e^{3t} and e^{2t} under the integral the integration variable can't be t.

14. That t and te^t are solutions of the homogeneous D.E. can be verified by direction substitution. Thus we assume $Y = tu_1(t) + te^tu_2(t)$. Following the pattern of earlier problems we find $tu_1'(t) + te^tu_2'(t) = 0$, Eq.(21), and $u_1'(t) + (t+1)e^tu_2' = 2t$, Eq.(25). [Note that $g(t) = 2t$, since the D.E. must be put into the form of Eq.(16)]. The solution of these equations gives $u_1'(t) = -2$ and $u_2'(t) = 2e^{-t}$. Hence, $u_1(t) = -2t$ and $u_2(t) = -2e^{-t}$, and $Y(t) = t(-2t) + te^t(-2e^{-t}) = -2t^2 - 2t$. However, since t is a solution of the homogeneous D.E. we can choose as our particular solution $Y(t) = -2t^2$.

18. For this problem, and for many others, it is probably easier to rederive Eqs.(26) without using the explicit form for $y_1(x)$ and $y_2(x)$ and then to substitute for $y_1(x)$ and $y_2(x)$ in Eqs.(26). In this case if we take $y_1 = x^{-1/2}\sin x$ and $y_2 = x^{-1/2}\cos x$, then $W(y_1,y_2) = -1/x$. If the D.E. is put in the form of Eq.(16), then $g(x) = 3x^{-1/2}\sin x$ and thus $u_1'(x) = 3\sin x\cos x$ and

$u_2'(x) = -3\sin^2 x = 3(-1 + \cos 2x)/2$. Hence
$u_1(x) = (3\sin^2 x)/2$ and $u_2(x) = -3x/2 + 3(\sin 2x)/4$, and

$$Y(x) = \frac{3\sin^2 x}{2}\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin 2x}{4}\right)\frac{\cos x}{\sqrt{x}}$$
$$= \frac{3\sin^2 x}{2}\frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3\sin x\cos x}{2}\right)\frac{\cos x}{\sqrt{x}}$$
$$= \frac{3\sin x}{2\sqrt{x}} - \frac{3\sqrt{x}\cos x}{2}.$$

The first term is a multiple of $y_1(x)$ and thus can be neglected for $Y(x)$.

22. Putting limits on the integrals of Eq.(28)

$$Y(t) = -y_1(t)\int_{t_0}^{t}\frac{y_2(s)g(s)ds}{W(y_1,y_2)(s)} + y_2(t)\int_{t_0}^{t}\frac{y_1(s)g(s)ds}{W(y_1,y_2)(s)}$$
$$= \int_{t_0}^{t}\frac{-y_1(t)y_2(s)g(s)ds}{W(y_1,y_2)(s)} + \int_{t_0}^{t}\frac{y_2(t)y_1(s)g(s)ds}{W(y_1,y_2)(s)}$$
$$= \int_{t_0}^{t}\frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]g(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}ds.$$

To show that $Y(t)$ satisfies $L[y] = g(t)$ we must take the derivative of Y using Leibnitz's rule, which says that if $Y(t) = \int_{t_0}^{t}G(t,s)ds$, then $Y'(t) = G(t,t) + \int_{t_0}^{t}\frac{\partial G}{\partial t}(t,s)ds$.
Letting $G(t,s)$ be the above integrand, then $G(t,t) = 0$ and $\frac{\partial G}{\partial t} = \frac{y_1(s)y_2'(t) - y_1'(t)y_2(s)}{W(y_1,y_2)(s)}g(s)$. Likewise

$$Y'' = \frac{\partial G(t,t)}{\partial t} + \int_{t_0}^{t}\frac{\partial^2 G}{\partial t^2}(t,s)ds$$
$$= g(t) + \int_{t_0}^{t}\frac{y_1(s)y_2''(t) - y_1''(t)y_2(s)}{W(y_1,y_2)(s)}g(s)ds.$$

Since y_1 amd y_2 are solutions of $L[y] = 0$, we have $L[Y] = g(t)$ since all the terms involving the integral will add to zero. Clearly $Y(t_0) = 0$ and $Y'(t_0) = 0$.

25. Note that $y_1 = e^{\lambda t}\cos\mu t$ and $y_2 = e^{\lambda t}\sin\mu t$ and thus $W(y_1,y_2) = \mu e^{2\lambda t}$. From Prob. 22 we then have:

$$Y(t) = \int_{t_0}^{t}\frac{e^{\lambda s}\cos\mu s e^{\lambda t}\sin\mu t - e^{\lambda t}\cos\mu t e^{\lambda s}\sin\mu s}{\mu e^{2\lambda s}}g(s)ds$$

$$= \mu^{-1}\int_{t_0}^{t} e^{\lambda(t-s)}[\cos\mu s\ \sin\mu t - \cos\mu t\ \sin\mu s]g(s)ds$$

$$= \mu^{-1}\int_{t_0}^{t} e^{\lambda(t-s)}[\sin\mu(t-s)]g(s)ds.$$

29. First, we put the D.E. in standard form by dividing by t^2: $y'' - 2y'/t + 2y/t^2 = 4$. Assuming that $y = tv(t)$ and substituting in the D.E. we obtain $tv'' = 4$. Hence $v'(t) = 4\ln t + c_2$ and $v(t) = 4\int \ln t\, dt + c_2 t + c_1 = 4(t\ln t - t) + c_2 t + c_1$, using integration by parts. Thus $y = 4t^2\ln t + c_3 t^2 + c_1 t$, where $c_3 = c_2 - 4$. Since $y_1 = c_1 t$, we can take $y_2 = 4t^2\ln t + c_3 t^2$, where $c_3 t^2$ represents the second fundamental solution of the related homogeneous equation and $4t^2\ln t$ is the particular solution.

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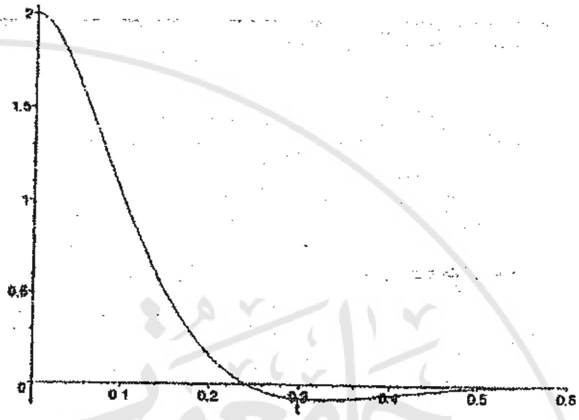
2. From Eq.(15) we have $R\cos\delta = -1$, and $R\sin\delta = \sqrt{3}$. Thus $R = \sqrt{1+3} = 2$ and $\delta = \tan^{-1}(-\sqrt{3}) + \pi = 2\pi/3 \cong 2.09440$. Note that we have to "add" π to the inverse tangent value since δ must be a second quadrant angle. Thus $u = 2\cos(t-2\pi/3)$.

6. The motion is an undamped free vibration. The units are in the CGS system. The spring constant [see Eq.(2)] is $k = (100\text{ gm})(980\text{cm/sec}^2)/5\text{cm}$. Hence the D.E. for the motion is $100u'' + [(100 \cdot 980)/5]u = 0$ where u is measured in cm and time in sec. We obtain $u'' + 196u = 0$ so $u = A\cos 14t + B\sin 14t$. The I.C. are $u(0) = 0 \Rightarrow A = 0$ and $u'(0) = 10$ cm/sec $\Rightarrow B = 10/14 = 5/7$. Hence $u(t) = (5/7)\sin 14t$, which first reaches equilibrium when $14t = \pi$, or $t = \pi/14$.

8. We use Eq.(33) without R and E(t) (there is no resistor or impressed voltage) and with L = 1 henry and $1/C = 4\times10^6$ since $C = .25\times10^{-6}$ farads. Thus the I.V.P. is $Q'' + 4\times10^6\, Q = 0$, $Q(0) = 10^{-6}$ coulombs and $Q'(0) = 0$.

9. The spring constant is $k = (20)(980)/5 = 3920$ dyne/cm. The I.V.P. for the motion is $20u'' + 400u' + 3920u = 0$ or $u'' + 20u' + 196u = 0$ and $u(0) = 2$, $u'(0) = 0$. Here u is measured in cm and t in sec. The general solution of the D.E. is $u = Ae^{-10t}\cos 4\sqrt{6}\,t + Be^{-10t}\sin 4\sqrt{6}\,t$. The I.C. $u(0) = 2 \Rightarrow A = 2$ and $u'(0) = 0 \Rightarrow -10A + 4\sqrt{6}B = 0$. The solution is $u = e^{-10t}[2\cos 4\sqrt{6}\,t + 5(\sin 4\sqrt{6}\,t)/\sqrt{6}]$cm.

The quasi frequency is $\mu = 4\sqrt{6}$, the quasi period is $T_d = 2\pi\mu = \pi/2\sqrt{6}$ and $T_d/T = 7/2\sqrt{6}$ since $T = 2\pi/14 = \pi/7$. To find an upper bound for τ, write u in the form of Eq.(26): $u(t) = \sqrt{4+25/6}\,e^{-10t}\cos(4\sqrt{6}\,t-\delta)$. Now, since $|\cos(4\sqrt{6}\,t-\delta)| \le 1$, we have $|u(t)| < .05 \Rightarrow \sqrt{4+25/6}\,e^{-10t} < .05$, which yields $\tau = .4046$. A more precise answer can be obtained with a computer algebra system, which in this case yields $\tau = .4045$. The original estimate was unusually close for this problem since $\cos(4\sqrt{6}\,t-\delta) = -0.9996$ for $t = .4046$.



12. Substituting the given values for L, C and R in Eq.(33), we obtain the D.E. $.2Q'' + 3\times10^2 Q' + 10^5 Q = 0$. The I.C. are $Q(0) = 10^{-6}$ and $Q'(0) = I(0) = 0$. The roots of the characteristic equation are $r_1 = -500$ and $r_2 = -1000$. Thus $Q = c_1e^{-500t} + c_2e^{-1000t}$ and hence $Q(0) = 10^{-6} \Rightarrow c_1 + c_2 = 10^{-6}$ and $Q'(0) = 0 \Rightarrow -500c_1 - 1000c_2 = 0$. Solving for c_1 and c_2 yields the solution.

17. The mass is 8/32 lb-sec^2/ft, and the spring constant is 8/(1/8) = 64 lb/ft. Hence $(1/4)u'' + \gamma u' + 64u = 0$ or $u'' + 4\gamma u' + 256u = 0$, where u is measured in ft, t in sec and the units of γ are lb-sec/ft. The characteristic equation is $r^2 + 4\gamma r + 256 = 0$, so $r_1, r_2 = [-4\gamma \pm \sqrt{16\gamma^2 - 1024}\,]/2$. The system will be overdamped, critically damped or underdamped as $(16\gamma^2 - 1024)$ is > 0, $= 0$, or < 0, respectively. Thus the system is critically damped when $\gamma = 8$ lb-sec/ft.

19. The general solution of the D.E. is $u = Ae^{r_1t} + Be^{r_2t}$ where $r_1, r_2 = [-\gamma \pm (\gamma^2 - 4km)^{1/2}]/2m$ provided $\gamma^2 - 4km \ne 0$, and where A and B are determined by the I.C. When the motion is overdamped, $\gamma^2 - 4km > 0$ and $r_1 > r_2$. Setting $u = 0$, we obtain $Ae^{r_1t} = -Be^{r_2t}$ or $e^{(r_1-r_2)t} = -B/A$. Since the exponential function is a monotone function, there is at most one value of t (when $B/A < 0$) for which this equation can be satisfied. Hence u can vanish at most once. If the system is critically damped, the general solution is

$u(t) = (A + Bt)e^{-\gamma t/2m}$. The exponential function is never zero; hence u can vanish only if $A + Bt = 0$. If $B = 0$ then u never vanishes; if $B \neq 0$ then u vanishes once at $t = -A/B$ provided $A/B < 0$.

20. The general solution of Eq.(21) for the case of critical damping is $u = (A + Bt)e^{-\gamma t/2m}$. The I.C. $u(0) = u_0 \Rightarrow$ $A = u_0$ and $u'(0) = v_0 \Rightarrow A(-\gamma/2m) + B = v_0$. Hence $u = [u_0 + (v_o + \gamma u_0/2m)t]e^{-\gamma t/2m}$. If $v_0 = 0$, then $u = u_0(1 + \gamma t/2m)e^{-\gamma t/2m}$, which is never zero since γ and m are postive. By L'Hospital's Rule $u \to 0$ as $t \to \infty$. Finally for $u_0 > 0$, we want the condition which will insure that $v = 0$ at least once. Since the exponential function is never zero we require $u_0 + (v_0 + \gamma u_0/2m)t = 0$ at a positive value of t. This requires that $v_0 + \gamma u_0/2m \neq 0$ and that $t = -u_0(v_0 + u_0\gamma/2m)^{-1} > 0$. We know that $u_0 > 0$ so we must have $v_0 + \gamma u_0/2m < 0$ or $v_0 < -\gamma u_0/2m$.

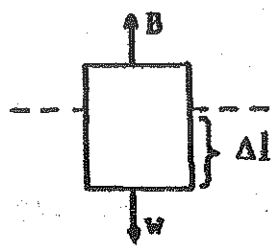
23. From Prob. 21: $\Delta = \dfrac{2\pi\gamma}{\mu(2m)} = T_d\gamma/2m$. Substituting the given values (and $m = 1/4$ from Prob.17) we find $\gamma = \dfrac{(1/2)\ (3)}{.3} = 5$ lb sec/ft.

24. From Eq.(13) $\omega_0^2 = \dfrac{2k}{3}$ so $P = 2\pi/\sqrt{2k/3} = \pi \Rightarrow k = 6$. Thus $u(t) = c_1\cos 2t + c_2\sin 2t$ and $u(0) = 2 \Rightarrow c_1 = 2$ and $u'(0) = v \Rightarrow c_2 = v/2$. Hence

$$u(t) = 2\cos 2t + \frac{v}{2}\sin 2t = \sqrt{4+\frac{v^2}{4}}\cos(2t-\gamma).$$

Thus $\sqrt{4+\dfrac{v^2}{4}} = 3$ and $v = \pm 2\sqrt{5}$.

27. First, consider the static case (which is the equilibrium position). Let Δl denote the length of the block below the surface of the water. The weight of the block, which is a downward force, is $w = \rho l^3 g$. This is balanced by an equal and opposite buoyancy force B, which is equal to the weight of the



displaced water. Thus $B = (\rho_0 l^2 \Delta l)g = \rho l^3 g$. Now let $u(t)$ be the displacement of the block from its equilibrium position. We take downward as the positive direction. In a displaced position the forces acting on the block are its weight, which acts downward and is unchanged, and the buoyancy force which is now $\rho_0 l^2(\Delta l + u)g$ and acts upward. The resultant force must be equal to the mass of the block times the acceleration, namely $\rho l^3 u''$. Hence $\rho l^3 g - \rho_0 l^2(\Delta l + u)g = \rho l^3 u''$. Hence the D.E. for the motion of the block is $\rho l^3 u'' + \rho_0 l^2 g u = 0$ or $u'' + \dfrac{\rho_0 g}{\rho l}u = 0$. This gives a simple harmonic motion with frequency $(\rho_0 g/\rho l)^{1/2}$ and natural period $T = 2\pi(\rho l/\rho_0 g)^{1/2}$.

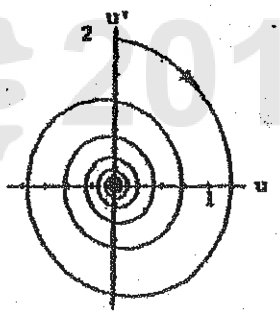
29a. The characteristic equation is $4r^2 + r + 8 = 0$, so $r = (-1\pm\sqrt{127})/8$ and hence

$$u(t) = e^{-t/8}\left(c_1\cos\frac{\sqrt{127}}{8}t + c_2\sin\frac{\sqrt{127}}{8}t\right).\ u(0) = 0 \Rightarrow c_1 = 0$$

and $u'(0) = 2 \Rightarrow \dfrac{\sqrt{127}}{8}c_2 = 2$. Thus

$$u(t) = \frac{16}{\sqrt{127}}e^{-t/8}\sin\frac{\sqrt{127}}{8}t.$$

29c. The phase plot is the spiral shown and the direction of motion is clockwise since the graph starts at (0,2) and u increases initially.



30a. The kinetic energy of a mass is given by $\frac{1}{2}mv^2$, so at $t = 0$ we have $v = u'(0) = b$ and thus $\frac{1}{2}mb^2$ is the initial kenetic energy. The work done deforming a spring an amount y from its undeformed state is stored in the spring and is known as the elastic potential energy. For our example, then, the potential energy is given by $\int_0^x F dy = \int_0^x ky dy = \frac{1}{2}kx^2$. For $x = u(0) = a$, this becomes $\frac{1}{2}ka^2$, as the initial potential energy.

30c. From part (a), the total energy in the system is $ku^2/2 + m(u')^2/2$. Using $u(t)$ as found in part(b), calculate u' and show that $ku^2/2 + m(u')^2/2 = (ka^2 + mb^2)/2$ for all t. This confirms the principle of conservation of energy when there is no damping.

Section 3.8, Page 215

1. We use the trigonometric identities

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

to obtain $\cos(A + B) - \cos(A - B) = -2\sin A \sin B$. If we choose $A + B = 9t$ and $A - B = 7t$, then $A = 8t$ and $B = t$. Substituting in the formula just derived, we obtain $\cos 9t - \cos 7t = -2\sin 8t \sin t$.

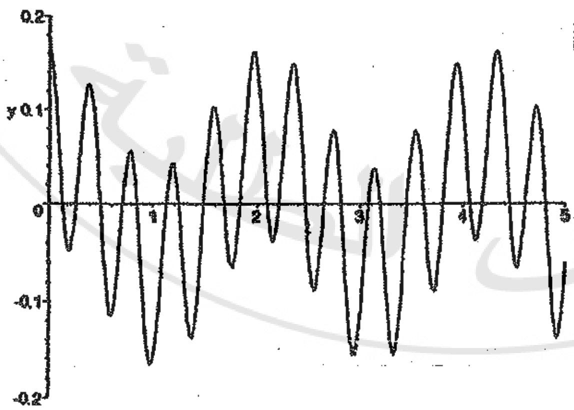
5. The mass $m = 4/32 = 1/8$ lb-sec^2/ft and the spring constant $k = 4/(1/8) = 32$ lb/ft. Since there is no damping, the I.V.P. is $(1/8)u'' + 32u = 2\cos 3t$, $u(0) = 1/6$, $u'(0) = 0$ where u is measured in ft and t in sec.

7a. From Prob. 5, we have $m = 1/8$, $F_0 = 2$, $\omega_0^2 = 256$, and $\omega^2 = 9$, so Eq.(18) becomes

$u = c_1\cos 16t + c_2\sin 16t + \dfrac{16}{247}\cos 3t$. The I.C.

$u(0) = 1/6 \Rightarrow c_1 + 16/247 = 1/6$ and $u'(0) = 0 \Rightarrow 16c_2 = 0$, and thus $u = (151/1482)\cos 16t + (16/247)\cos 3t$ ft.

7b.



7c. Resonance occurs when the frequency ω of the forcing function $4\sin\omega t$ is the same as the natural frequency ω_0 of the system. Since $\omega_0 = 16$, the system will resonate when $\omega = 16$ rad/sec.

10. The I.V.P. is $.25u'' + 16u = 8\sin 8t$, or $u'' + 64u = 32\sin 8t$, $u(0) = \frac{1}{4}$ and $u'(0) = 0$. Since $u_c = c_1\cos 8t + c_2\sin 8t$, we assume the form $U(t) = t(A\cos 8t + B\sin 8t)$ and resonance occurs. Substituting U into the D.E. we find $U'' + 64U = -16A\sin 8t + 16B\cos 8t$ and thus $A = -2$, $B = 0$. Therefore $u(t) = c_1\cos 8t + c_2\sin 8t - 2t\cos 8t$. The I.C. yield $c_1 = 1/4$ and $8c_2 - 2 = 0$ and hence $u(t) = (\cos 8t + \sin 8t - 8t\cos 8t)/4$. The velocity will be zero when $u'(t) = 8\sin 8t(8t-1) = 0$. Note that $8t-1 = 0$ gives the first zero and $\sin 8t$ gives all the others.

11a. For this problem the mass $m = 8/32$ lb-sec^2/ft and the spring constant $k = 8/(1/2) = 16$ lb/ft, so the D.E. is $0.25u'' + 0.25u' + 16u = 4\cos 2t$ where u is measured in ft and t in sec. To determine the steady state response we need only compute a particular solution of the nonhomogeneous D.E. since the solutions of the homogeneous D.E. decay to zero as $t \to \infty$. We assume $u(t) = A\cos 2t + B\sin 2t$, and substitute in the D.E.: $- A\cos 2t - B\sin 2t + (1/2)(-A\sin 2t + B\cos 2t) + 16(A\cos 2t + B\sin 2t) = 4\cos 2t$. Hence $15A + (1/2)B = 4$ and $-(1/2)A + 15B = 0$, from which we obtain $A = 240/901$ and $B = 8/901$. Thus $U(t) = (240\cos 2t + 8\sin 2t)/901$.

11b. In order to determine the value of m that maximizes the steady state response, we note that the present problem has exactly the form of Eq.(8) considered in the text. Referring to Eqs.(11) and (12), the magnitude of the response, R, is a maximum when Δ is a minimum since F_0 is constant. Δ, as given in Eq.(12), will be a minimum when $f(m) = m^2(\omega_0{}^2 - \omega^2)^2 + \gamma^2\omega^2$, where $\omega_0{}^2 = k/m$, is a minimum. We calculate df/dm and set this quantity equal to zero to obtain $m = k/\omega^2$. We verify that this value of m gives a minimum of f(m) by the second derivative test. For this problem $k = 16$ lb/ft and $\omega = 2$ rad/sec so the value of m that maximizes the response of the system is $m = 4$ slugs.

14. Since $U(t) = R\cos(\omega t-\delta)$ we have $U'(t) = \frac{-F_0\omega}{\Delta}\sin(\omega t-\delta)$, where Δ is given by Eq.(12). Since F_0 is a constant, differentiate $\frac{\omega}{\Delta}$ with respect to ω and set it equal to zero. Alternatively, you can minimize $(\Delta/\omega)^2$, which simplifies the differentiation.

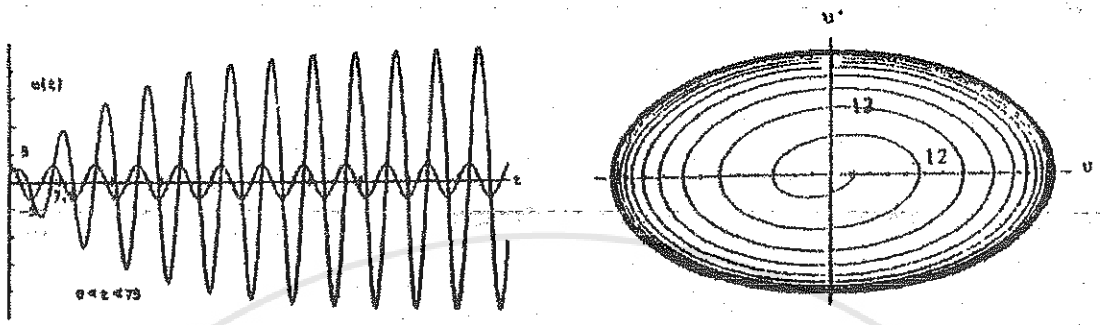
15. We must solve the three I.V.P.: (1) $u_1'' + u_1 = F_0t$, $0 < t < \pi$, $u_1(0) = u_1'(0) = 0$; (2) $u_2'' + u_2 = F_0(2\pi - t)$, $\pi < t < 2\pi$, $u_2(\pi) = u_1(\pi)$, $u_2'(\pi) = u_1'(\pi)$; and (3) $u_3'' + u_3 = 0$, $2\pi < t$, $u_3(2\pi) = u_2(2\pi)$, $u_3'(2\pi) = u_2'(2\pi)$. The conditions at π and 2π insure the continuity of u and u' at those points. The general solutions of the D.E. are $u_1 = b_1\cos t + b_2\sin t + F_0t$, $u_2 = c_1\cos t + c_2\sin t + F_0(2\pi - t)$, and $u_3 = d_1\cos t + d_2\sin t$. The I.C. and matching conditions, in order, give $b_1 = 0$, $b_2 + F_0 = 0$, $-b_1 + \pi F_0 = -c_1 + \pi F_0$, $-b_2 + F_0 = -c_2 - F_0$, $c_1 = d_1$, and $c_2 - F_0 = d_2$. [Note that the form of the particular solution for the first two cases is not the most general, but they do yield a solution]. Solving these equations we obtain

$$u = F_0 \begin{cases} t - \sin t & , 0 \le t \le \pi \\ (2\pi - t) - 3\sin t & , \pi < t \le 2\pi \\ -4\sin t & , 2\pi < t. \end{cases}$$

16. From Eq.(33) of Sect. 3.7, the I.V.P. is $Q'' + 5\text{x}10^3 Q' + 4\text{x}10^6 Q = 12$, $Q(0) = 0$, and $Q'(0) = 0$. The particular solution is of the form $Q = A$, so that upon substitution into the D.E. we obtain $4\text{x}10^6 A = 12$ or $A = 3\text{x}10^{-6}$. The general solution of the D.E. is $Q = c_1e^{r_1t} + c_2e^{r_2t} + 3\text{x}10^{-6}$, where r_1 and r_2 satisfy $r^2 + 5\text{x}10^3 r + 4\text{x}10^6 = 0$ and thus $r_1 = -1000$ and $r_2 = -4000$. The I.C. yield $c_1 = -4\text{x}10^{-6}$ and $c_2 = 10^{-6}$ and thus $Q = 10^{-6}(e^{-4000t} - 4e^{-1000t} + 3)$ coulombs. Substituting $t = .001$ sec we obtain $Q(.001) = 10^{-6}(e^{-4} - 4e^{-1} + 3) = 1.5468 \times 10^{-6}$ coulombs. Since the exponentials are to a negative power $Q(t) \rightarrow 3\text{x}10^{-6}$ coulombs as $t \rightarrow \infty$, which is the steady state charge.

22. The steady-state response is $12\sin 2t$ and thus the amplitude of the steady state response is four times the amplitude of the forcing term. This large an increase is due to the fact that the forcing function has the same frequency as the natural frequency, $\omega_0 = 2$, of the system. The graph also shows a phase lag of approximately 1/4 of a period. That is, the maximum of the response occurs 1/4 of a period after the maximum of the forcing

function. Both these results are substantially different than those of either Probs. 21 or 23.



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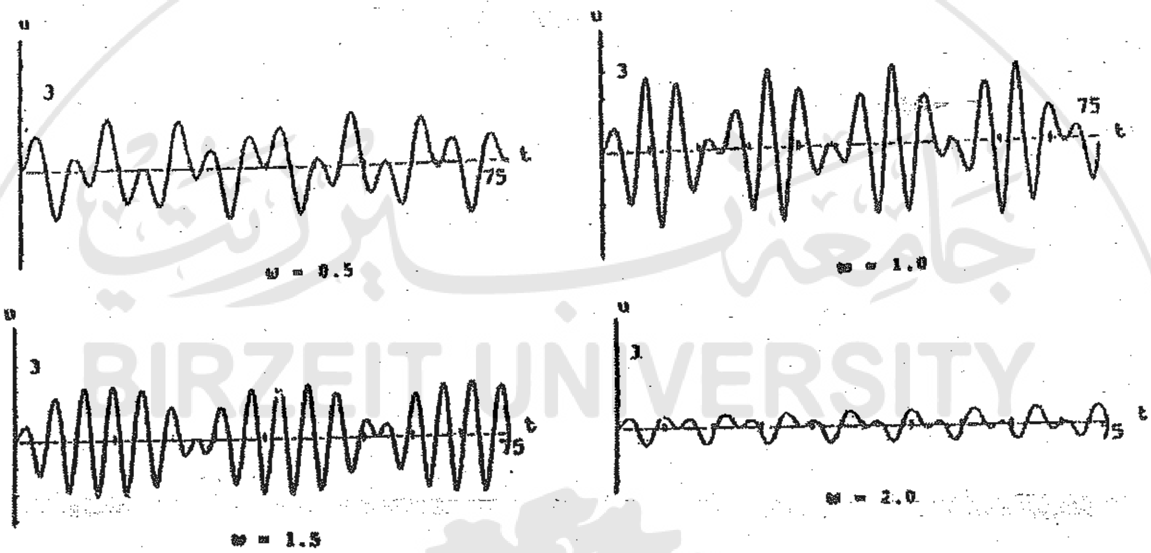

From viewing the above graphs, it appears that the system exhibits a beat near ω = 1.5, while the pattern for ω = 1.0 is more irregular. However, the system exhibits the resonance characteristic of the linear system for ω near 1, as the amplitude of the response is the largest here.

CHAPTER 4

Section 4.1, Page 224

2. Writing the equation in standard form of Eq.(2), we obtain $y''' + [(\sin t)/t]y'' + (3/t)y = \cos t/t$. The functions $p_1(t) = \sin t/t$, $p_3(t) = 3/t$ and $g(t) = \cos t/t$ each have a discontinuity at $t = 0$. Hence Theorem 4.1.1 guarantees that a solution exists for $t < 0$ or for $t > 0$.

4. The equation is in standard form with $p_1(t)$, $p_2(t)$ and $p_3(t)$ being continuous for all t. However, $g(t) = \ln t$ is defined and continuous only for $t > 0$.

8. We have $W(f_1,f_2,f_3) = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix} = 0$ for all t.
Thus by the extension of Theorem 3.3.1 (or by the discussion following Eq.(8), the given functions are linearly dependent. To find a linear relation we have $c_1(2t-3) + c_2(2t^2+1) + c_3(3t^2+t) =$ $(2c_2+3c_3)t^2 + (2c_1+c_3)t + (-3c_1+c_2)$ which is zero when $(2c_2+3c_3) = 0$, $2c_1+c_3 = 0$ and $-3c_1+c_2 = 0$. Solving, we find $c_1 = 1$, $c_2 = 3$ and $c_3 = -2$ and hence $(2t-3)+3(2t^2+1)-2(3t^2+t) = 0$.

13. That e^t, e^{-t}, and e^{-2t} are solutions can be verified by direct substitution. For $y = e^{-2t}$, $y' = -2e^{-2t}$, $y'' = 4e^{-2t}$, and $y''' = -8e^{-2t}$ and thus $(-8e^{-2t}) + 2(4e^{-2t}) - (-2e^{-2t}) - 2(e^{-2t}) = 0$, verifying that e^{-2t} is a solution. Computing the Wronskian we obtain,

$$W(e^t,e^{-t},e^{-2t}) = \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} = e^{-2t}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{-2t}$$

17. To show that the given Wronskian is zero, it is helpful, in evaluating the Wronskian, to note that $(\sin^2 t)' = 2\sin t\cos t = \sin 2t$.
The result can also be obtained directly since $\sin^2 t = (1 - \cos 2t)/2 = \frac{1}{10}(5) + (-1/2)\cos 2t$ and hence

$\sin^2 t$ is a linear combination of 5 and cos2t. Thus the functions are linearly dependent and their Wronskian is zero.

19a. If $y = t^n$ then $y' = nt^{n-1}$, $y'' = n(n-1)t^{n-2}$, $\cdots$ $y^{n-1} = [n(n-1)(n-2)\cdots 2]t$, so $y^{(n)} = [n(n-1)(n-2)\cdots(2)(1)]$.

19b. The nth derivative of e^{rt} is $y^{(n)} = r^n e^{rt}$.

19c. If we let $L[y] = y^{(4)} - 5y'' + 4y$ and if we use the result of Prob. 19b, we have $L[e^{rt}] = (r^4 - 5r^2 + 4)e^{rt}$. Thus e^{rt} will be a solution of the D.E. provided $(r^2-4)(r^2-1) = 0$. Solving for r, we obtain the four solutions e^t, e^{-t}, e^{2t} and e^{-2t}. Since $W(e^t, e^{-t}, e^{2t}, e^{-2t}) \neq 0$, the four functions form a fundamental set of solutions.

23. Writing the equation in the form of Eq.(2), we have $p_1(t) = \dfrac{2}{t}$ and from Prob. 20, $W = ce^{-\int \frac{2dt}{t}} = \dfrac{c}{t^2}$.

25a. On $0 < t < 1$, $f(t) = t^3$ and $g(t) = t^3$. Hence there are nonzero constants, $c_1 = 1$ and $c_2 = -1$, such that $c_1 f(t) + c_2 g(t) = 0$ for each t in (0,1). On $-1 < t < 0$, $f(t) = -t^3$ and $g(t) = t^3$; thus $c_1 = c_2 = 1$ defines constants such that $c_1 f(t) + c_2 g(t) = 0$ for each t in (-1,0). Thus f and g are linearly dependent on $0 < t < 1$ and on $-1 < t < 0$.

25b. We will show that f(t) and g(t) are linearly independent on $-1 < t < 1$ by demonstrating that it is impossible to find constants c_1 and c_2, not both zero, such that $c_1 f(t) + c_2 g(t) = 0$ for all t in (-1,1). Assume that there are two such nonzero constants and choose two points t_0 and t_1 in $-1 < t < 1$ such that $t_0 < 0$ and $t_1 > 0$. Then $-c_1 t_0^3 + c_2 t_0^3 = 0$ and $c_1 t_1^3 + c_2 t_1^3 = 0$. These equations have a nontrivial solution for c_1 and c_2 only if the determinant of coefficients is zero. But the determinant of coefficients is $-2t_0^3 t_1^3 \neq 0$ for t_0 and t_1 as specified. Hence f(t) and g(t) are linearly independent on $-1 < t < 1$.

25c. For $-1 < t < 0$, $W(f,g) = \begin{vmatrix} -t^3 & t^3 \\ -3t^2 & 3t^2 \end{vmatrix} = 0$ and for $0 \le t < 1$,

$W(f,g) = \begin{vmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{vmatrix} = 0$. This shows that f and g cannot be solutions of an equation $y'' + p(t)y' + q(t)y = 0$ with p and q continuous on $-1 < t < 1$.

27. Differentiating e^t and substituting in the D.E. we verify that $y = e^t$ is a solution: $(2-t)e^t + (2t-3)e^t - te^t + e^t = 0$. Now, as in Prob. 26, we let $y = v(t)e^t$. Differentiating three times and substituting into the D.E. yields $(2-t)e^t v''' + (3-t)e^t v'' = 0$. Dividing by $(2-t)e^t$ and letting $w = v''$ we obtain the first order separable equation $w' = -\frac{t-3}{t-2}w = (-1 + \frac{1}{t-2})w$. Separating t and w, integrating, and then solving for w yields $w = v'' = c_1(t-2)e^{-t}$. Integrating this twice then gives $v = c_1te^{-t} + c_2t + c_3$ so that $y = ve^t = c_1t + c_2te^t + c_3e^t$, which is the complete solution, since it contains the given $y_1(t)$ and three constants.

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2. If $-1 + i\sqrt{3} = R(\cos\theta + i\sin\theta) = Re^{i\theta}$, then $R\cos\theta = -1$ and $R\sin\theta = \sqrt{3}$. Thus $R^2 = (-1)^2 + (\sqrt{3})^2 = 4$ and the angle θ is given by $R\cos\theta = 2\cos\theta = -1$ and $R\sin\theta = 2\sin\theta = \sqrt{3}$. Hence $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$ which has the solution $\theta = 2\pi/3$. The angle θ is only determined up to an additive integer multiple of $\pm 2\pi$.

8. Writing $(1-i)$ in the form $Re^{i\theta}$, we have $R\cos\theta = 1$ and $R\sin\theta = -1$, which yield $R = \sqrt{2}$ and $\theta = -\pi/4$. Thus $(1-i) = \sqrt{2}e^{i(-\pi/4+2m\pi)}$ (where m is any integer) and hence $(1-i)^{1/2} = [2^{1/2}e^{i(-\pi/4+2m\pi)}]^{1/2} = 2^{1/4}e^{i(-\pi/8+m\pi)}$. We obtain the two square roots by setting $m = 0,1$. They are $2^{1/4}e^{-i\pi/8}$ and $2^{1/4}e^{i7\pi/8}$. Note that any other integer value of m gives one of these two values. Also note that $1-i$ could be written as $1-i = \sqrt{2}e^{i(7\pi/4 + 2m\pi)}$.

12. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E., we obtain the characteristic equation $r^3 - 3r^2 + 3r - 1 = 0$ which has roots $r = 1,1,1$. Since the roots are repeated, the general solution is $y = c_1e^t + c_2te^t + c_3t^2e^t$.

15. We look for solutions of the form $y = e^{rt}$. Substituting in the D.E. we obtain the characteristic equation $r^6 + 1 = 0$. The six roots of $-1 = e^{i\pi}$ are obtained by setting $m = 0,1,2,3,4,5$ in $(-1)^{1/6} = e^{i(\pi+2m\pi)/6}$. They are $e^{i\pi/6} = (\sqrt{3} + i)/2$, $e^{i\pi/2} = i$, $e^{i5\pi/6} = (-\sqrt{3} + i)/2$, $e^{i7\pi/6} = (-\sqrt{3} - i)/2$, $e^{i3\pi/2} = -i$, and $e^{i11\pi/6} = (\sqrt{3} - i)/2$. Note that there are three pairs of conjugate roots. The general solution is

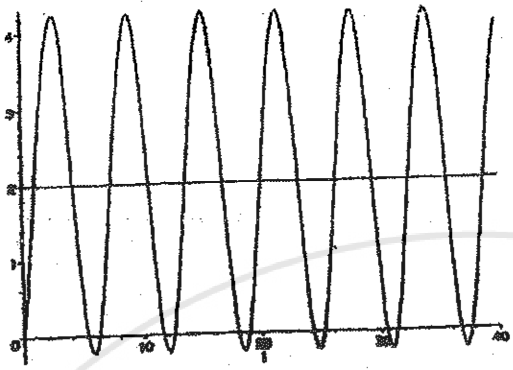
$$y = e^{\sqrt{3}t/2}[c_1\cos(t/2) + c_2\sin(t/2)] + e^{-\sqrt{3}t/2}[c_3\cos(t/2) + c_4\sin(t/2)] + c_5\cos t + c_6\sin t.$$

23. The characteristic equation is $r^3 - 5r^2 + 3r + 1 = 0$. Using the procedure suggested following Eq. (12) we try, since $a_0 = a_n = 1$, $r = 1$ as a root and find that indeed it is. Factoring out $(r-1)$ we are then left with $r^2 - 4r - 1 = 0$, which has the roots $2 \pm \sqrt{5}$.

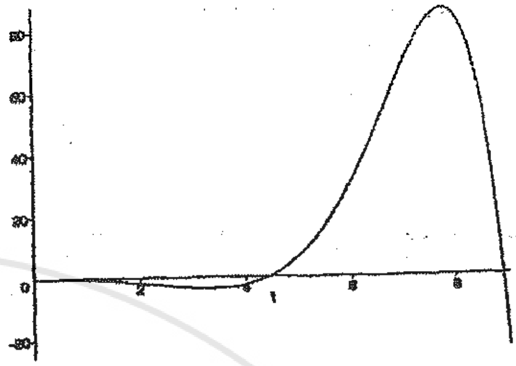
27. The characteristic equation in this case is $12r^4 + 31r^3 + 75r^2 + 37r + 5 = 0$. Using an equation solver we find $r = -\frac{1}{4}, -\frac{1}{3}, -1 \pm 2i$. Thus $y = c_1e^{-t/4} + c_2e^{-t/3} + e^{-t}(c_3\cos 2t + c_4\sin 2t)$. As in Prob. 23, it is possible to find the first two of these roots without using an equation solver. Factoring then reduces the characteristic equation to a quadratic, which can be solved for the other two roots.

29. The characteristic equation is $r^3 + r = 0$ and hence $r = 0, +i, -i$ are the roots and the general solution is $y(t) = c_1 + c_2\cos t + c_3\sin t$. $y(0) = 0$ implies $c_1 + c_2 = 0$, $y'(0) = 1$ implies $c_3 = 1$ and $y''(0) = 2$ implies $-c_2 = 2$. Use this last equation in the first to find $c_1 = 2$ and thus $y(t) = 2 - 2\cos t + \sin t$, which continues to oscillate about $y = 2$ as $t \to \infty$.

29.



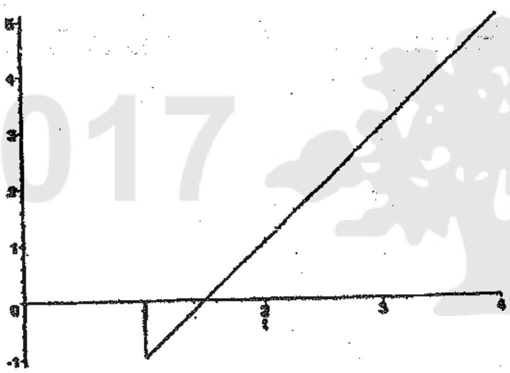
30.



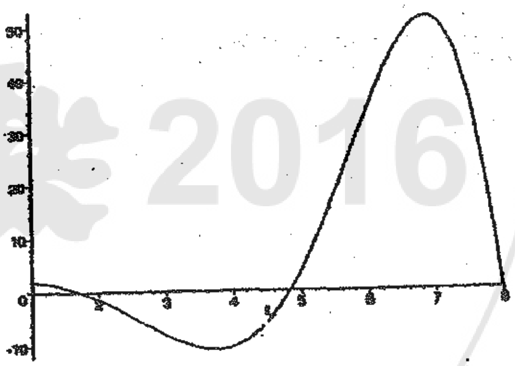
30. The general solution is given by Eq. (21).

31. The characteristic equation is $r^4 - 4r^3 + 4r^2 = 0$, which has the roots $r = 0,0,2,2$. Thus the general solution would normally be written $y(t) = c_1 + c_2t + c_3e^{2t} + c_4te^{2t}$. However, in order to evaluate the c's when the initial conditions are given at $t = 1$, it is advantageous to rewrite this as $y(t) = c_1 + c_2t + c_5e^{2(t-1)} + c_6(t-1)e^{2(t-1)}$, which also satisfies the given D.E.

31.



34.



34. The characteristic equation is $4r^3 + r + 5 = 0$, which has roots $-1, \frac{1}{2} \pm i$, where $r_1 = -1$ can be found as in Prob.23. Thus $y(t) = c_1e^{-t} + e^{t/2}(c_2\cos t + c_3\sin t)$,
$y'(t) = -c_1e^{-t} + e^{t/2}[(c_2/2 + c_3)\cos t + (-c_2 + c_3/2)\sin t]$
and
$y''(t) = c_1e^{-t} + e^{t/2}[(-3c_2/4 + c_3)\cos t + (-c_2 - 3c_3/4)\sin t]$.
The I.C. then yield $c_1 + c_2 = 2$, $-c_1 + c_2/2 + c_3 = 1$ and $c_1 - 3c_2/4 + c_3 = -1$. Solving these last three equations give $c_1 = 2/13$, $c_2 = 24/13$ and $c_3 = 3/13$.

37. The approach for solving the D.E. would normally yield $y(t) = c_1\cos t + c_2\sin t + c_5e^t + c_6e^{-t}$ as the solution.

Since $\cosh t = (e^t+e^{-t})/2$ and $\sinh t = (e^t-e^{-t})/2$, $y(t)$ can be written as $y(t) = c_1\cos t + c_2\sin t + c_3\cosh t + c_4\sinh t$, where c_3 and c_4 can be written in terms of c_5 and c_6. It is convenient to use cosht and sinht rather than e^t and e^{-t} because the I.C. are given at $t = 0$. Since cosht and sinht and all of their derivatives are either 0 or 1 at $t = 0$, the algebra in satisfying the I.C. is greatly simplified.

38a. Since $p_1(t) = 0$, $W = ce^{-\int 0dt} = c$.

39a. As in Sect.3.7, the force that the spring designated by k_1 exerts on mass m_1 is $-3u_1$. By an analysis similar to that shown in Sect.3.7, the middle spring exerts a force of $-2(u_1-u_2)$ on mass m_1 and a force of $-2(u_2-u_1)$ on mass m_2. Thus Newton's Law gives $m_1u_1'' = -3u_1 - 2(u_1-u_2)$ and $m_2u_2'' = -2(u_2-u_1)$, where u_1 and u_2 are measured from their equilibrium positions. Setting the masses equal to 1 and rewriting each equation yields Eqs.(i). In all cases the positive direction is taken in the direction shown in Figure 4.2.4.

39b. The characteristic equation for (ii) is $r^4 + 7r^2 + 6 = 0$, or $(r^2+1)(r^2+6) = 0$. Thus the general solution of Eq.(ii) is $u_1(t) = c_1\cos t + c_2\sin t + c_3\cos\sqrt{6}\,t + c_4\sin\sqrt{6}\,t$.

39c. From Eq.(iii) and u_1 from 39b we have $u_1(0) = c_1 + c_3 = 1$ and $u_1'(0) = c_2 + \sqrt{6}\,c_4 = 0$. From Eq.(i) we have $u_1''(0) = 2u_2(0) - 5u_1(0) = -1$ and $u_1'''(0) = 2u_2'(0) - 5u_1'(0) = 0$, hence $-c_1 - 6c_3 = -1$ and $-c_2 - 6\sqrt{6}\,c_4 = 0$. Solving the four equations for the c_i we find $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$, so that $u_1 = \cos t$. The first of Eqs.(i) then gives $2u_2 = u_1'' + 5u_1 = 4\cos t$ and thus $u_2 = 2\cos t$.

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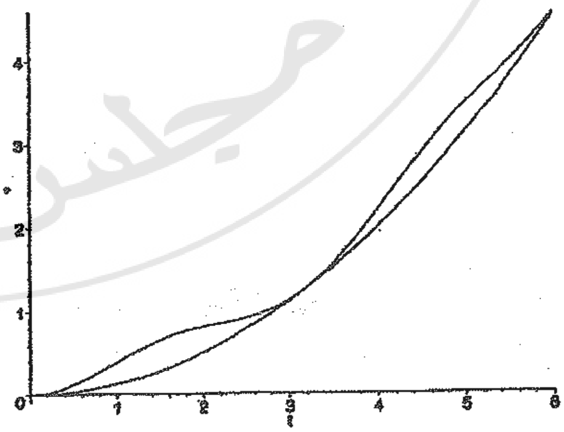
1. First solve the homogeneous D.E. The characteristic equation is $r^3 - r^2 - r + 1 = 0$, and the roots are $r = -1, 1, 1$; hence $y_c(t) = c_1e^{-t} + c_2e^t + c_3te^t$. Using the

superposition principle, we can write a particular solution as the sum of particular solutions corresponding to the D.E. $y'''-y''-y'+y = 2e^{-t}$ and $y'''-y''-y'+y = 3$. Our initial choice for a particular solution, Y_1, of the first equation is Ae^{-t}; but e^{-t} is a solution of the homogeneous equation so we multiply by t. Thus, $Y_1(t) = Ate^{-t}$. For the second equation we choose $Y_2(t) = B$, and there is no need to modify this choice. The constants are determined by substituting into the individual equations. We obtain A = 1/2, B = 3. Thus, the general solution is

$y = c_1e^{-t} + c_2e^t + c_3te^t + (te^{-t})/2 + 3.$

5. The characteristic equation is $r^4 - 4r^2 = r^2(r^2-4) = 0$, so $y_c(t) = c_1 + c_2t + c_3e^{-2t} + c_4e^{2t}$. For the particular solution correspnding to t^2 we assume $Y_1 = t^2(At^2 + Bt + C)$ and for the particular solution corresponding to e^t we assume $Y_2 = De^t$. Substituting Y_1, in the D.E. yields $-48A = 1$, $B = 0$ and $24A-8C = 0$ and substituting Y_2 yields $-3D = 1$. Solving for A, B, C and D gives the desired solution.

9. The characteristic equation for the related homogeneous D.E. is $r^3 + 4r = 0$ with roots $r = 0, +2i, -2i$. Hence $Y_c(t) = c_1 + c_2\cos 2t + c_3\sin 2t$. The initial choice for $Y(t)$ is $At + B$, but since B is a solution of the homogeneous equation we must multiply by t and assume $Y(t) = t(At+B)$. A and B are found by substituting in the D.E., which gives $A = 1/8$, $B = 0$, and thus the general solution is $y(t)=c_1+c_2\cos 2t+c_3\sin 2t+(1/8)t^2$. Applying the I.C. we have $y(0) = 0 \Rightarrow c_1 + c_2 = 0$, $y'(0) = 0 \Rightarrow 2c_3 = 0$, and $y''(0) = 1 \Rightarrow -4c_2 + 1/4 = 1$, which have the solution



$c_1 = 3/16$, $c_2 = -3/16$, $c_3 = 0$. For $t = \pi, 2\pi\ldots$ the graph will be tangent to $t^2/8$ and for large t the graph will be approximated by $t^2/8$.

13. The characteristic equation for the homogeneous D.E. is $r^3 - 2r^2 + r = 0$ with roots $r = 0,1,1$. Hence the complementary solution is $y_c(t) = c_1 + c_2e^t + c_3te^t$. We consider the differential equations $y''' - 2y'' + y' = t^3$ and $y''' - 2y'' + y' = 2e^t$ separately. Our initial choice for a particular solution, Y_1, of the first equation is $A_0t^3 + A_1t^2 + A_2t + A_3$; but since a constant is a solution of the homogeneous equation we must multiply by t. Thus $Y_1(t) = t(A_0t^3 + A_1t^2 + A_2t + A_3)$. For the second equation we first choose $Y_2(t) = Be^t$, but since both e^t and te^t are solutions of the homogeneous equation, we multiply by t^2 to obtain $Y_2(t) = Bt^2e^t$. Then $Y(t) = Y_1(t) + Y_2(t)$ by the superposition principle and $y(t) = y_c(t) + Y(t)$.

17. The characteristic equation is $r^4 - r^3 - r^2 + r = r(r-1)(r^2-1) = 0$, so the complementary solution is $y_c(t) = c_1 + c_2e^{-t} + c_3e^t + c_4te^t$. The superposition principle allows us to consider separately the D.E. $y^{(4)} - y''' - y'' +y' = t^2 + 4$ and $y^{(4)}s - y''' - y'' + y' = t\sin t$. For the first equation our initial choice is $Y_1(t) = A_0t^2 + A_1t + A_2$; but this must be multiplied by t since a constant is a solution of the homogeneous D.E. Hence $Y_1(t) = t(A_0t^2 + A_1t + A_2)$. For the second equation our initial choice that $Y_2 = (B_0t + B_1)\cos t + + (C_0t + C_1)\sin t$ does not need to be modified. Hence

$$Y(t) = t(A_0t^2 + A_1t + A_2) + (B_0t + B_1)\cos t + (C_0t + C_1)\sin t.$$

20. $(D-a)(D-b)f = (D-a)(Df-bf) = D^2f - (a+b)Df + abf$ and $(D-b)(D-a)f = (D-b)(Df-af) = D^2f - (b+a)Df + baf$. Since $a+b = b+a$ and $ab = ba$, we find the given equation holds for any function f.

22a. The D.E. of Prob.13 can be written as $D(D-1)^2y = t^3 + 2e^t$. Since D^4 annihilates t^3 and $(D-1)$ annihilates $2e^t$, we have $D^5(D-1)^3y = 0$, which corresponds to Eq.(ii) of Prob. 21. The solution of this equation is $y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + A_5 + (B_1t^2 + B_2t + B_3)e^{-t}$. Since A_5 and $(B_2t + B_3)e^{-t}$ are solutions of the homogeneous equation related to the original D.E., they may be deleted and thus

$Y(t) = A_1t^4 + A_2t^3 + A_3t^2 + A_4t + B_1t^2e^{-t}$.

22b. If $y = te^{-t}$ then $Dy = -te^{-t} + e^{-t}$ and $D^2y = te^{-t} - 2e^{-t}$, so $(D+1)^2y = (D^2+2D+1)y = 0$ and thus $(D+1)^2$ annihilates te^{-t}. Likewise D^2-1 annihilates 2cost. Thus $(D+1)^2(D^2+1)$ annihilates the right side of the D.E. of Prob.14.

22e. $D^3(D^2+1)^2$ annihilates the right side of the D.E.of Prob.17.

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1. The complementary solution is $y_c = c_1 + c_2\cos t + c_3\sin t$ and thus we assume a particular solution of the form $Y = u_1(t) + u_2(t)\cos t + u_3(t)\sin t$. Differentiating and assuming Eq.(5), we obtain $Y' = -u_2\sin t + u_3\cos t$ and

$$u_1' + u_2'\cos t + u_3'\sin t = 0 \qquad (a).$$

Continuing this process we obtain $Y'' = -u_2\cos t - u_3\sin t$, and

$$-u_2'\sin t + u_3'\cos t = 0 \qquad (b)$$

and $Y''' = u_2\sin t - u_3\cos t - u_2'\cos t - u_3'\sin t$.

Substituting Y and its derivatives, as given above, into the D.E. we obtain the third equation:

$$-u_2'\cos t - u_3'\sin t = \tan t \qquad (c).$$

Equations (a), (b) and (c) constitute Eqs.(10) of the text for this problem and may be solved to give $u_1' = \tan t$, $u_2' = -\sin t$, and $u_3' = -\sin^2 t/\cos t$. Thus $u_1 = -\ln\cos t$, $u_2 = \cos t$ and $u_3 = \sin t - \ln(\sec t + \tan t)$ and substitution into Y above gives $Y = -\ln\cos t + 1 - (\sin t)\ln(\sec t + \tan t)$, since $\sin^2 t + \cos^2 t = 1$ Note that the constant 1 can be absorbed in c_1 in y_c above.

4. Replace tant in Eq.(c) of Prob. 1 by sect and use Eqs.(a) and (b) as in Prob.1 to obtain $u_1' = \sec t$, $u_2' = -1$ and $u_3' = -\sin t/\cos t$.

5. Replace sect in Prob. 7 with $e^{-t}\sin t$.

7. Since e^t, $\cos t$ and $\sin t$ are solutions of the related homogenous equation we have

$Y(t) = u_1e^t + u_2\cos t + u_3\sin t$. Eqs. (10) then are

$u_1'e^t + u_2'\cos t + u_3'\sin t = 0$

$u_1'e^t - u_2'\sin t + u_3'\cos t = 0$

$u_1'e^t - u_2'\cos t - u_3'\sin t = \sec t$.

Using Abel's identity, $W(t) = c\exp(-\int p_1(t)dt) = ce^t$.

Using the above equations, $W(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 2$, so

$c = 2$ and $W(t) = 2e^t$. From Eq.(11), we have

$$u_1'(t) = \frac{\sec t\, W_1(t)}{2e^t}, \text{ where } W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1$$

and thus

$u_1'(t) = \frac{1}{2}e^{-t}/\cos t$. Likewise

$$u_2' = \frac{\sec t\, W_2(t)}{2e^t} = -\frac{1}{2}\sec t(\cos t - \sin t) \text{ and}$$

$$u_3' = \frac{\sec t\, W_3(t)}{2e^t} = -\frac{1}{2}\sec t(\sin t + \cos t). \text{ Thus}$$

$$u_1 = \frac{1}{2}\int_{t_0}^{t}\frac{e^{-s}ds}{\cos(s)},\ u_2 = -\frac{1}{2}t - \frac{1}{2}\ln(\cos t) \text{ and } u_3 = -\frac{1}{2}t + \frac{1}{2}\ln(\cos t)$$

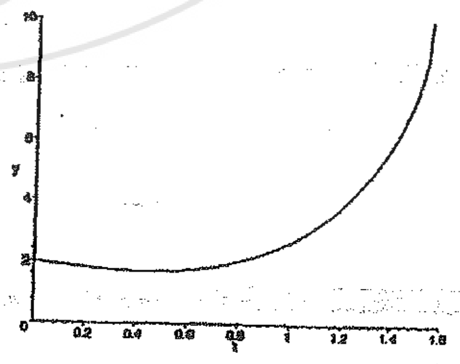
which, when substituted into the assumed form for Y, yields the desired solution.

11. Since the D.E. is the same as in Prob. 7, we may use the complete solution from that, with $t_0 = 0$. Thus

$y(0) = c_1 + c_2 = 2$, $y'(0) = c_1 + c_3 - \frac{1}{2} + \frac{1}{2} = -1$ and

$y''(0) = c_1 - c_2 + \frac{1}{2} - 1 + \frac{1}{2} = 1$.

A computer algebra system may be used to find the respective derivatives. Note that the solution is valid only for $0 \le t < \frac{\pi}{2}$, where we see the vertical asymptote.



14. Since a fundamental set of solutions of the homogeneous D.E. is $y_1 = e^t$, $y_2 = \cos t$, $y_3 = \sin t$, a particular solution is of the form
$Y(t) = e^t u_1(t) + (\cos t)u_2(t) + (\sin t)u_3(t)$. Differentiating and making the same assumptions that lead to Eqs.(10), we obtain

$$u_1'e^t + u_2'\cos t + u_3'\sin t = 0$$
$$u_1'e^t - u_2'\sin t + u_3'\cos t = 0$$
$$u_1'e^t - u_2'\cos t - u_3'\sin t = g(t)$$

Solving these equations using either determinants or by elimination, we obtain $u_1' = (1/2)e^{-t}g(t)$,
$u_2' = (1/2)(\sin t - \cos t)g(t)$, $u_3' = -(1/2)(\sin t + \cos t)g(t)$.
Integrating these and substituting into Y yields

$$Y(t) = \frac{1}{2}\{e^t\int_{t_0}^t e^{-s}g(s)ds + \cos t\int_{t_0}^t[\sin(s) - \cos(s)]g(s)ds$$
$$-\sin t\int_{t_0}^t[\sin(s) + \cos(s)]g(s)ds\}.$$

Putting e^t, $\cos t$ and $\sin t$ inside the respective integrals yields

$$Y(t) = (1/2)\int_{t_0}^t[e^{t-s} + \cos t\sin(s) - \cos t\cos(s) - \sin t\sin(s) - \sin t\cos(s)]g(s)ds.$$

If we use the trigonometric identities
$\sin(A-B) = \sin A\cos B - \cos A\sin B$ and
$\cos(A-B) = \cos A\cos B + \sin A\sin B$, we obtain the desired result. Note: Eqs.(11) and (12) of this section give the same result, but it is not recommended to memorize these equations.

16. The characteristic equation has the repeated roots $r = 1,1,1$ and thus the particular solution has the form $Y = e^t u_1(t) + te^t u_2(t) + t^2e^t u_3(t)$. Differentiating, making the same assumptions as in the earlier problems, and solving the three linear equations for u_1', u_2', and u_3' yields
$u_1' = (1/2)t^2e^{-t}g(t)$, $u_2' = -te^{-t}g(t)$ and $u_3' = (1/2)e^{-t}g(t)$.
Integrating and substituting into Y yields the desired solution. For instance

$$te^t u_2 = -te^t\int_{t_0}^t se^{-s}g(s)ds = -\frac{1}{2}\int_{t_0}^t 2tse^{(t-s)}g(s)ds, \text{ and}$$

likewise for u_1 and u_3. If $g(t) = t^{-2}e^t$ then $g(s) = e^s/s^2$, and thus $e^{(t-s)}g(s) = \dfrac{e^t}{s^2}$ and the integration with repect to s is accomplished using the power rule. Note that terms involving t_0 become part of the complimentary solution.

CHAPTER 5

Section 5.1, Page 249

2. Use the ratio test:

$$\lim_{n\to\infty} \frac{|(n+1)x^{n+1}/2^{n+1}|}{|nx^n/2^n|} = \lim_{n\to\infty} \frac{n+1}{n}\,\frac{1}{2}\,|x| = \frac{|x|}{2}.$$

Therefore the series converges absolutely for $|x| < 2$. For $x = 2$ and $x = -2$ the n^{th} term does not approach zero as $n \to \infty$ so the series diverge. Hence the radius of convergence is $\rho = 2$.

5. Use the ratio test:

$$\lim_{n\to\infty} \frac{|(2x+1)^{n+1}/(n+1)^2|}{|(2x+1)^n/n^2|} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}|2x+1| = |2x+1|.$$

Therefore the series converges absolutely for $|2x+1| < 1$, or $|x+1/2| < 1/2$. At $x = 0$ and $x = -1$ the series also converge absolutely. However, for $|x+1/2| > 1/2$ the series diverges by the ratio test. The radius of convergence is $\rho = 1/2$.

9. For this problem $f(x) = \sin x$, so $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{iv}(x) = \sin x \ldots$, and thus $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = -1, \ldots$. The even terms in the series will vanish and the odd terms will alternate in sign. We obtain $\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$.

Now, $$\lim_{n\to\infty} \frac{|(-1)^{n+1}x^{2n+3}/(2n+3)!|}{|(-1)^n x^{2n+1}/(2n+1)!|} = \lim_{n\to\infty} x^2 \frac{1}{(2n+3)(2n+2)} = 0,$$

so the series converges for all x and hence $\rho = \infty$.

12. For this problem $f(x) = x^2$. Hence $f'(x) = 2x$, $f''(x) = 2$, and $f^{(n)}(x) = 0$ for $n > 2$. Then $f(-1) = 1$, $f'(-1) = -2$, $f''(-1) = 2$ and $x^2 = 1 - 2(x+1) + 2(x+1)^2/2! = 1 - 2(x+1) + (x+1)^2$. Since the series terminates after a finite number of terms, it converges for all x. Thus $\rho = \infty$.

13. For this problem $f(x) = \ln x$. Hence $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 1\cdot 2/x^3, \ldots$, and $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 1\cdot 2, \ldots , f^{(n)}(1) = (-1)^{n+1}(n-1)!$ The Taylor series

is $\ln x = (x-1) - (x-1)^2/2 + (x-1)^3/3 - \ldots =$
$\sum_{n=1}^{\infty}(-1)^{n+1}(x-1)^n/n$. It follows from the ratio test that the series converges absolutely for $|x-1| < 1$. However, the series diverges at $x = 0$ so $\rho = 1$.

18. Writing the individual terms of y, we have
$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$, so
$y' = a_1 + 2a_2x + 3a_3x^2 + \ldots + (n+1)a_{n+1}x^n + \ldots$, and
$y'' = 2a_2 + 3\cdot2a_3x + 4\cdot3a_4x^2 + \ldots + (n+2)(n+1)a_{n+2}x^n + \ldots$.
If $y'' = y$, we then equate coefficients of like powers of x to obtain $2a_2 = a_0$, $3\cdot2a_3 = a_1$, $4\cdot3a_4 = a_2$, $\ldots$ $(n+2)(n+1)a_{n+2} = a_n$.
Thus $a_2 = \dfrac{a_0}{2}$, $a_3 = \dfrac{a_1}{6}$, $a_4 = \dfrac{a_2}{4\cdot3} = \dfrac{a_0}{4!}$, $\ldots$ $a_{n+2} = \dfrac{a_n}{(n+2)(n*1)}$.
These yield the desired results for $n = 0,1,2,3, \ldots$.

19. Set $m = n-1$ on the right hand side of the equation. Then $n = m+1$ and when $n = 1$, $m = 0$. Thus the right hand side becomes $\sum_{m=0}^{\infty}a_m(x-1)^{m+1}$, which is the same as the left hand side when m is replaced by n.

23. Multiplying each term of the first series by x yields
$x\sum_{n=1}^{\infty}na_nx^{n-1} = \sum_{n=1}^{\infty}na_nx^n = \sum_{n=0}^{\infty}na_nx^n$, where the last equality can be verified by writing out the first few terms (or noting that $na_n = 0$ for $n = 0$). Changing the index from k to n ($n=k$) in the second series then yields
$$\sum_{n=0}^{\infty}na_nx^n + \sum_{n=0}^{\infty}a_nx^n = \sum_{n=0}^{\infty}(n+1)a_nx^n.$$

25. $\sum_{m=2}^{\infty} m(m-1)a_mx^{m-2} + x\sum_{k=1}^{\infty} ka_kx^{k-1} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} ka_kx^k$
$= \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + na_n]x^n$. In the first case we have let $n = m - 2$ in the first summation and multiplied each term of the second summation by x. In the second case we have let $n = k$ and noted that for $n = 0$, $na_n = 0$.

28. If we shift the index of summation in the first sum by letting $m = n-1$, we have

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1}x^m.$$ Substituting this into the given equation and letting $m = n$ again, we obtain:

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0, \text{ or}$$

$$\sum_{n=0}^{\infty} [(n+1)a_{n+1} + 2a_n]x^n = 0.$$

Hence $a_{n+1} = -2a_n/(n+1)$ for $n = 0,1,2,3,\ldots$. Thus $a_1 = -2a_0$, $a_2 = -2a_1/2 = 2^2a_0/2$, $a_3 = -2a_2/3 = -2^3a_0/2\cdot3 = -2^3a_0/3!\ldots$ and $a_n = (-1)^n2^na_0/n!$. Notice that for $n = 0$ this formula reduces to a_0 so we can write

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n 2^n\, a_0 x^n/n! = a_0\sum_{n=0}^{\infty} (-2x)^n/n! = a_0e^{-2x}.$$

<u>Section 5.2, Page 259</u>

2a. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and since we must multiply y' by x in the D.E. we do not shift the index; and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$ Substituting in the D.E., we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$ In order to have the starting point the same in all three summations, we let $n = 0$ in the first and third terms to obtain the following

$$(2\cdot1\; a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0.$$

Thus $a_{n+2} = a_n/(n+2)$ for $n = 1,2,3,\ldots$. Note that the recurrence relation is also correct for $n = 0$.

2b. From the recurrence relation we have $a_2 = a_0/2$, $a_4 = a_2/4 = a_0/2\cdot4$, $a_6 = a_4/6 = a_0/2\cdot4\cdot6$, so $y_1 = 1 + x^2/2 + x^4/2\cdot4 + x^6/2\cdot4\cdot6 + \cdots$, and $a_3 = a_1/3$, $a_5 = a_3/5 = a_1/3\cdot5$, $a_7 = a_5/7 = a_1/3\cdot5\cdot7$, so $y_2 = (x + x^3/3 + x^5/3\cdot5 + x^7/3\cdot5\cdot7 + \cdots$.

2c. $W(y_1,y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and thus y_1,y_2 form a fundamental set of solutions.

2d. From Part(b) we see the even coefficients can be written as $a_{2m} = a_0/2^m\, m!$. For the odd coefficients notice that $a_3 = 2a_1/(2\cdot3) = 2a_1/3!$, that $a_5 = 2\cdot4a_1/(2\cdot3\cdot4\cdot5) = 2^2\cdot2a_1/5!$, and that $a_7 = 2\cdot4\cdot6a_1/(2\cdot3\cdot4\cdot5\cdot6\cdot7) = 2^3\cdot3!\ a_1/7!$. Likewise $a_9 = a_7/9 = 2^3\cdot3!\ a_1/(7!)9 = 2^3\cdot3!\ 8a_1/9! = 2^4\cdot4!\ a_1/9!$. Continuing we have $a_{2m+1} = 2^m m!\ a_1/(2m+1)!$.

Thus $y = a_0\sum_{m=0}^{\infty}\dfrac{x^{2m}}{2^m m!} + a_1\sum_{m=0}^{\infty}\dfrac{2^m m!\ x^{2m+1}}{(2m+1)!}$.

3a. $y = \sum_{n=0}^{\infty}a_n(x-1)^n$; $y' = \sum_{n=1}^{\infty}na_n(x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n$, and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-1)^n.$$

Substituting in the D.E. and setting $x = 1 + (x-1)$ we obtain

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n - \sum_{n=1}^{\infty}na_n(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^n = 0,$$

where the third term comes from:

$$-(x-1)y' = -\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^{n+1} = -\sum_{n=1}^{\infty}na_n(x-1)^n.$$

Letting $n = 0$ in the first, second, and the fourth sums, we obtain

$(2\cdot1\cdot a_2 - 1\cdot a_1 - a_0)(x-1)^0 +$

$$\sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - (n+1)a_n](x-1)^n = 0.$$

Setting the terms in the square brackets equal to zero and dividing by (n-1) yields $(n+2)a_{n+2} - a_{n+1} - a_n = 0$ for $n = 1,2,3,\ldots,$ (which also holds for $n = 0$). This recurrance relation can be used to solve for a_2 in terms of a_0 and a_1, then for a_3 in terms of a_0 and a_1, etc.

3b. In many cases it is easier to first take $a_0 = 0$ and generate one solution and then take $a_1 = 0$ and generate a second solution. Thus, choosing $a_0 = 0$ we find that $a_2 = a_1/2$, $a_3 = (a_2+a_1)/3 = a_1/2$, $a_4 = (a_3+a_2)/4 = a_1/4$, $a_5 = (a_4+a_3)/5 = 3a_1/20,\ldots$. This yields the solution $y_2(x) = (x-1) + (x-1)^2/2 + (x-1)^3/2 + (x-1)^4/4 + \ldots$. The second solution may be obtained by choosing $a_1 = 0$. Then $a_2 = a_0/2$, $a_3 = (a_2+a_1)/3 = a_0/6$, $a_4 = (a_3+a_2)/4 = a_0/6$, $a_5 = (a_4+a_3)/5 = a_0/15,\ldots$. This yields the solution $y_1(x) = 1+(x-1)^2/2+(x-1)^3/6+(x-1)^4/6+(x-1)^5/15+\ldots$.

3c. $W(y_1,y_2)(1) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ and thus y_1, y_2 form a fundamental set of solutions.

3d. A general term is not easily found in this case.

5. $y = \sum_{n=0}^{\infty} a_n x^n$; $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$; and $y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substituting in the D.E. and shifting the index in both summations for y″ gives

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} (n+1)n\, a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n =$$

$$(2\cdot1\cdot a_2 + a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n]x^n = 0.$$

Thus $a_2 = -a_0/2$ and $a_{n+2} = na_{n+1}/(n+2) - a_n/(n+2)(n+1)$, $n = 1,2,\ldots$. Choosing $a_0 = 0$ yields $a_2 = 0$, $a_3 = -a_1/6$, $a_4 = 2a_3/4 = -a_1/12$, $a_5 = 3a_4/5 - a_3/20 = -a_1/24 \ldots,$ and hence $y_2(x) = a_1(x - x^3/6 - x^4/12 - x^5/24 + \ldots)$. A second linearly independent solution is obtained by choosing $a_1 = 0$. Then

$a_2 = -a_0/2,\ a_3 = a_2/3 = -a_0/6,\ a_4 = 2a_3/4 - a_2/12 = -a_0/24, \ldots$
which gives $y_1(x) = a_0(1 - x^2/2 - x^3/6 - x^4/24 + \ldots)$.

8. If $y = \sum_{n=0}^{\infty} a_n(x-1)^n$ then

$$xy = [1+(x-1)]y = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1},$$

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1}, \text{ and}$$

$$xy'' = [1+(x-1)]y''$$
$$= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}.$$

14. You will need to rewrite x+1 as $3 + (x-2)$ in order to multiply x+1 times y′ as a power series about $x_0 = 2$.

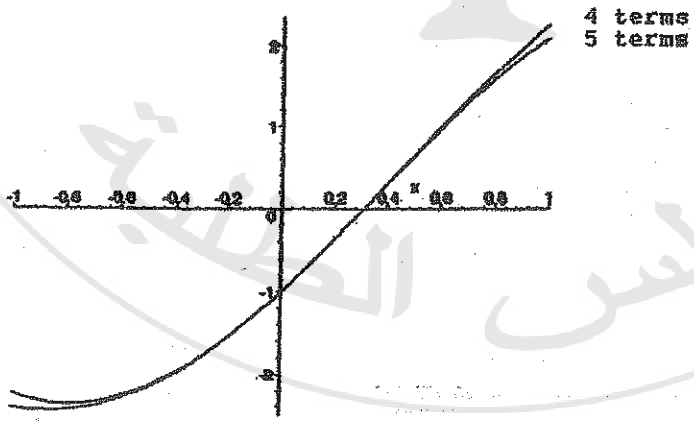
16a. From Prob. 6 we have

$$y(x) = c_1(1 - x^2 + \frac{1}{6}x^4 + \ldots) + c_2(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots).$$

Now $y(0) = c_1 = -1$ and $y'(0) = c_2 = 3$ and thus

$$y(x) = -1 + x^2 - \frac{1}{6}x^4 + \ldots + 3x - \frac{3}{4}x^3 + \cdots$$
$$= -1 + 3x + x^2 - \frac{3}{4}x^3 - \frac{1}{6}x^4 + \ldots .$$

16b.



16c. It appears that f is a reasonable approximation for $|x| < 0.7$. In fact, the magnitude of the difference in the two graphs is .02 for $|x| = .6$ and .04 for $|x| = .7$

19. Letting $t = x-1$ yields $(x-1)^2 = t^2$ and $(x^2-1) = t^2+2t$. Now let $u(t) = y(t+1)$ and hence $u' = y'$ and $u'' = y''$. Thus the D.E. transforms into $u''(t) + t^2u'(t) + (t^2+2t)u(t) = 0$.

Assuming that $u(t) = \sum_{n=0}^{\infty} a_nt^n$, we have $u'(t) = \sum_{n=1}^{\infty} na_nt^{n-1}$ and

$u''(t) = \sum_{n=2}^{\infty} n(n-1)a_nt^{n-2}$. Substituting in the D.E. and shifting indices yields

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0,$$

$$2\cdot1\cdot a_2t^0 + (3\cdot2\cdot a_3 + 2\cdot a_0)t^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}]t^n = 0.$$

It follows that $a_2 = 0$, $a_3 = -a_0/3$ and $a_{n+2} = -a_{n-1}/(n+2) - a_{n-2}/[(n+2)(n+1)]$, $n = 2,3,4\ldots$. We obtain one solution by choosing $a_1 = 0$. Then $a_4 = -a_0/12$, $a_5 = -a_2/5 - a_1/20 = 0$, $a_6 = -a_3/6 - a_2/30 = a_0/18, \ldots$. Thus one solution is $u_1(t) = a_0(1 - t^3/3 - t^4/12 + t^6/18 + \ldots)$ so $y_1(x) = u_1(x-1) = 1 - (x-1)^3/3 - (x-1)^4/12 + (x-1)^6/18 + \ldots$. We obtain a second solution by choosing $a_0 = 0$. Then $a_4 = -a_1/4$, $a_5 = -a_2/5 - a_1/20 = -a_1/20$, $a_6 = -a_3/6 - a_2/30 = 0$, $a_7 = -a_4/7 - a_3/42 = a_1/28, \ldots$. Thus $u_2(t) = t - t^4/4 - t^5/20 + t^7/28 + \ldots$ or

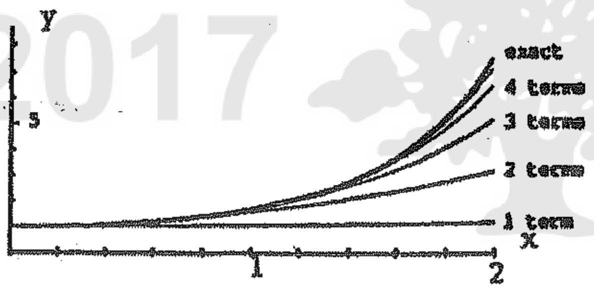
$$y_2(x) = u_2(x-1) = (x-1) - (x-1)^4/4 - (x-1)^5/20 + (x-1)^7/28 + \ldots .$$

The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = 1 + (x-1)$ so $x^2 = 1 + 2(x-1) + (x-1)^2$ and $x^2 - 1 = 2(x-1) + (x-1)^2$. The D.E. now appears as $y'' + (x-1)^2y' + [(x-1)^2 + 2(x-1)]y = 0$ which is identical to the transformed equation with $t = x - 1$.

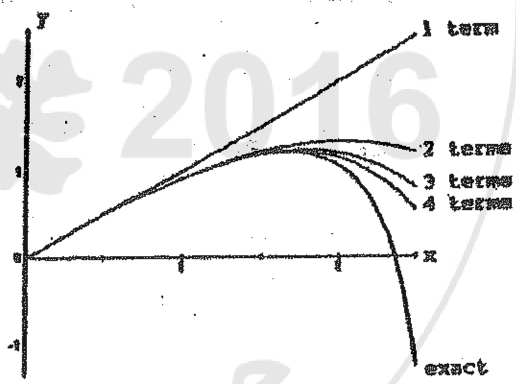
22b. $y = a_0 + a_1x + a_2x^2 + \dots$, $y^2 = a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 + \dots$, $y' = a_1 + 2a_2x + 3a_3x^2 + \dots$, and $(y')^2 = a_1^2 + 4a_1a_2x + (6a_1a_3 + 4a_2^{\ 2})x^2 + \dots$. Substituting these into $(y')^2 = 1 - y^2$ and collecting coefficients of like powers of x yields $(a_1^2 + a_0^2 - 1) + (4a_1a_2 + 2a_0a_1)x + (6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2)x^2 + \dots = 0$. As in the earlier problems, each coefficient must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_0^2 - 1 = 0$ gives $a_1^2 = 1$. However, the D.E. indicates that y' is always positive, so $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1a_2 + 2a_0a_1 = 0$ implies that $a_2 = 0$; and $6a_1a_3 + 4a_2^2 + 2a_0a_2 + a_1^2 = 6a_1a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus $y = x - x^3/3! + \dots$, which are the first two terms of the Taylor series for sinx.

23. We have $y(x) = a_0y_1 + a_1y_2$, where y_1 and y_2 are found in Prob.2. Now, $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$ and thus $y(x) = 1 + x^2/2! + x^4/4! + x^6/6! + \cdots$.

23.



26.



26. Again, $y(x) = a_0y_1 + a_1y_2$, where y_1 and y_2 are found in Prob.10, and $y(0) = a_0 = 0$ and $y'(0) = a_1 = 1$.

Thus $y(x) = x - \dfrac{x^3}{12} - \dfrac{x^5}{240} - \dfrac{x^7}{2240}$.

<u>Section 5.3, Page 265</u>

1. The D.E. can be solved for y'' to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then $\phi''(x) = -x\phi'(x) - \phi(x)$ and thus setting $x = 0$ we obtain $\phi''(0) = -0 - 1 = -1$.

Differentiating the equation for y'' yields $y''' = -xy'' - 2y'$ and hence setting $y = \phi(x)$ again yields $\phi'''(0) = -0 - 0 = 0$. In a similar fashion $y^{(4)} = -xy''' - 3y''$ and thus $\phi^{(4)}(0) = -0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

3. We have $y'' = -\frac{1+x}{x}y' - \frac{3\ln x}{x^2}y$, so $\phi''(1) = 2\phi'(1) - 0\phi(1) = 0$, and $y''' = -\frac{1+x}{x}y'' + (\frac{1-3\ln x}{x^2})y' + (\frac{6\ln x-3}{x^3})y$ so $\phi'''(1) = -2\phi''(1) + \phi'(1) - 3\phi(1) = -6$.

6. The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is 1,3, and 1, respectively. Thus a lower bound for the radius of convergence for series solutions in powers of $(x-4)$, $(x+4)$, and x is $\rho = 1$, $\rho = 3$, and $\rho = 1$, respectively.

7. If $x^3 = -1$, then $x = e^{i(\pi+2k\pi)/3}$, $k = 0,1,2$. Thus $x_1 = e^{i\pi/3} = (1+i\sqrt{3})/2$, $x_2 = -1$ and $x_3 = (1-i\sqrt{3})/2$. Thus for $x_0 = 0$ we have $\rho = 1$ and for $x_0 = 2$ we have $\rho = \sqrt{(3/2)^2 + 3/4} = \sqrt{3}$.

9a. Since $P(x) = 1$ has no zeros, the radius of convergence about $x_0 = 0$ is $\rho = \infty$.

9f. Since $P(x) = 2 + x^2$ has zeros at $x = \pm\sqrt{2}\,i$, the lower bound for the radius of convergence of the series solution about $x_0 = 0$ is $\rho = \sqrt{2}$.

9h. $P(x) = x$ has a zero at $x = 0$ and since $x_0 = 1$, $\rho = 1$.

10a. If we assume that $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$ and

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$. Substituting in the D.E. gives:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} na_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shifting indices of summation and collecting coefficients of like powers of x yields the equation:

$$(2\cdot1\cdot a_2 + \alpha^2 a_0)x^0 + [3\cdot2\cdot a_3 + (\alpha^2-1)a_1]x^1 + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (\alpha^2-n^2)a_n]x^n = 0.$$

Hence the recurrence relation is $a_{n+2} = (n^2-\alpha^2)a_n/(n+2)(n+1)$, $n = 0, 1, 2, \ldots$. For the first solution we choose $a_1 = 0$. We find that $a_2 = -\alpha^2 a_0/2\cdot1$, $a_3 = 0$, $a_4 = (2^2-\alpha^2)a_2/4\cdot3 = -(2^2-\alpha^2)\alpha^2 a_0/4!$ $\ldots$, $a_{2m} = -[(2m-2)^2 - \alpha^2]\ldots(2^2-\alpha^2)\alpha^2 a_0/(2m)!$, and $a_{2m+1} = 0$, so $y_1(x) = 1 - \frac{\alpha^2}{2!}x^2 - \frac{(2^2-\alpha^2)\alpha^2}{4!}x^4 - \ldots$

$$- \frac{[(2m-2)^2-\alpha^2]\ldots(2^2-\alpha^2)\alpha^2}{(2m)!}x^{2m} - \ldots,$$

where we have set $a_0 = 1$. For the second solution we take $a_0 = 0$ and $a_1 = 1$ in the recurrence relation to obtain the desired solution.

10b. If α is an even integer 2k then $(2m-2)^2-\alpha^2 = 4(m-1)^2-4k^2$. Thus when m = k+1 all terms in the series for $y_1(x)$ are zero after the x^{2k} term. A similar argument shows that if $\alpha = 2k+1$ then all terms in $y_2(x)$ are zero after the x^{2k+1} term.

11. The Taylor series about x = 0 for sinx is $\sin x = x - x^3/3! + x^5/5! - \ldots$. Assuming that $y = \sum_{n=2}^{\infty} a_n x^n$ we find $y'' + (\sin x)y = 2a_2 + 6a_3x + 12a_4x^2$

$+ 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \ldots$

$+ (x-x^3/3!+x^5/5!-\ldots)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots)$

$= 2a_2 + (6a_3+a_0)x + (12a_4+a_1)x^2 + (20a_5+a_2-a_0/6)x^3 +$

$(30a_6+ a_3-a_1/6)x^4 + (42a_7+a_4-a_2/3!+a_0/5!)x^5 + \ldots = 0.$

Hence $a_2 = 0$, $a_3 = -a_0/6$, $a_4 = -a_1/12$, $a_5 = a_0/120$, $a_6 = (a_1+a_0)/180$, $a_7 = -a_0/7! + a_1/504$, $\ldots$. We set $a_0 = 1$ and $a_1 = 0$ and obtain

$y_1(x) = (1 - x^3/6 + x^5/120 + x^6/180 + \ldots)$. Next we set

$a_0 = 0$ and $a_1 = 1$ and obtain

$y_2(x) = (x - x^4/12 + x^6/180 + x^7/504 + \ldots)$. Since $p(x) = 1$ and $q(x) = \sin x$ both have $\rho = \infty$, the solution in this case converges for all x, that is, $\rho = \infty$

18. We know that $e^x = 1 + x + x^2/2! + x^3/3! + \ldots$, and therefore $e^{x^2} = 1 + x^2 + x^4/2! + x^6/3! + \ldots$. Hence, if $y = \sum_{n=0}^{\infty} a_n x^n$, we have $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, so

$$a_1 + 2a_2x + 3a_3x^2 + \ldots = (1+x^2 + x^4/2+\ldots)(a_0+a_1x+a_2x^2+\ldots)$$
$$= a_0 + a_1x + (a_0+a_2)x^2 + \ldots .$$

Thus, $a_1 = a_0$ $2a_2 = a_1$ and $3a_3 = a_0 + a_2$, which yield the desired solution.

20. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the D.E. we obtain $\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = x^2$. Shifting indices in the summation yields $\sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = x^2$. Equating coefficients of both sides then gives: $a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 1$ and $(n+1)a_{n+1} = a_n$ for $n = 3,4,\ldots$. Thus $a_1 = a_0$, $a_2 = a_1/2 = a_0/2$, $a_3 = 1/3 + a_2/3 = 1/3 + a_0/2\cdot3$, $a_4 = a_3/4 = 1/3\cdot4 + a_0/2\cdot3\cdot4 = 2/4! + a_0/4!$, and in general $a_n = a_{n-1}/n = 2/n! + a_0/n!$. Hence

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} \ldots\right) + 2\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots\right).$$

Using the power series for e^x, the first and second sums can be rewritten as $a_0e^x + 2(e^x - 1 - x - x^2/2)$, which is the same solution as found using methods of Chapt.2.

22. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the Legendre equation, shifting indices, and collecting coefficients of like powers of x yields

$[2\cdot 1\cdot a_2 + \alpha(\alpha+1)a_0]x^0 + \{3\cdot 2\cdot a_3 - [2\cdot 1 - \alpha(\alpha+1)]a_1\}x^1 +$

$\sum_{n=2}^{\infty}\{(n+2)(n+1)a_{n+2} - [n(n+1) - \alpha(\alpha+1)]a_n\}x^n = 0.$ Thus

$a_2 = -\alpha(\alpha+1)a_0/2!$, $a_3 = [2\cdot 1 - \alpha(\alpha+1)]a_1/3! =$
$-(\alpha-1)(\alpha+2)a_1/3!$ and the recurrence relation is
$(n+2)(n+1)a_{n+2} = -[\alpha(\alpha+1) - n(n+1)]a_n = -(\alpha-n)(\alpha+n+1)a_n$,
$n = 2,3,\ldots$. Setting $a_1 = 0$, $a_0 = 1$ yields a solution with $a_3 = a_5 = a_7 = \ldots = 0$ and
$a_4 = \alpha(\alpha-2)(\alpha+1)(\alpha+3)/4!,\ldots,$ and
$a_{2m} = (-1)^m[\alpha(\alpha-2)\ldots(\alpha-2m+2)][(\alpha+1)\ldots(\alpha+2m-1)]/(2m)!$.
The second linearly independent solution is obtained by setting $a_0 = 0$ and $a_1 = 1$. The coefficients are
$a_2 = a_4 = a_6 = \ldots = 0$ and $a_3 = -(\alpha-1)(\alpha+2)/3!$, and
$a_5 = -(\alpha-3)(\alpha+4)a_3/5\cdot 4 = (\alpha-1)(\alpha-3)(\alpha+2)(\alpha+4)/5!$.

26. Using the chain rule we have:

$$\frac{dF(\phi)}{d\phi} = \frac{dF[\phi(x)]}{dx}\frac{dx}{d\phi} = -f'(x)\sin\phi(x) = -f'(x)\sqrt{1-x^2},$$

$$\frac{d^2F(\phi)}{d\phi^2} = \frac{d}{dx}[-f'(x)\sqrt{1-x^2}]\frac{dx}{d\phi} = (1-x^2)f''(x) - xf'(x),$$

which when substituted into the D.E. yields the desired result.

28. Since $[(1-x^2)y']' = (1-x^2)y'' - 2xy'$, the Legendre Equation, from Prob. 22, can be written as shown. Thus, carrying out the multiplications indicated yields the two equations:

$$P_m[(1-x^2)P_n']' = -n(n+1)P_nP_m$$

$$P_n[(1-x^2)P_m']' = -m(m+1)P_nP_m.$$

As long as $n \neq m$ the second equation can be subtracted from the first and the result integrated from -1 to 1 to obtain

$$\int_{-1}^{1}\{P_m[(1-x^2)P_n']' - P_n[(1-x^2)P_m']'\}dx = [m(m+1)-n(n+1)]\int_{-1}^{1}P_nP_m dx$$

The left side may be integrated by parts to yield

$$[P_m(1-x^2)P_n' - P_n(1-x^2)P_m']_{-1}^{1} + \int_{-1}^{1}[P_m'(1-x^2)P_n' - P_n'(1-x^2)P_m']dx,$$

which is zero. Thus $\int_{-1}^{1}P_n(x)P_m(x)dx = 0$ for $n \neq m$.

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2. This equation is of the form of an Euler equation with x replaced by x + 1, so we seek solutions of the form $y = (x+1)^r$ for $x + 1 > 0$. Substitution of y into the D.E. yields $F(r) = [r(r-1) + 3r + 3/4](x+1)^r = 0$. Thus $r^2 + 2r + 3/4 = 0$, which gives $r = -3/2, -1/2$. The general solution of the D.E. is then $y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}$, $x \neq -1$.

4. If $y = x^r$ then $F(r) = r(r-1) + 3r + 5 = 0$. So $r^2 + 2r + 5 = 0$ and $r = (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i$. Thus the general solution of the D.E. is $y = c_1x^{-1}\cos(2\ln|x|) + c_2x^{-1}\sin(2\ln|x|)$, $x \neq 0$.

9. Again let $y = x^r$ to obtain $F(r) = r(r-1) - 5r + 9 = 0$, or $(r-3)^2 = 0$. Thus the roots are $x = 3,3$ and $y = c_1x^3 + c_2x^3\ln|x|$, $x \neq 0$, is the solution of the D.E.

13. In this csae $F(r) = 2r(r-1) + r - 3 = 2r^2 - r - 3 = (2r-3)(r+1) = 0$, so $y = c_1x^{3/2} + c_2x^{-1}$ (since $x_0 = 1$, we don't need $|x|$) and $y' = \frac{3}{2}c_1x^{1/2} - c_2x^{-2}$. Setting $x = 1$ in y and y′ we obtain $c_1 + c_2 = 1$ and $\frac{3}{2}c_1 - c_2 = 4$, which yield $c_1 = 2$ and $c_2 = -1$. Hence $y = 2x^{3/2} - x^{-1}$. As $x \to 0^+$ we have $y \to -\infty$ due to the second term.

16. We have $F(r) = r(r-1) + 3r + 5 = r^2 + 2r + 5 = 0$. Thus $r_1, r_2 = -1 \pm 2i$ and $y = x^{-1}[c_1\cos(2\ln x) + c_2\sin(2\ln x)]$. Then $y(1) = c_1 = 1$ and $y' = -x^{-2}[\cos(2\ln x) + c_2\sin(2\ln x)] + x^{-1}[-\sin(2\ln x)2/x + c_2\cos(2\ln x)2/x]$ so that $y'(1) = -1+2c_2 = -1$, or $c_2 = 0$. Hence $y = x^{-1}\cos(2\ln x)$ for $x > 0$. As $x \to 0^+$ this will oscillate rapidly, with large amplitudes.

17. Since the coefficients of y, y′ and y″ have no common factors and since P(x) vanishes only at x = 0 we conclude that x = 0 is a singular point. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we obtain $p(x) = (1-x)/x$ and $q(x) = 1$. Thus for the singular point we have $\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} 1-x = 1$, $\lim_{x\to 0} x^2q(x) = 0$ and thus x = 0

is a regular singular point.

21. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we find $p(x) = x/(1-x)(1+x)^2$ and $q(x) = (1+x)/(1-x^2)$. Therefore $x = \pm 1$ are singular points. Since $\lim_{x\to 1} (x-1)p(x)$ and $\lim_{x\to 1} (x-1)^2 q(x)$ both exist, we conclude $x = 1$ is a regular singular point. Finally, since $\lim_{x\to -1} (x+1)p(x)$ does not exist, we conclude that $x = -1$ is an irregular singular point.

28. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = e^x/x$ and $q(x) = (3\cos x)/x$. Thus $x = 0$ is a singular point. Since $xp(x) = e^x$ is analytic at $x = 0$ and $x^2q(x) = 3x\cos x$ is analytic at $x = 0$ the point $x = 0$ is a regular singular point.

33. Writing the D.E. in the form $y'' + p(x)y' + q(x)y = 0$, we see that $p(x) = \dfrac{x}{\sin x}$ and $q(x) = \dfrac{4}{\sin x}$. Since $\lim_{x\to 0} q(x)$ does not exist, the point $x_0 = 0$ is a singular point and since neither $\lim_{x\to \pm n\pi} p(x)$ nor $\lim_{x\to \pm n\pi} q(x)$ exist, either, the points $x_0 = \pm n\pi$ are also singular points. To determine whether the singular points are regular or irregular we must use Eq.(31) and the result #7 of multiplication and division of power series from Section 5.1. For $x_0 = 0$, we have

$$xp(x) = \frac{x^2}{\sin x} = \frac{x^2}{x - \frac{x^3}{6} + \dots} = x[1 + \frac{x^2}{6} + \dots]$$

$$= x + \frac{x^3}{6} + \dots,$$

which converges about $x_0 = 0$ and thus $xp(x)$ is analytic at $x_0 = 0$. $x^2q(x)$, by similar steps, is also analytic at $x_0 = 0$ and thus $x_0 = 0$ is a regular singular point. For $x_0 = n\pi$, we have

$$(x-n\pi)p(x) = \frac{(x-n\pi)x}{\sin x} = \frac{(x-n\pi)[(x-n\pi) + n\pi]}{\pm(x-n\pi) \mp \frac{(x-n\pi)^3}{6} \pm \dots}$$

$$= \pm[(x-n\pi) + n\pi][1 + \frac{(x-n\pi)^2}{6} + \dots], \text{ which}$$

converges about $x_0 = n\pi$ and thus $(x-n\pi)p(x)$ is analytic at $x= n\pi$. Similarly $(x+n\pi)p(x)$ and $(x\pm n\pi)^2q(x)$ are analytic and thus $x_0 = \pm n\pi$ are regular singular points.

35. Substituting $y = x^r$, we find that $r(r-1) + \alpha r + 5/2 = 0$ or $r^2 + (\alpha-1)r + 5/2 = 0$. Thus $r_1, r_2 = [-(\alpha-1) \pm \sqrt{(\alpha-1)^2-10}\,]/2$. In order for solutions to approach zero as $x\to 0$ it is necessary that the real parts of r_1 and r_2 be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of r_1 and r_2 will be negative. Suppose that $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose that $\alpha < 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $|\alpha-1| = 1 - \alpha$; hence the real parts of r_1 and r_2 will be positive. Thus if $\alpha < 1$ the solutions of the D.E. will approach zero as $x\to 0$.

39. In all cases the roots of $F(r) = 0$ are given by Eq.(6) and the forms of the solution are given in Eqs.(25),(26) and (27).

39a. The real part of the root must be positive so, from Eq.(6), $\alpha < 1$. Also $\beta > 0$, since the $\sqrt{(\alpha-1)^2-4\beta}$ term must be less than $|\alpha-1|$.

39d. The real part of the root must be negative, so $\alpha > 1$, with $\beta \geq 0$ (for $\beta = 0$ one root is zero, which is bounded as $x \to \infty$). If $\alpha = 1$, then the roots are $\pm\sqrt{-4\beta}$, so $\beta > 0$ will yield oscillatory solutions as $x \to \infty$, which are bounded.

40. Assume that $y = v(x)x^{r_1}$. Then $y' = v(x)r_1x^{r_1-1} + v'(x)x^{r_1}$ and $y'' = v(x)r_1(r_1-1)x^{r_1-2} + 2v'(x)r_1x^{r_1-1} + v''(x)x^{r_1}$. Substituting in the D.E. and collecting terms yields $x^{r_1+2}\, v'' + (\alpha + 2r_1)x^{r_1+1}\, v' + [r_1(r_1-1) + \alpha r_1 + \beta]x^{r_1}\, v = 0$. Now we make use of the fact that r_1 is a double root of $f(r) = r(r-1) + \alpha r + \beta$. This means that $f(r_1) = 0$ and $f'(r_1) = 2r_1 - 1 + \alpha = 0$. Hence the D.E. for v reduces to $x^{r_1+2}\, v'' + x^{r_1+1}\, v'$. Since $x > 0$ we may divide by x^{r_1+1}

to obtain $xv'' + v' = 0$. Thus $v(x) = \ln x$ and a second solution is $y = x^{r_1}\ln x$.

41. Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into the D.E. yields

$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + 3\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$. The last sum

becomes $\sum_{n=2}^{\infty} a_{n-2}x^{n-1}$ (let $m = n+2$ and then replace m by n), the first term of the middle sum is $3a_1$, and thus we have

$3a_1 + \sum_{n=2}^{\infty}\{[2n(n-1)+3n]a_n + a_{n-2}\}x^{n-1} = 0$. Hence $a_1 = 0$ and

$a_n = \dfrac{-a_{n-2}}{n(2n+1)}$, which is the desired recurrance relation. Thus all even coefficients are found in terms of a_0 and all odd coefficients are zero, thereby yielding only one solution of the desired form.

43. If $\xi = 1/x$ then

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = -\frac{1}{x^2}\frac{dy}{d\xi} = -\xi^2\frac{dy}{d\xi},$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\xi}\left(-\xi^2\frac{dy}{d\xi}\right)\frac{d\xi}{dx} = \left(-2\xi\frac{dy}{d\xi} - \xi^2\frac{d^2y}{d\xi^2}\right)\left(-\frac{1}{x^2}\right)$$

$$= \xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}.$$

Substituting in the D.E. we have

$P(1/\xi)\left[\xi^4\dfrac{d^2y}{d\xi^2} + 2\xi^3\dfrac{dy}{d\xi}\right] + Q(1/\xi)\left[-\xi^2\dfrac{dy}{d\xi}\right] + R(1/\xi)y = 0$, or

$$\xi^4P(1/\xi)\frac{d^2y}{d\xi^2} + [2\xi^3P(1/\xi) - \xi^2Q(1/\xi)]\frac{dy}{d\xi} + R(1/\xi)y = 0.$$

The result then follows from the theory of singular points at $\xi = 0$.

45. Since $P(x) = x^2$, $Q(x) = x$ and $R(x) = -4$ we have $f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4P(1/\xi) = -4/\xi^2$. Thus the point at infinity is a singular point. Since both $\xi f(\xi)$ and $\xi^2 g(\xi)$ are analytic at $\xi = 0$, the point at infinity is a

regular singular point.

47. Since $P(x) = x^2$, $Q(x) = x$, and $R(x) = x^2 - \upsilon^2$, $f(\xi) = [2P(1/\xi)/\xi - Q(1/\xi)/\xi^2]/P(1/\xi) = 2/\xi - 1/\xi = 1/\xi$ and $g(\xi) = R(1/\xi)/\xi^4 P(1/\xi) = (1/\xi^2 - \upsilon^2)/\xi^2 = 1/\xi^4 - \upsilon^2/\xi^2$. Thus the point at infinity is a singular point. Although $\xi f(\xi) = 1$ is analytic at $\xi = 0$, $\xi^2 g(\xi) = 1/\xi^2 - \upsilon^2$ is not, so the point at infinity is an irregular singular point.

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2a. If the D.E. is put in the standard form $y'' + p(x)y + q(x)y = 0$, then $p(x) = x^{-1}$ and $q(x) = 1 - 1/9x^2$. Thus $x = 0$ is a singular point. Since $xp(x) \to 1$ and $x^2q(x) \to -1/9$ as $x \to 0$ it follows that $x = 0$ is a regular singular point.

2b. In determining a series solution of the D.E. it is more convenient to leave the equation in the form given rather than divide by x^2, the coefficient of y''. If we substitute

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$, we have

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (x^2 - \frac{1}{9})\sum_{n=0}^{\infty} a_n x^{n+r} = 0.$$

Note that $x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$. Thus we have $[r(r-1) + r - \frac{1}{9}]a_0 x^r + [(r+1)r + (r+1) - \frac{1}{9}]a_1 x^{r+1} + \sum_{n=2}^{\infty}\{[(n+r)(n+r-1) + (n+r) - \frac{1}{9}]a_n + a_{n-2}\} x^{n+r} = 0$. From the first term, the indicial equation is $r^2 - 1/9 = 0$ with roots $r_1 = 1/3$ and $r_2 = -1/3$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $a_n = -a_{n-2}/[(n+r)^2 - 1/9]$.

2c. For $r = 1/3$ we have

$$a_n = \frac{-a_{n-2}}{(n + \frac{1}{3})^2 - (\frac{1}{3})^2} = -\frac{a_{n-2}}{(n + \frac{2}{3})n}, \quad n = 2,3,4,\ldots .$$

Since $a_1 = 0$ it follows from the recurrence relation that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients it is convenient to let $n = 2m$, $m = 1,2,3,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m(m + \frac{1}{3})$. The first few coefficients are given by

$$a_2 = \frac{(-1)a_0}{2^2(1 + \frac{1}{3})1}, \quad a_4 = \frac{(-1)a_2}{2^2(2 + \frac{1}{3})2} = \frac{a_0}{2^4(1 + \frac{1}{3})(2 + \frac{1}{3})2!}$$

$$a_6 = \frac{(-1)a_4}{2^2(3 + \frac{1}{3})3} = \frac{(-1)a_0}{2^6(1 + \frac{1}{3})(2 + \frac{1}{3})(3 + \frac{1}{3})3!},$$ and the

coefficent of x^{2m} for $m = 1, 2, \ldots$ is

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(1 + \frac{1}{3})(2 + \frac{1}{3}) \ldots (m + \frac{1}{3})}.$$ Thus one

solution (on setting $a_0 = 1$) is

$$y_1(x) = x^{1/3}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(1 + \frac{1}{3})(2 + \frac{1}{3})\ldots(m + \frac{1}{3})}(\frac{x}{2})^{2m}].$$

2d. Since $r_2 = -1/3 \neq r_1$ and $r_1 - r_2 = 2/3$ is not an integer, we can calculate a second series solution corresponding to $r = -1/3$. The recurrence relation is $n(n-2/3)a_n = -a_{n-2}$, which yields the desired solution following the steps in Part c. Note that $a_1 = 0$, as in the first solution, and thus all the odd coefficients are zero.

4a. Putting the D.E. in the form $y''+p(x)y'+q(x)y = 0$, we see that $p(x) = 1/x$ and $q(x) = -1/x$. Thus $x = 0$ is a singular point, and since $xp(x) \to 1$ and $x^2q(x) \to 0$, as $x \to 0$, $x = 0$ is a regular singular point.

4b. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in $xy'' + y' - y = 0$ and shifting indices we obtain

$$\sum_{n=-1}^{\infty} a_{n+1}(r+n+1)(r+n)x^{n+r} + \sum_{n=-1}^{\infty} a_{n+1}(r+n+1)x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0,$$

or $[r(r-1) + r]a_0 x^{-1+r} + \sum_{n=0}^{\infty} [(r+n+1)^2 a_{n+1} - a_n]x^{n+r} = 0.$

From the first coefficient we find $r^2 = 0$ is the indicial equation and from the coefficient of x^{n+r} we find the recurrance relation is $a_{n+1} = a_n/(n+1+r)^2$.

4c. Setting $r = 0$ in the recurrance relation we find $(n+1)^2 a_{n+1} = a_n$, $n = 0,1,2,\ldots$. The coefficients are $a_1 = a_0$, $a_2 = a_1/2^2 = a_0/2^2$, $a_3 = a_2/3^2 = a_0/3^2 2^2$, $a_4 = a_3/4^2 = a_0/4^2 3^2 2^2, \ldots$ and $a_n = a_0/(n!)^2$. Thus one solution (on setting $a_0 = 1$) is $y = \sum_{n=0}^{\infty} x^n/(n!)^2$.

4d. Since the the indicial equation has only one root, we only have one solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^{n+r}$.

11a. If we make the change of variable $t = x-1$ and let $y = u(t)$, then the Legendre equation transforms to $(t^2 + 2t)u''(t) + 2(t+1)u'(t) - \alpha(\alpha+1)u(t) = 0$. Since $x = 1$ is a regular singular point of the original equation, we know that $t = 0$ is a regular singular point of the transformed equation. Substituting $u = \sum_{n=0}^{\infty} a_n t^{n+r}$ in the transformed equation and shifting indices, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + 2\sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}t^{n+r}$$

$$+ 2\sum_{n=0}^{\infty}(n+r)a_n t^{n+r} + 2\sum_{n=-1}^{\infty}(n+r+1)a_{n+1}t^{n+r} - \alpha(\alpha+1)\sum_{n=0}^{\infty}a_n t^{n+r} = 0,$$

$$\text{or } [2r(r-1) + 2r]a_0 t^{r-1} + \sum_{n=0}^{\infty}\{2(n+r+1)^2 a_{n+1}$$

$$+ [(n+r)(n+r+1) - \alpha(\alpha+1)]a_n\}t^{n+r} = 0.$$

The indicial equation is $2r^2 = 0$ so $r = 0$ is a double root. Thus there will be only one series solution of the form $y = \sum_{n=0}^{\infty} a_n t^{n+r}$.

11b. The recurrence relation is $2(n+1)^2 a_{n+1} = [\alpha(\alpha+1) - n(n+1)]a_n, n = 0,1,2,\ldots$. We have $a_1 = [\alpha(\alpha+1)]a_0/2\cdot1^2$, $a_2 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2]a_0/2^2\cdot2^2\cdot1^2$, $a_3 = [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2][\alpha(\alpha+1) - 2\cdot3]a_0/2^3\cdot3^2\cdot2^2\cdot1^2,\ldots,$ and $a_n = [\alpha(\alpha+1)][\alpha(\alpha+1)-1\cdot2]\ldots[\alpha(\alpha+1)-(n-1)n]a_0/2^n(n!)^2$. Reverting to the variable x it follows that one solution of the Legendre equation in powers of x-1 is

$$y_1(x) = \sum_{n=0}^{\infty} [\alpha(\alpha+1)][\alpha(\alpha+1) - 1\cdot2] \ldots$$

$[\alpha(\alpha+1) - (n-1)n](x-1)^n/2^n(n!)^2$ where we have set $a_0 = 1$, which is equivalent to the answer in the text if a (-1) is taken out of each square bracket.

14a. The standard form is $y'' + p(x)y' + q(x)y = 0$, with $p(x) = 1/x$ and $q(x) = 1$. Thus $x = 0$ is a singular point; and since $xp(x) \to 1$ and $x^2q(x) \to 0$ as $x \to 0$, $x = 0$ is a regular singular point.

14b. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + x^2y = 0$ and shifting indices appropriately, we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=2}^{\infty}a_{n-2}x^{n+r} = 0, \text{ or}$$

$$[r(r-1)+r]a_0x^r + [(1+r)r+1+r]a_1x^{r+1} + \sum_{n=2}^{\infty}[(n+r)^2a_n + a_{n-2}]x^{n+r} = 0.$$

The indicial equation is $r^2 = 0$ so $r = 0$ is a double root. It is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero.

14c. The recurrence relation is $n^2a_n = -a_{n-2}$, $n = 2,3,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients we let $n = 2m$, $m = 1,2,\ldots$. Then $a_{2m} = -a_{2m-2}/2^2m^2$ so $a_2 = -a_0/2^2 \cdot 1^2$, $a_4 = a_0/2^2 \cdot 2^2 \cdot 1^2 \cdot 2^2, \ldots$, and $a_{2m} = (-1)^m a_0/2^{2m}(m!)^2$. Thus one solution of the Bessel equation of order zero is $J_0(x) = 1 + \sum_{m=1}^{\infty}(-1)^m x^{2m}/2^{2m}(m!)^2$ where we have set $a_0 = 1$.

14d. Using the ratio test it can be shown that the series converges for all x. Also note that $J_0(x) \to 1$ as $x \to 0$.

15. In order to determine the form of the integral for x near zero we must study the integrand for x small. Using the above series for J_0, we have

$$\frac{1}{x[J_0(x)]^2} = \frac{1}{x[1 - x^2/2 + \ldots]^2} = \frac{1}{x[1 - x^2 + \ldots]}$$

$$= \frac{1}{x}[1 + x^2 + \ldots] \text{ for } x \text{ small. Thus}$$

$$y_2(x) = J_0(x)\int\frac{dx}{x[J_0(x)]^2} = J_0(x)\int[\frac{1}{x} + x + \ldots]dx$$

$$= J_0(x)[\ln x + \frac{x^2}{x} + \ldots],$$ and it is clear that $y_2(x)$ will contain a logarithmic term.

16a. Putting the D.E. in the standard form $y'' + p(x)y' + q(x)y = 0$ we see that $p(x) = 1/x$ and $q(x) = (x^2-1)/x^2$. Thus $x = 0$ is a singular point and since $xp(x) \to 1$ and $x^2q(x) \to -1$ as $x \to 0$, $x = 0$ is a regular singular point.

16b. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ into $x^2y'' + xy' + (x^2-1)y = 0$, shifting indices appropriately, and collecting coefficients of common powers of x we obtain

$$[r(r-1) + r - 1]a_0x^r + [(1+r)r + 1 + r - 1]a_1x^{r+1}$$

$$+ \sum_{n=2}^{\infty}\{[(n+r)^2 - 1]a_n + a_{n-2}\}x^{n+r} = 0.$$

The indicial equation is $r^2-1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$.

16c. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recurrence relation is $[(n+r)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,4\ldots$. For $r = 1$ we have $a_n = -a_{n-2}/[n(n+2)]$, $n = 2,3,4,\ldots$. Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. Let $n = 2m$. Then $a_{2m} = -a_{2m-2}/2^2m(m+1)$, $m = 1,2,\ldots,$ so $a_2 = -a_0/2^2 \cdot 1 \cdot 2$, $a_4 = -a_2/2^2 \cdot 1 \cdot 2 \cdot 3 = a_0/2^2 \cdot 2^2 \cdot 1 \cdot 2 \cdot 2 \cdot 3, \ldots,$ and $a_{2m} = (-1)^m a_0/2^{2m} m!(m+1)!$. Thus one solution (set $a_0 = 1/2$) of the Bessel equation of order one is

$$J_1(x) = (x/2) \sum_{n=0}^{\infty} (-1)^n x^{2n}/(n+1)!n!2^{2n}.$$

16d. The ratio test shows that the series converges for all x. Also note that $J_1(x) \to 0$ as $x \to 0$.

16e. For $r = -1$ the recurrence relation is $[(n-1)^2 - 1]a_n = -a_{n-2}$, $n = 2,3,\ldots$, so for $n = 2$ the coefficient of a_2 is zero and we cannot calculate a_2. Consequently it is not possible to find a series solution of the form $x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

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1a. The D.E. has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x$, $Q(x) = 2x$, and $R(x) = 6e^x$. From this we find $p(x) = Q(x)/P(x) = 2$ and $q(x) = R(x)/P(x) = 6e^x/x$ and thus $x = 0$ is a singular point. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are analytic at $x = 0$ we conclude that $x = 0$ is a regular singular point.

1b. We have $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$ and thus Eq.(7), the indicial equation, is $F(r) = r(r-1) = 0$, which has the roots $r_1 = 1$ and $r_2 = 0$. These are the exponents of the singularity at $x = 0$.

3a. The equation has the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1)$, $Q(x) = 6x^2$ and $R(x) = 3$. Since $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors and $P(0) = 0$ and $P(1) = 0$, we conclude that $x = 0$ and $x = 1$

are singular points. The first point, $x = 0$, can be shown to be a regular singular point using steps similar to those shown in Prob. 1. For $x = 1$, we must put the D.E. in the form seen in Ex.(1). To do this, divide the D.E. by x and multiply by $(x-1)$ to obtain

$(x-1)^2y'' + 6x(x-1)y + \frac{3}{x}(x-1)y = 0$. Comparing this to Ex.(1) we find that $(x-1)p(x) = 6x$ and $(x-1)^2q(x) = 3(x-1)/x$ which are both analytic at $x = 1$ and hence $x = 1$ is a regular singular point.

3b. These last two expressions approach $p_0 = 6$ and $q_0 = 0$ respectively as $x \to 1$, and thus the indicial equation is $F(r) = r(r-1) + 6r + 0 = r(r+5) = 0$.

9a. For this D.E., $p(x) = \frac{-(1+x)}{x^2(1-x)}$ and $q(x) = \frac{2}{x(1-x)}$ and thus $x = 0,1$ are singular points. Since $xp(x)$ is not analytic at $x = 0$, $x = 0$ is not a regular singular point. Looking at $(x-1)p(x) = \frac{1+x}{x^2}$ and $(x-1)^2q(x) = \frac{2(1-x)}{x}$ we see that $x = 1$ is a regular singular point.

9b. As in Ex.(1) $p_0 = \lim_{x\to1}(x-1)p(x) = 2$ and $q_0 = \lim_{x\to1}(x-1)^2q(x) = 0$. Thus the indicial equation is $F(r) = r^2 + r$ and $r_1 = 0$ and $r_2 = -1$.

13a. We have $p(x) = Q/P = 1/x$ and $q(x) = R/P = 1/x$ and thus $x = 0$ is a singular point. Now $xp(x) = 1$ and $x^2q(x) = x$ are analytic at $x = 0$, so $x = 0$ is a regular singular point.

13b. From Part(a) we have $p_0 = 1$ and $q_0 = 0$ so the indicial equation is $F(r) = r(r-1) + r = r^2$ and thus the exponents of the singularity are $r_1 = r_2 = 0$.

13c. From Eq.(4) we have $y(x) = \sum_{n=0}^{\infty} a_nx^n$. Differentiating and substituting into the D.E. yields

$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} + \sum_{n=1}^{\infty} na_nx^{n-1} - \sum_{n=0}^{\infty} a_nx^n = 0$. Adjusting the exponents and the index appropriately then gives

$(a_1 - a_0) + \sum_{n=1}^{\infty}[(n+1)^2 a_{n+1} - a_n]x^n = 0$. Thus $a_1 = a_0$ and

$a_{n+1} = a_n/(n+1)^2$. Setting n = 1,2,3 we obtain

$y_1(x) = 1 + x + x^2/4 + x^3/36 + \cdots$.

For the second solution we use Eq.(18):

$y = y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n$. When this is substituted in the D.E.

we obtain $2y_1' + \sum_{n=2}^{\infty} n(n-1)b_n x^{n-1} + \sum_{n=1}^{\infty} nb_n x^{n-1} - \sum_{n=0}^{\infty} b_n x^n = 0$,

where we have used the fact that y_1 is a solution so that all terms involving lnx add to zero. Condensing the terms, as above, we get

$2(1+x/2+x^2/12+ \cdots) + b_1 + \sum_{n=1}^{\infty}[(n+1)^2 b_{n+1} - b_n]x^n = 0$. The coefficient of each power of x must be zero so $b_1+2 = 0$, $1+4b_2-b_1 = 0$ and $(1/6)+9b_3-b_2 = 0$. Solving we get $b_1 = -2$, $b_2 = (b_1-1)/4 = -3/4$ and $b_3 = (b_2-1/6)/9 = -11/108$, which give the desired solution.

17a. We have $p(x) = \dfrac{\sin x}{x^2}$ and $q(x) = -\dfrac{\cos x}{x^2}$, so that x = 0 is a singular point. Note that $xp(x) = (\sin x)/x \to 1 = p_0$ as $x \to 0$ and $x^2 q(x) = -\cos x \to -1 = q_0$ as $x \to 0$. In order to assert that x = 0 is a regular singular point we must demonstrate that $xp(x)$ and $x^2q(x)$, with $xp(x) = 1$ at $x = 0$ and $x^2q(x) = -1$ at $x = 0$, have convergent power series (are analytic) about x = 0. We know that cosx is analytic so we need only consider (sinx)/x. Now

$\sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ for $-\infty < x < \infty$ so

$(\sin x)/x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n+1)!$ and hence is analytic.

Thus we conclude that x = 0 is a regular singular point.

17b. From part a) it follows that the indicial equation is $r(r-1) + r - 1 = r^2 - 1 = 0$ and the roots are $r_1 = 1$, $r_2 = -1$.

17c. To find the first few terms of the solution corresponding to $r_1 = 1$, assume that

$$y(x) = x\sum_{n=0}^{\infty} a_n x^n$$

$$= x(a_0 + a_1x + a_2x^2 + \ldots) = a_0x + a_1x^2 + a_2x^3 + \ldots .$$

Substituting this series for y in the D.E. and expanding sinx and cosx about x = 0 yields

$x^2(2a_1 + 6a_2x + 12a_3x^2 + 20a_4x^3 + \ldots) +$
$(x - x^3/3! + x^5/5! - \ldots)(a_0 + 2a_1x + 3a_2x^2 + 4a_3x^3 + 5a_4x^4 + \ldots) - (1 - x^2/2! + x^4/4! - \ldots)(a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + \ldots) = 0$. Collecting terms we have $(a_0 - a_0)x +$
$(2a_1 + 2a_1 - a_1)x^2 + (6a_2 + 3a_2 - a_0/6 - a_2 + a_0/2)x^3 +$
$(12a_3 + 4a_3 - 2a_1/6 - a_3 + a_1/2)x^4 +$
$(20a_4 + 5a_4 - 3a_2/6 + a_0/120 - a_4 + a_2/2 - a_0/24)x^5 + \ldots = 0$.
Simplifying yields, $3a_1x^2 + (8a_2 + a_0/3)x^3 + (15a_3 + a_1/6)x^4 + (24a_4 - a_0/30)x^5 + \ldots = 0$. Thus, $a_1 = 0$, $a_2 = -a_0/4!$, $a_3 = 0$, $a_4 = a_0/6!, \ldots$. Hence
$y_1(x) = x - x^3/4! + x^5/6! + \ldots$ where we have set $a_0 = 1$.

For the second solution we use a variation of Eq.(24) similar to Eq.(18):

$$y_2(x) = ay_1(x)\ln x + x^{-1}(1+\sum_{n=1}^{\infty} c_n x^n)$$

$$= ay_1(x)\ln x + \frac{1}{x} + c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots, \text{ so}$$

$y_2' = ay_1'\ln x + ay_1x^{-1} - x^{-2} + c_2 + 2c_3x + 3c_4x^2 + \ldots$, and
$y_2'' = ay_1''\ln x + 2ay_1'x^{-1} - ay_1x^{-2} + 2x^{-3} + 2c_3 + 3c_4x + \ldots$.
When these are substituted in the given D.E. the terms including lnx will appear as
$a[x^2y_1'' + (\sin x)y_1' - (\cos x)y_1]$, which is zero since y_1 is a solution. For the remainder of the terms, use
$y_1 = x - x^3/24 + x^5/720$ and the cosx and sinx series as shown earlier to obtain

$-c_1 + (2/3+2a)x + (3c_3+c_1/2)x^2 + (4/45+c_2/3+8c_4)x^3 + ... = 0$. These yield $c_1 = 0$, $a = -1/3$, $c_3 = 0$, and $c_4 = -c_2/24 - 1/90$. We may take $c_2 = 0$, since this term will simply generate $y_1(x)$ over again. Thus

$y_2(x) = -\frac{1}{3}y_1(x)\ln x + x^{-1} - \frac{1}{90}x^3$. If a computer algebra system is used, then additional terms in each series may be obtained without much additional effort. The next terms, in each case, are shown here:

$$y_1(x) = x - \frac{x^3}{24} + \frac{x^5}{720} - \frac{43x^7}{1451520} + \ldots \text{ and}$$

$$y_2(x) = -\frac{1}{3}y_1(x)\ln x + \frac{1}{x}[1 - \frac{x^4}{90} + \frac{41x^6}{120960} - \ldots].$$

18a. We first write the D.E. in the standard form as given for Theorem 5.6.1 except that we are expanding in powers of $(x-1)$ rather than powers of x:

$(x-1)^2y'' + (x-1)[(x-1)/2\ln x]y' + [(x-1)^2/\ln x]y = 0$. Since $\ln 1 = 0$, $x = 1$ is a singular point. To show it is a regular singular point of this D.E. we must show that $(x-1)/\ln x$ is analytic at $x = 1$; it will then follow that $(x-1)^2/\ln x = (x-1)[(x-1)/\ln x]$ is also analytic at $x = 1$. If we expand $\ln x$ in a Taylor series about $x = 1$ we find that $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$.

Thus

$$(x-1)/\ln x = [1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - \ldots]^{-1} = 1 + \frac{1}{2}(x-1) + \ldots$$

has a power series expansion about $x = 1$, and hence is analytic.

18b. We can use the above result to obtain the indicial equation at $x = 1$. We have

$$(x-1)^2y'' + (x-1)[\frac{1}{2} + \frac{1}{4}(x-1) + \ldots]y' + [(x-1) + \frac{1}{2}(x-1)^2 + \ldots]y = 0.$$

Thus $p_0 = 1/2$, $q_0 = 0$ and the indicial equation is $r(r-1) + r/2 = 0$. Hence $r = 1/2$ and $r = 0$.

18c. In order to find the first three non-zero terms in a series solution corresponding to $r = 1/2$, it is better to keep the differential equation in its original form and to substitute the above power series for $\ln x$:

$$[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots]y'' + \frac{1}{2}y' + y = 0.$$

Next we substitute $y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + \ldots$ and collect coefficients of like powers of $(x-1)$ which are then set equal to zero. This requires some algebra before we find that $6a_1/4 + 9a_0/8 = 0$ and $5a_2 + 5a_1/8 - a_0/12 = 0$. These equations yield $a_1 = -3a_0/4$ and $a_2 = 53a_0/480$. With $a_0 = 1$ we obtain the solution

$$y_1(x) = (x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + \ldots .$$

18d. Since the radius of convergence of the Taylor Series for $(x-1)/\ln x$ is 1, we would expect $\rho = 1$.

20a. If we write the D.E. in the standard form as given in Theorem 5.6.1 we obtain $x^2y'' + x[\alpha/x]y' + [\beta/x]y = 0$ where $xp(x) = \alpha/x$ and $x^2q(x) = \beta/x$. Neither of these terms are analytic at $x = 0$ so $x = 0$ is an irregular singular point.

20b. Substituting $y = x^r\sum_{n=0}^{\infty}a_nx^n$ in $x^3y'' + \alpha xy' + \beta y = 0$ gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_nx^{n+r} + \beta\sum_{n=0}^{\infty}a_nx^{n+r} = 0.$$

Shifting the index in the first series and collecting coefficients of common powers of x we obtain

$$(\alpha r + \beta)a_0x^r + \sum_{n=1}^{\infty}\{(n+r-1)(n+r-2)a_{n-1} + [\alpha(n+r) + \beta]a_n\}x^{n+r} = 0.$$

Thus the indicial equation is $\alpha r + \beta = 0$ with the single root $r = -\beta/\alpha$.

20c. From part b, the recurrence relation is

$$a_n = -\frac{(n+r-1)(n+r-2)a_{n-1}}{\alpha(n+r) + \beta}, \quad n = 1,2,\ldots$$

$$= -\frac{(n - \frac{\beta}{\alpha} - 1)(n - \frac{\beta}{\alpha} - 2)a_{n-1}}{\alpha n}, \text{ for } r = -\beta/\alpha.$$

For $\frac{\beta}{\alpha} = -1$, $a_n = -\frac{n(n-1)a_{n-1}}{\alpha n}$, so that $a_1 = 0 \cdot a_0 = 0$. Since all other a_n are multiples of a_1, and hence are zero, $y(x) = x$ is the solution. Similarly for $\frac{\beta}{\alpha} = 0$, $a_n = -\frac{(n-1)(n-2)}{\alpha n}a_{n-1}$ and again for $n = 1$ $a_1 = 0$ and $y(x) = 1$ is the solution. Continuing in this fashion, we see that the series solution will terminate for β/α any positive integer as well as 0 and -1. For other values of β/α, we have $\left|\frac{a_n}{a_{n-1}}\right| = \frac{(n-\frac{\beta}{\alpha}-1)(n-\frac{\beta}{\alpha}-2)}{\alpha n}$, which approaches ∞ as $n \to \infty$ and thus the ratio test yields a zero radius of convergence.

21b. Substituting $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E. gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1-s} + \beta\sum_{n=0}^{\infty}a_n x^{n+r+2-t} = 0.$$

If $s = 2$ and $t = 2$ the first term in each of the three series is $r(r-1)a_0x^r$, $\alpha r a_0 x^{r-1}$, and $\beta a_0 x^r$, respectively. Thus the indicial equation is $F(r) = \alpha r a_0 = 0$, which requires $r = 0$. Hence there is at most one solution of the assumed form.

21d. In order for the indicial equation to be quadratic in r it is necessary that the first term in the first series contribute to the indicial equation. This means that the first term in the second and the third series cannot have powers less than x^r. The first terms are $r(r-1)a_0x^r$, $\alpha r a_0 x^{r+1-s}$, and $\beta a_0 x^{r+2-t}$, respectively. Thus if $s \le 1$ and $t \le 2$ the quadratic term will appear in the indicial equation.

Section 5.7, Page 301

1. It is clear that $x = 0$ is a singular point. The D.E. is in the standard form given in Theorem 5.6.1 with $xp(x) = 2$ and $x^2q(x) = x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 2r]a_0x^r + \sum_{n=1}^{\infty} [(r+n)(r+n+1)a_n + a_{n-1}]x^{r+n} = 0.$$

The indicial equation is $F(r) = r(r+1) = 0$ with roots $r_1 = 0$, $r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$, $n = 1,2,\ldots$ if $F(r+n) \neq 0$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2),\ldots,$ and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\ldots F(r+n)$, provided $F(r+n) \neq 0$ for $n = 1,2,\ldots$. For the case $r_1 = 0$, we have $a_n(0) = (-1)^n a_0/F(1)F(2) \ldots F(n) = (-1)^n a_0/n!(n+1)!$ so one solution is $y_1(x) = \sum_{n=0}^{\infty}(-1)^n x^n/n!(n+1)!$ where we have set $a_0 = 1$.

If we try to use the above recurrence relation for the case $r_2 = -1$ we find that $a_n(-1) = -a_{n-1}/n(n-1)$, which is undefined for $n = 1$. Thus we must follow the procedure described at the end of Sect. 5.6 to calculate a second solution of the form given in Eq.(24). Specifically, we use Eqs.(19) and (20) of Sect.5.6 to calculate a and $c_n(r_2)$, where $r_2 = -1$. Since $r_1 - r_2 = 1 = N$, we have $a_N(r) = a_1(r) = -1/F(r+1)$, with $a_0 = 1$. Hence

$$a = \lim_{r \to -1} [(r+1)(-1)/F(r+1)] = \lim_{r \to -1}[-(r+1)/(r+1)(r+2)] = -1.$$

Next

$$c_n(-1) = \frac{d}{dr}[(r+1)a_n(r)]\Big|_{r=-1} = (-1)^n \frac{d}{dr}\left[\frac{(r+1)}{F(r+1)\ \ldots\ F(r+n)}\right]\Big|_{r=-1},$$

where we again have set $a_0 = 1$. Observe that $(r+1)/F(r+1)\ldots F(r+n)=1/[(r+2)^2(r+3)^2\ldots(r+n)^2(r+n+1)]=1/G_n(r)$. Hence $c_n(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. Notice that $G_n(-1) = 1^2 \cdot 2^2 \cdot 3^2 \ldots (n-1)^2 n = (n-1)!n!$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \ldots + 1/(n-1)] + 1/n = H_n + H_{n-1}$. Thus $c_n(-1) = (-1)^{n+1}(H_n + H_{n-1})/(n-1)!n!$. From Eq.(24) of Sect.5.6 we obtain the second solution

$$y_2(x) = -y_1(x)\ln x + x^{-1}[1 - \sum_{n=1}^{\infty}(-1)^n(H_n + H_{n-1})x^n/n!(n-1)!].$$

2. It is clear that $x = 0$ is a singular point. The D.E. is in the standard form given in Theorem 5.6.1 with $xp(x) = 3$ and $x^2q(x) = 1+x$. Both are analytic at $x = 0$, so $x = 0$ is a regular singular point. Substituting

$y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shifting indices

appropriately, and collecting coefficients of like powers of x yields

$$[r(r-1) + 3r + 1]a_0x^r + \sum_{n=1}^{\infty}\{[(r+n)(r+n+2) + 1]a_n + a_{n-1}\}x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 + 2r + 1 = (r+1)^2 = 0$ with the double root $r_1 = r_2 = -1$. Treating a_n as a function of r, we see that $a_n(r) = -a_{n-1}(r)/F(r+n)$, $n = 1,2,\ldots$. Thus $a_1(r) = -a_0/F(r+1)$, $a_2(r) = a_0/F(r+1)F(r+2),\ldots$, and $a_n(r) = (-1)^n a_0/F(r+1)F(r+2)\ldots F(r+n)$. Setting $r = -1$ we find that $a_n(-1)=(-1)^n a_0/(n!)^2$, $n = 1,2,\ldots$. Hence one solution is $y_1(x) = x^{-1}\sum_{n=0}^{\infty}(-1)^n x^n/(n!)^2$ where we have set $a_0 = 1$. To find a second solution we follow the procedure described in Sect.5.6 for the case when the roots of the indicial equation are equal. Specifically, the second solution will have the form given in Eq.(17) of that section. We must calculate $a_n'(-1)$. If we let $G_n(r) = F(r+1)\ldots F(r+n) = (r+2)^2(r+3)^2\ldots(r+n+1)^2$ and take $a_0 = 1$, then $a_n'(-1) = (-1)^n[1/G_n(r)]'$ evaluated $r = -1$. Hence $a_n'(-1) = (-1)^{n+1}G_n'(-1)/G_n^2(-1)$. But $G_n(-1) = (n!)^2$ and $G_n'(-1)/G_n(-1) = 2[1/1 + 1/2 + 1/3 + \ldots\ 1/n] = 2H_n$. Thus a second solution is $y_2(x) = y_1(x)\ln x - 2x^{-1}\sum_{n=1}^{\infty}(-1)^n H_n x^n/(n!)^2$.

3. The roots of the indicial equation are r_1 and $r_2 = 0$ and thus the analysis is similar to that for Prob. 2.

4. The roots of the indicial equation are $r_1 = -1$ and $r_2 = -2$ and thus the analysis is similar to that for Prob.1.

5. Since $x = 0$ is a regular singular point, substitute $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ in the D.E., shift indices appropriately, and collect coefficients of like powers of x to obtain

$$[r^2 - 9/4]a_0x^r + [(r+1)^2 - 9/4]a_1x^{r+1} + \sum_{n=2}^{\infty} \{[(r+n)^2 - 9/4]a_n + a_{n-2}\} x^{n+r} = 0.$$

The indicial equation is $F(r) = r^2 - 9/4 = 0$ with roots $r_1 = 3/2$, $r_2 = -3/2$. Treating a_n as a function of r we see that $a_n(r) = -a_{n-2}(r)/F(r+n)$, $n = 2,3,..$ if $F(r+n) \neq 0$. For the case $r_1 = 3/2$, $F(r_1+1)$, which is the coefficient of x^{r_1+1} is $\neq 0$ so we must set $a_1 = 0$. It follows that $a_3 = a_5 = \ldots = 0$. For the even coefficients, set $n = 2m$ so $a_{2m}(3/2) = -a_{2m-2}(3/2)/F(3/2 + 2m) = -a_{2m-2}/2^2m(m+3/2)$, $m = 1,2\ldots$. Thus $a_2(3/2) = -a_0/2^2\cdot 1(1 + 3/2)$, $a_4(3/2) = a_0/2^4\cdot 2!(1 + 3/2)(2 + 3/2),\ldots,$ and $a_{2m}(3/2) = (-1)^m/2^{2m}m!\cdot(1 + 3/2)\ldots(m + 3/2)$. Hence one solution is

$$y_1(x) = x^{3/2}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1 + 3/2)(2 + 3/2)\ldots(m + 3/2)}(\frac{x}{2})^{2m}],$$

where we have set $a_0 = 1$. For this problem, the roots r_1 and r_2 of the indicial equation differ by an integer: $r_1 - r_2 = 3/2 - (-3/2) = 3$. Hence we can anticipate that there may be difficulty in calculating a second solution corresponding to $r = r_2$. This difficulty will occur in calculating $a_3(r) = -a_1(r)/F(r+3)$ since when $r = r_2 = -3/2$ we have $F(r_2+3) = F(r_1) = 0$. However, in this problem we are fortunate because $a_1 = 0$ and it will not be necessary to use the theory described at the end of Sect.5.6. Notice for

$r = r_2 = -3/2$ that the coefficient of x^{r_2+1} is $[(r_2+1)^2 - 9/4]a_1$, which does not vanish unless $a_1 = 0$. Thus the recurrence relation for the odd coefficients yields $a_5 = -a_3/F(7/2)$, $a_7 = -a_5/F(11/2) = a_3/F(11/2)F(7/2)$ and so forth. Substituting these terms into the assumed form we see that a multiple of $y_1(x)$ has been obtained and thus we may take $a_3 = 0$ without loss of generality. Hence $a_3 = a_5 = a_7 = \ldots = 0$. The even coefficients are given by $a_{2m}(-3/2) = -a_{2m-2}(-3/2)/F(2m - 3/2)$, $m = 1,2\ldots$.
Thus $a_2(-3/2) = -a_0/2^2\cdot1\cdot(1 - 3/2)$,
$a_4(-3/2) = a_0/2^4 2!(1 - 3/2)(2 - 3/2),\ldots,$ and
$a_{2m}(-3/2) = (-1)^m a_0/2^{2m}m!(1 - 3/2)(2 - 3/2)\ \ldots\ (m - 3/2)$.
Thus a second solution is

$$y_2(x) = x^{-3/2}[1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m!(1 - 3/2)(2 - 3/2)\ \ldots\ (m - 3/2)}(\frac{x}{2})^{2m}].$$

7. Apply the ratio test:

$$\lim_{m\to\infty}\frac{|(-1)^{m+1}\ x^{2m+2}/2^{2m+2}[(m+1)!]^2|}{|(-1)^m\ x^{2m}/2^{2m}(m!)^2|} = |x^2|\ \lim_{m\to\infty}\frac{1}{2^2(m+1)^2} = 0$$

for every x. Thus the series for $J_0(x)$ converges absolutely for all x.

12. If $\xi = \alpha x^\beta$, then $dy/dx = \frac{1}{2}x^{-1/2}f + x^{1/2}f'\alpha\beta x^{\beta-1}$ where f' denotes $df/d\xi$. Find d^2y/dx^2 in a similar fashion and use algebra to show that f satisfies the D.E. $\xi^2 f'' + \xi f' + [\xi^2 - \upsilon^2]f = 0$, which is the Bessel Equation of order υ.

13. To compare $y'' - xy = 0$ with the D.E. of Prob.12, we must multiply by x^2 to get $x^2y'' - x^3y = 0$. Thus $2\beta = 3$, $\alpha^2\beta^2 = -1$ and $1/4 - \upsilon^2\beta^2 = 0$. Hence $\beta = 3/2$, $\alpha = 2i/3$ and $\upsilon^2 = 1/9$ which yields the desired result.

14. First we verify that $J_0(\lambda_j x)$ satisfies the D.E. We know that $J_0(t)$ is a solution of the Bessel equation of order zero:

$$t^2J_0''(t) + tJ_0'(t) + t^2J_0(t) = 0 \text{ or}$$

$$J_0''(t) + t^{-1}J_0'(t) + J_0(t) = 0.$$

Let $t = \lambda_j x$. Then

$$\frac{d}{dx} J_0(\lambda_j x) = \frac{d}{dt} J_0(t)\frac{dt}{dx} = \lambda_j J_0'(t)$$

$$\frac{d^2}{dx^2} J_0(\lambda_j x) = \lambda_j \frac{d}{dt}[J_0'(t)]\frac{dt}{dx} = \lambda_j^2 J_0''(t).$$

Substituting $y = J_0(\lambda_j x)$ in the given D.E. and making use of these results, we have

$$\lambda_j^2 J_0''(t) + (\lambda_j/t)\, \lambda_j J_0'(t) + \lambda_j^2 J_0(t) =$$

$$\lambda_j^2 [J_0''(t) + t^{-1}J_0'(t) + J_0(t)] = 0.$$

Thus $y = J_0(\lambda_j x)$ is a solution of the given D.E. For the second part of the problem we follow the hint. First, rewrite the D.E. by multiplying by x to yield $xy'' + y' + \lambda_j^2 xy = 0$, which can be written as $(xy')' = -\lambda_j^2 xy$. Now let $y_i(x) = J_0(\lambda_i x)$ and $y_j(x) = J_0(\lambda_j x)$ and we have, respectively:

$$(xy_i')' = -\lambda_i^2 xy_i$$

$$(xy_j')' = -\lambda_j^2 xy_j.$$

Now, multiply the first equation by y_j, the second by y_i, integrate each from 0 to 1, and subtract the second from the first:

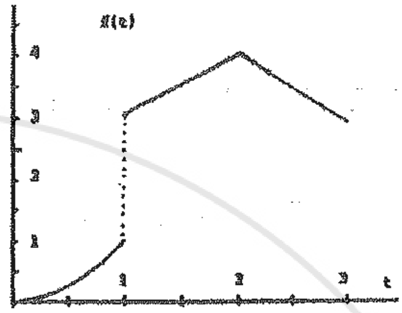
$$\int_0^1 [y_j(xy_i')' - y_i(xy_j')']dx = -(\lambda_i^2 - \lambda_j^2)\int_0^1 xy_i y_j dx.$$

If we integrate each term on the left side once by parts and note that $y_i = y_j = 0$ at $x = 1$, we find that the left side of this equation is identically zero. Hence the right side is identically zero and for $\lambda_i \neq \lambda_j$ this gives the desired result.

CHAPTER 6

Section 6.1, Page 311

1. The graph of f(t) is shown. Since the function is continuous on each interval, but has a jump discontinuity at t = 1, f(t) is piecewise continuous.



2. Note that $\lim_{t\to 1^+} (t-1)^{-1} = \infty$.

5b. Since t^2 is continuous for $0 \le t \le A$ for any positive A and since $t^2 \le e^{at}$ for any a > 0 and for t sufficiently large, it follows from Theorem 6.1.2 that $\mathcal{L}\{t^2\}$ exists for s > 0. $\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}t^2dt = \lim_{M\to\infty}\int_0^M e^{-st}t^2dt$

$$= \lim_{M\to\infty}\left[\frac{-t^2}{s}e^{-st}\Big|_0^M + \frac{2}{s}\int_0^M e^{-st}t\,dt\right]$$

$$= \lim_{M\to\infty}\frac{-M^2}{s}e^{-sM} + \frac{2}{s}\lim_{M\to\infty}\left[-\frac{1}{s}te^{-st}\Big|_0^M + \frac{1}{s}\int_0^M e^{-st}dt\right]$$

$$= 0 + 2\lim_{M\to\infty}\frac{-M}{s^2}e^{-sM} + \frac{2}{s^2}\lim_{M\to\infty} -\frac{1}{s}e^{-st}\Big|_0^M = \frac{2}{s^3}.$$

6. That f(t) = cosat satisfies the hypotheses of Theorem 6.1.2 can be verified by recalling that $|\cos at| \le 1$ for all t. To determine $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$ we must integrate by parts twice to get

$$\int_0^\infty e^{-st}\cos at\,dt = \lim_{M\to\infty}\left[(-s^{-1}e^{-st}\cos at + as^{-2}e^{-st}\sin at)\Big|_0^M - (a^2/s^2)\int_0^M e^{-st}\cos at\,dt\right].$$

Evaluating the first two terms and letting $M \to \infty$ yields

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{1}{s} - (a^2/s^2)\int_0^\infty e^{-st}\cos at\,dt$$ and hence

$[1+a^2/s^2]\int_0^\infty e^{-st}\cos at\,dt = 1/s$, s > 0. Division by $[1+a^2/s^2]$ and simplification yields the desired solution.

9. From the definition for coshbt we have $\pounds\{e^{at}\cosh bt\} = \pounds\{\frac{1}{2}[e^{(a+b)t} + e^{(a-b)t}]\}$. Using the linearity property of $\pounds$, Eq.(6), the right side becomes $\frac{1}{2}\pounds\{e^{(a+b)t}\} + \frac{1}{2}\pounds\{e^{(a-b)t}\}$ which can be evaluated using the result of Ex. 5 and thus

$$\pounds\{e^{at}\cosh bt\} = \frac{1/2}{s-(a+b)} + \frac{1/2}{s-(a-b)}, \text{ for } s-a > |b|$$
$$= \frac{s-a}{(s-a)^2-b^2}.$$

13. We write $\sin at = (e^{iat} - e^{-iat})/2i$, then the linearity of the Laplace transform operator allows us to write $\pounds\{e^{at}\sin bt\} = (1/2i)\pounds\{e^{(a+ib)t}\}-(1/2i)\pounds\{e^{(a-ib)t}\}$. Each of these two terms can be evaluated by using the result of Ex. 5, where we now have to require s to be greater than the real part of the complex numbers $a \pm ib$ in order for the integrals to converge. Complex algebra then gives the desired result. An alternate method of evaluation would be to use integration on the integral appearing in the definition of $\pounds\{e^{at}\sin bt\}$, but that method requires integration by parts twice.

16. Before starting note that both tsinat and tcosat satisfy Condition 2 of Theorem 6.1.2 and thus the $\lim_{M\to\infty}$ in the follwing are both zero. Using integration by parts twice we have $F(s) = \int_0^\infty (t\sin at)e^{-st}dt$

$$= \lim_{M\to\infty}(t\sin at)\Big|_0^M + \frac{1}{s}\int_0^\infty(\sin at + at\cos at)e^{-st}dt$$

$$= \lim_{M\to\infty}(\sin at + at\cos at)\Big|_0^M + \frac{1}{s^2}\int_0^\infty(2at\cos at - at\sin at)e^{-st}dt$$

$$= \frac{2a}{s^s}\pounds\{\cos at\} - \frac{a}{s^2}F(s) \text{ and thus } (1 + \frac{a}{s^2})F(s) = \frac{2as}{s^2(s^2+a^2)}.$$

Solving for F(s) we find $F(s) = \dfrac{2as}{(s^2+a^2)^2}$.

21. The integral $\int_0^A (t^2 + 1)^{-1}dt$ can be evaluated in terms of the arctan function and then Eq. (1) can be used. To illustrate Theorem 6.1.1, however, consider that

$\frac{1}{t^2+1} < \frac{1}{t^2}$ for $t \geq 1$ and, from Ex. 3, $\int_1^\infty t^{-2}dt$ converges

and hence $\int_1^\infty (t^2 + 1)^{-1}dt$ also converges.

$\int_0^1 (t^2 + 1)^{-1}dt$ is finite and hence does not affect the

convergence of $\int_0^\infty (t^2 + 1)^{-1}dt$ at infinity.

25. If we let $u = f$ and $dv = e^{-st}dt$ then

$$F(s) = \int_0^\infty e^{-st}f(t)dt = \lim_{M \to \infty} -\frac{1}{s}e^{-st}f(t)\Big|_0^M + \frac{1}{s}\int_0^\infty e^{-st}f'(t)dt$$

$$= \frac{1}{s}f(0) + \frac{1}{s}\int_0^\infty e^{-st}f'(t)dt.$$ This last integral converges (and is thus finite) using an argument similar to that given to establish Theoremm 6.1.2. Hence $\lim_{s \to \infty} F(s) = 0$.

27a. let $x = st$ and so that $dx = sdt$. Then use the definition of $\Gamma(P+1)$ from Prob. 26.

27b. From Part a, $\mathcal{L}\{t^n\} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}x^n dx = \frac{n}{s^{n+1}}\int_0^\infty e^{-x}x^{n-1}dx$

$= \frac{n!}{s^{n+1}}\int_0^\infty e^{-x}dx$, using integration by parts successively. Evaluation of the last integral yields the desired answer.

27c. From part a, $\mathcal{L}\{t^{-1/2}\} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-x}x^{-1/2}dx$. Let $x = y^2$, then

$2dy = x^{-1/2}dx$ and thus $\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-y^2}dy$.

27d. Use the definition of $\mathcal{L}\{t^{1/2}\}$ and integrate by parts once to get $\mathcal{L}\{t^{1/2}\} = (1/2s)\mathcal{L}\{t^{-1/2}\}$. The result follows from part c.

<u>Section 6.2, Page 320</u>

Problems 1 through 10 are solved by using partial fractions and algebra to manipulate the given function into a form matching one of the functions appearing in the middle column of Table 6.2.1.

2. We have $\frac{4}{(s-1)^3} = 2\frac{2!}{(s-1)^{2+1}}$ and thus the inverse Laplace transform is $2t^2e^t$, using line 11.

4. We have $\frac{3s}{s^2-s-6} = \frac{3s}{(s-3)(s+2)} = \frac{9/5}{s-3} + \frac{6/5}{s+2}$ using partial fractions. Thus $(9/5)e^{3t} + (6/5)e^{-2t}$ is the inverse transform, from line 2.

7. We have $\frac{2s+1}{s^2-2s+2} = \frac{2s+1}{(s-1)^2+1} = \frac{2(s-1)}{(s-1)^2+1} + \frac{3}{(s-1)^2+1}$, where we first used the concept of completing the square (in the denominator) and then added and subtracted appropriately to put the numerator in the desired form. Lines 9 and 10 may now be used to find the desired result.

In each of the Problems 11 through 23 it is assumed that the I.V.P. has a solution $y = \phi(t)$ which, with its first two derivatives, satisfies the conditions of the Corollary 6.2.2.

11. Take the Laplace transform of the D.E., using Eq.(1) and Eq.(2), to get
$s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = 0.$
Using the I.C. and solving for Y(s) we obtain
$Y(s) = \frac{s-2}{s^2-s-6}$. Following the pattern of Eq.(12) we have
$\frac{s-2}{s^2-s-6} = \frac{a}{s+2} + \frac{b}{s-3} = \frac{a(s-3)+b(s+2)}{(s+2)(s-3)}$. Equating like powers in the numerators we find $a+b = 1$ and $-3a + 2b = -2$. Thus $a = 4/5$ and $b = 1/5$ and
$Y(s) = \frac{4/5}{s+2} + \frac{1/5}{s-3}$, which yields the desired solution using Table 6.2.1.

14. Taking the Laplace transform we have
$s^2Y(s) - sy(0) - y'(0) - 4[sY(s)-y(0)] + 4Y(s) = 0$. Using the I.C. and solving for Y(s) we find $Y(s) = \frac{s-3}{s^2-4s+4}$. Since the denominator is a perfect square, the partial fraction form is
$\frac{s-3}{s^2-4s+4} = \frac{a}{(s-2)^2} + \frac{b}{s-2}$. Solving for a and b, as shown in examples of this section or in Prob. 11, we find $a = -1$ and

$b = 1$. Thus $Y(s) = \dfrac{1}{s-2} - \dfrac{1}{(s-2)^2}$, from which we find $y(t) = e^{2t} - te^{2t}$ (lines 2 and 11 in Table 6.2.1).

15. Note that $Y(s) = \dfrac{2s-4}{s^2-2s+4} = \dfrac{2s-4}{(s-1)^2+3} = \dfrac{2(s-1)}{(s-1)^2+3} - \dfrac{2}{(s-1)^2+3}$. Three formulas in Table 6.2.1 are now needed: $F(s-c)$ (with $c = 1$) in line 14 in conjunction with the ones for cosat and sinat (with $a = \sqrt{3}$), lines 5 and 6.

17. The Laplace transform of the D.E. is $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] + 6[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + Y(s) = 0$. Using the I.C. and solving for $Y(s)$ we find $Y(s) = \dfrac{s^2 - 4s + 7}{s^4-4s^3+6s^2-4s+1}$. The correct partial fraction form for this is $\dfrac{a}{(s-1)^4} + \dfrac{b}{(s-1)^3} + \dfrac{c}{(s-1)^2} + \dfrac{d}{s-1}$. Setting this equal to $Y(s)$ above and equating the numerators we have $s^2-4s+7 = a + b(s-1) + c(s-1)^2 + d(s-1)^3$. Solving for a,b,c, and d and use of line 11 in Table 6.2.1 yields the desired solution.

20. The Laplace transform of the D.E. is $s^2Y(s) - sy(0) - y'(0) + \omega^2Y(s) = s/(s^2+4)$. Applying the I.C. and solving for $Y(s)$ we get $Y(s) = s/[(s^2+4)(s^2+\omega^2)] + s/(s^2+\omega^2)$. Decomposing the first term by partial fractions we have

$$Y(s) = \frac{s}{(\omega^2-4)(s^2+4)} - \frac{s}{(\omega^2-4)(s^2+\omega^2)} + \frac{s}{s^2+\omega^2}$$

$$= (\omega^2-4)^{-1}\left[\frac{(\omega^2-5)s}{s^2+\omega^2} + \frac{s}{s^2+4}\right].$$

Then, using lines 5 and 6 of Table 6.1.2, we have $y = (\omega^2-4)^{-1}[(\omega^2-5)\cos\omega t + \cos 2t]$.

22. Solving for $Y(s)$ we find $Y(s) = \dfrac{1}{(s-1)^2+1} + \dfrac{1}{(s+1)[(s-1)^2+1]}$. Using partial fractions on the second term we obtain $Y(s) = \dfrac{1}{(s-1)^2+1} + \dfrac{1}{5}\left[\dfrac{1}{s+1} - \dfrac{s-3}{(s-1)^2+1}\right]$. Combining the

first and third terms we have

$$Y(s) = \frac{1}{5}\left[\frac{1}{s+1} - \frac{s-1}{(s-1)^2+1} + \frac{7}{(s-1)^2+1}\right].$$

Hence, $y = (1/5)(e^{-t} - e^{t}\cos t + 7e^{t}\sin t)$.

24. Under the standard assumptions, the Lapace transform of the left side of the D.E. is $s^2Y(s) - sy(0) - y'(0) + 4Y(s)$, or $(s^2 + 4)Y(s) - s$. To transform the right side we must revert to the definition of the Laplace transform to determine $\int_0^\infty e^{-st}f(t)dt$. Since $f(t)$ is piecewise continuous we are able to calculate $\mathcal{L}\{f(t)\}$ by

$$\int_0^\infty e^{-st}f(t)dt = \int_0^\pi e^{-st}\,dt + \lim_{M\to\infty}\int_\pi^M (e^{-st})(0)dt$$

$$= \int_0^\pi e^{-st}dt = (1 - e^{-\pi s})/s.$$

Hence, the Laplace transform $Y(s)$ of the solution is given by $Y(s) = s/(s^2+4) + (1 - e^{-\pi s})/s(s^2+4)$.

27b. The Taylor series for f about $t = 0$ is

$$f(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!},$$ which is obtained from part(a) by dividing each term of the sine series by t. Also, f is continuous for $t > 0$ since $\lim_{t\to 0+}\frac{\sin t}{t} = 1$. Assuming that we can compute the Laplace transform of f term by term, we obtain $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n+1)!}\}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathcal{L}\{t^{2n}\} = \sum_{n=0}^{\infty}\frac{(-1)^n 2n!}{(2n+1)!s^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,\frac{1}{s^{2n+1}},$$

which converges for $s > 1$. The Taylor series for arctan x is given by $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, for $|x| < 1$. Comparing $\mathcal{L}\{f(t)\}$ with the Taylor series for arctanx, we conclude that $\mathcal{L}\{f(t)\} = \arctan(1/s)$, $s > 1$.

30. Setting $n = 2$ in Prob. 28b, we have

$$\mathcal{L}\{t^2\sin bt\} = \frac{d^2}{ds^2}\left[\frac{b}{s^2+b^2}\right] = \frac{d}{ds}\left[\frac{-2bs}{(s^2+b^2)^2}\right] = \frac{-2b}{(s^2+b^2)^2} + \frac{8bs^2}{(s^2+b^2)^3}$$

$$= \frac{2b(3s^2-b^2)}{(s^2+b^2)^3}.$$

32. Using the result of Prob. 28a, repeatedly, we have

$$\mathcal{L}\{te^{at}\} = -\frac{d}{ds}(s-a)^{-1} = (s-a)^{-2},$$

$$\mathcal{L}\{t^2e^{at}\} = -\frac{d}{ds}(s-a)^{-2} = 2(s-a)^{-3}, \text{ and}$$

$$\mathcal{L}\{t^3e^{at}\} = -\frac{d}{ds}2(s-a)^{-3} = 3!(s-a)^{-4}.$$ Continuing in this fashion, or using induction, we obtain the desired result.

36a. Taking the Laplace transform of the D.E. we obtain

$$\mathcal{L}\{y''\} - \mathcal{L}\{ty\} = \mathcal{L}\{y''\} + \mathcal{L}\{-ty\}$$

$$= s^2Y(s) - sy(0) - y'(0) + Y'(s) = 0.$$

Hence, Y satisfies $Y' + s^2Y = s$.

38a. From Eq(i) we have $A_k = \lim\limits_{s\to r_k} (s-r_k)\dfrac{P(r_k)}{Q(r_k)}$, since Q has distinct zeros. Thus $A_k = P(r_k)\lim\limits_{s\to r_k}\dfrac{s-r_k}{Q(r_k)} = \dfrac{P(r_k)}{Q'(r_k)}$, by L'Hospital's Rule.

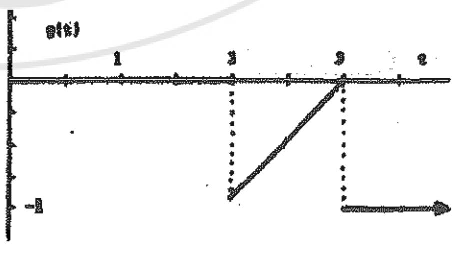
38b. Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s-r_k}\right\} = e^{r_k t}$, the result follows.

Section 6.3, Page 328

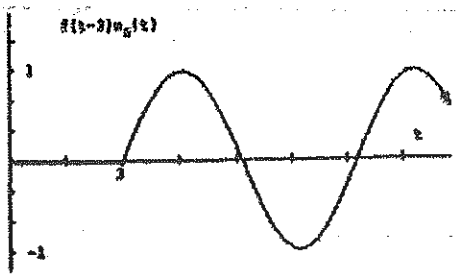
2. From the definition of $u_c(t)$ we have:

$$g(t) = (t-3)u_2(t) - (t-2)u_3(t)$$

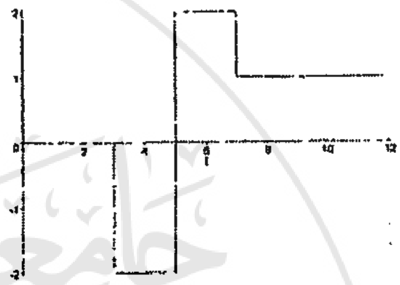
$$= \begin{cases} 0 - 0 = 0, & 0 \le t < 2 \\ (t-3) - 0 = t-3, & 2 \le t < 3. \\ (t-3) - (t-2) = -1, & 3 \le t \end{cases}$$



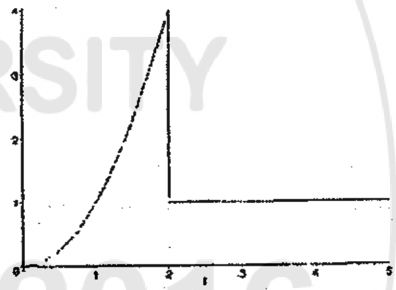
4. As indicated in the discussion following Eq.(2), the unit step function can be used to translate a given function f, with domain $t\geq 0$, a distance c to the right by the multiplication $u_c(t)f(t-c)$. Hence the required graph of $y = u_3(t)f(t-3)$ for $f(t) = \sin t$ is shown.



7. There are step changes at $t = 3,5,7$. Thus we use $u_3(t)$, $u_5(t)$ and $u_7(t)$ multiplied by the appropriate step size to obtain $f(t) = -2u_3(t)+4u_5(t)-u_7(t)$.



10. There is a discontinuity at $t = 2$ so we use $u_2(t)$ multiplied by $(1-t^2)$, which subtracts the existing t^2 function. Thus $f(t) = t^2 + (1-t^2)u_2(t)$.



14. In order to use Theorem 6.3.1 we must write f(t) in terms of $u_c(t)$. Since $t^2 - 2t + 2 = (t-1)^2 + 1$ (by completing the square), we can write $f(t) = u_1(t)g(t-1)$, where $g(t) = t^2+1$. Now applying Theorem 6.3.1 we have $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)g(t-1)\} = e^{-s}\mathcal{L}\{g(t)\} = e^{-s}(2/s^3 + 1/s)$.

20. Use partial fractions to write $F(s) = e^{-2s}\frac{1}{3}[\frac{1}{s-1} - \frac{1}{s+2}]$. For ease in calculations let us define $G(s) = (s-1)^{-1}$ and $H(s) = (s+2)^{-1}$. Then $F(s) = [e^{-2s}G(s) - e^{-2s}H(s)]/3$. Using the fact that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$ and applying Theorem 6.3.1, we have $F(s) = [e^{-2s}\mathcal{L}\{e^t\} - e^{-2s}\mathcal{L}\{e^{-2t}\}]/3$. Thus $F(s) = [\mathcal{L}\{u_2(t)e^{(t-2)}\} - \mathcal{L}\{u_2(t)e^{-2(t-2)}\}]/3$. Using the linearity of the Laplace transform, we have $\mathcal{L}\{f(t)\} = \mathcal{L}\{u_2(t)[e^{t-2} - e^{-2(t-2)}]/3\}$. Hence, $f(t) = [u_2(t)(e^{t-2} - e^{-2(t-2)})]/3$. An alternate method is to complete the square in the denominator:

$F(s) = \dfrac{e^{-2s}}{(s+1/2)^2 - 9/4}$. From line 7, Table 6.2.1, this gives $f(t) = (2/3)u_2(t)e^{-(t-2)/2}\sinh\frac{3}{2}(t-2)$, which can be shown to be the same as that found above.

21. Completing the square in the denominator we have $F(s) = \dfrac{2e^{-2s}(s-1)}{(s-1)^2 + 1}$. Since the inverse of $\dfrac{s-1}{(s-1)^2 + 1}$ is $e^t\cos t$ we conclude that $f(t) = 2u_2(t)e^{t-2}\cos(t-2)$ since e^{-2s} causes a shift of 2 units on the t axis (Theorem 6.3.1).

27. By completing the square in the denominator of F we can write $F(s) = \dfrac{2s+1}{(2s+1)^2+4}$. This has the form $G(2s+1)$ where $G(u) = \dfrac{u}{u^2+4}$. We must find $\mathcal{L}^{-1}\{G(2s+1)\}$. Applying the results of Prob.25(c), with a = 2 and b = 1, we have $\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}e^{-t/2}\cos(\frac{2t}{2})$, since $\mathcal{L}^{-1}\{G(s)\} = \cos 2t$.

28. If the approach of Prob.27 is used we find $f(t) = (1/3)e^{2t/3}\sinh(t/3)$, which is equivalent to the given answer using the definition of sinht.

33. Assuming that term-by-term integration of the infinite series is permissible and recalling that $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$ for $s > 0$, we have $\mathcal{L}\{f(t)\} = (1/s) + \sum_{k=1}^{\infty}(-1)^k \mathcal{L}\{u_k(t)\}$

$= (1/s) + \sum_{k=1}^{\infty}(-1)^k (e^{-ks}/s) = \frac{1}{s}\sum_{k=0}^{\infty}(-e^{-s})^k$. We recognize the last series as the geometric series, $\sum_{k=0}^{\infty}ar^k$, with $a = 1$ and $r = -e^{-s}$. This series converges to $[1/(1+e^{-s})]$ if $|r| < 1$ (or $s > 0$). Hence,

$\mathcal{L}\{f(t)\} = (\frac{1}{s})\dfrac{1}{1+e^{-s}}$, $s > 0$.

34. Using the definition of the Laplace transform we have $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$. Since f is periodic with period T, we have $f(t+T) = f(t)$. This suggests that we rewrite the improper integral as

$$\int_0^\infty e^{-st}f(t)dt = \sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)dt.$$

The periodicity of f also suggests that we make the change of variable $t = r + nT$. Hence,

$$F(s) = \sum_{n=0}^{\infty}\int_0^T e^{-s(r+nT)}f(r+nT)dr = \sum_{n=0}^{\infty}(e^{-sT})^n\int_0^T e^{-rs}f(r)dr,$$

where we have used the fact that $f(r+nT) = f(r+(n-1)T) = \ldots = f(r+T) = f(r)$, since f is periodic. We recognize this last series as the geometric series, $\sum_{n=0}^{\infty} au^n$, with $a = \int_0^T e^{-rs}f(r)dr$ and $u = e^{-sT}$. The geometric series converges to $a/(1-u)$ for $|u| < 1$ and consequently we obtain

$$F(s) = (1 - e^{-sT})^{-1}\int_0^T e^{-rs}f(r)dr, \quad s > 0.$$

36. The function f is periodic with period 2. The result of Prob.34 gives us $\mathcal{L}\{f(t)\} = \int_0^2 e^{-st}f(t)dt/(1-e^{-2s})$. Calculating the integral we have

$$\begin{aligned}\int_0^2 e^{-st}f(t)dt &= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt\\ &= (1-e^{-s})/s + (e^{-2s}-e^{-s})/s\\ &= (e^{-2s}-2e^{-s}+1)/s\\ &= (1-e^{-s})^2/s.\end{aligned}$$

Since the denominator of $\mathcal{L}\{f(t)\}$, $1 - e^{-2s}$, may be written as $(1-e^{-s})(1+e^{-s})$ we obtain the desired answer.

<u>Section 6.4, Page 336</u>

1a. f(t) can be written in the form $f(t) = 1 - u_{3\pi}(t)$ and thus the Laplace transform of the D.E. is $(s^2+1)Y(s) - sy(0) - y'(0) = (1/s) - e^{-3\pi s}/s$. Using the I.C. and solving for Y(s), we obtain $Y(s) = (s^2+1)^{-1} + [s(s^2+1)]^{-1} - e^{-3\pi s}/s(s^2+1)$. Using partial fractions on the second and third terms we find $Y(s)=(s^2+1)^{-1}+(1/s)-s/(s^2+1)-e^{-3\pi s}/s+e^{-3\pi s}s/(s^2+1)$. The inverse transform of the first three terms can be obtained directly from Table 6.2.1. Using Theorem 6.3.1 to find the inverse transform of the last two terms, we have $\mathcal{L}^{-1}\{e^{-3\pi s}/s\} = u_{3\pi}(t)g(t-3\pi)$ where

$g(t) = \mathcal{L}^{-1}\{1/s\} = 1$ and

$\mathcal{L}^{-1}\{e^{-3\pi s}s/(s^2+1)\} = u_{3\pi}(t)h(t-3\pi)$ where

$h(t) = \mathcal{L}^{-1}\{s/(s^2+1)\} = \cos t$. Hence,

$$y = 1 + \sin t - \cos t + u_{3\pi}(t)[\cos(t-3\pi) - 1]$$
$$= 1 + \sin t - \cos t - u_{3\pi}(t)[1 + \cos t].$$

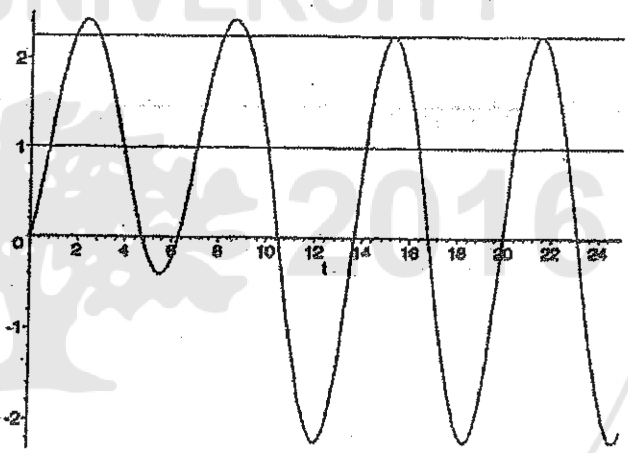
1b. The graph of the forcing function is a unit pulse for $0 \le t < 3\pi$ and 0 thereafter. The graph of the solution is composed of two segments. The first, for $0 \le t < 3\pi$, is a sinusoid oscillating about 1, which represents the system response to a unit forcing function and the given initial conditions. For $t \ge 3\pi$, the forcing function, $f(t)$, is zero and the "initial" conditions are

$$y(3\pi) = \lim_{t\to 3\pi^-}(1 + \sin t - \cos t) = 2$$

and

$$y'(3\pi) = \lim_{t\to 3\pi^-}(\cos t + \sin t) = -1.$$

For $t \ge 3\pi$ the system response is $y(t) = \sin t - 2\cos t$, which is a sinusoid of magnitude $\sqrt{5}$ oscillating about zero.



3a. According to Theorem 6.3.1,

$\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = e^{-2\pi s}/(s^2+1)$.

Transforming the D.E., we have

$(s^2+4)Y(s) - sy(0) - y'(0) = 1/(s^2+1) - e^{-2\pi s}/(s^2+1)$.

Using the I.C. and solving for $Y(s)$, we obtain

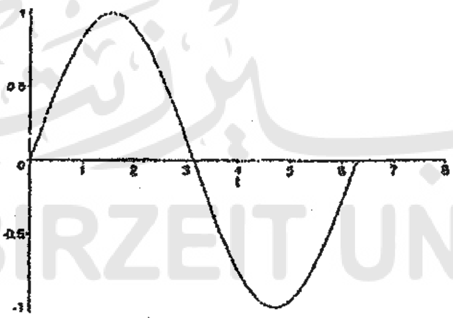
$Y(s) = (1-e^{-2\pi s})/(s^2+1)(s^2+4)$. We apply partial fractions to write

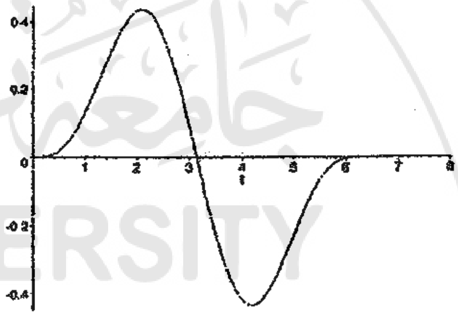
$Y(s) = [(s^2+1)^{-1} - (s^2+4)^{-1} - e^{-2\pi s}(s^2+1)^{-1} + e^{-2\pi s}(s^2+4)^{-1}]/3$.

We compute the inverse transform of the first two terms directly from Table 6.2.1 after noting that $(s^2+4)^{-1} = (1/2)[2/(s^2+4)]$. We apply Theorem 6.3.1 to the

last two terms to obtain the solution,
$y = (1/3)\{\sin t - (1/2)\sin 2t - u_{2\pi}(t)[\sin(t-2\pi) - (1/2)\sin 2(t-2\pi)]\}$.
This may be simplified, using trigonometric identities, to $y = [(2\sin t - \sin 2t)(1-u_{2\pi}(t))]/6$.

3b. Note that the forcing function is $\sin t - \sin(t-2\pi) = 0$ for $t \geq 2\pi$. The solution is $y(t) = 2\sin t - \sin 2t$ for $0 \leq t < 2\pi$. Thus $y(2\pi^-) = 0$ and $y'(2\pi^-) = 2\cos 2\pi - 2\cos 4\pi = 0$. Hence the "initial" value problem for $t \geq 2\pi$ is $y'' + 4y = 0$, $y(2\pi) = 0$, $y'(2\pi) = 0$, which has the trivial solution $y = 0$ for $t \geq 2\pi$ [Note that $1 - u_{2\pi}(t) = 0$ for $t \geq 2\pi$, so this agrees with the above solution].





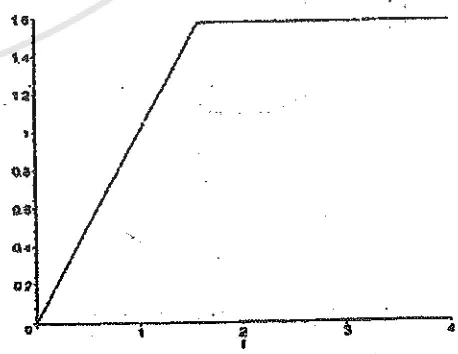
8a. Taking the Laplace transform, applying the I.C. and using Theorem 6.3.1 we have $(s^2+s+5/4)Y(s) = (1-e^{-\pi s/2})/s^2$. Thus

$$Y(s) = \frac{1-e^{-\pi s/2}}{s^2(s^2+s+5/4)}$$

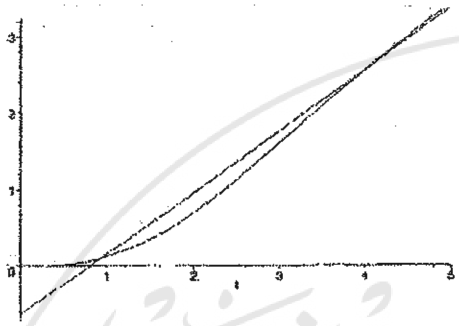
$$= (1-e^{-\pi s/2})\left[\frac{4/5}{s^2} - \frac{16/25}{s} + \frac{(16/25)s-4/25}{(s+1/2)^2+1}\right]$$

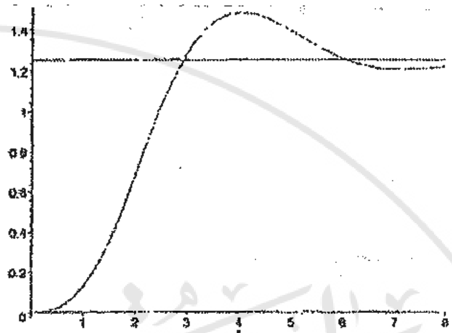
$= (1-e^{-\pi s/2})H(s)$, where we have used partial fractions and completed the square in the denominator of the last term. Since the numerator of the last term of H can be written as $\frac{16}{25}[(s+1/2) - 3/4]$, we see that $\mathcal{L}^{-1}\{H(s)\} = (4/25)(5t - 4 + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t)$, which yields the desired solution.

8b. The graph of the forcing function is a ramp ($f(t) = t$) for $0 \leq t < \pi/2$ and a constant ($f(t) = \pi/2$) for $t \geq \pi/2$.



The solution will be a damped sinusoid oscillating about the "ramp" $(20t-16)/25$ [which is the first two terms in the answer of Part a.] for $0 \le t < \pi/2$ and oscillating about $2\pi/5$ for $t \ge \pi/2$. The first part is shown on the left over an 'expanded' interval to detect this behaviour. The graph of the solution is on the right.





10. Note that $g(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$. Proceeding as in Prob. 8 we find

$Y(s) = (1+e^{-\pi s})\dfrac{1}{(s^2+1)(s^2+s+5/4)}$. The correct partial fraction expansion of the quotient is $\dfrac{as+b}{s^2+1} + \dfrac{cs+d}{s^2+s+5/4}$, where $a+c = 0$, $a+b+d = 0$, $(5/4)a+b+c = 0$ and $(5/4)b+d = 1$ by equating coefficients. Solving for a,b,c,d and following the steps of Prob. 8 yields the desired solution.

16b. Taking the Laplace transform of the D.E. we obtain $U(s^2 + s/4 + 1) = k(e^{-3s/2}-e^{-5s/2})/s$, since the I.C. are zero. Solving for U and using partial fractions yields $U(s) = k(e^{-3s/2}-e^{-5s/2})(\dfrac{1}{s} - \dfrac{s+1/4}{s^2+s/4+1})$. Thus, if

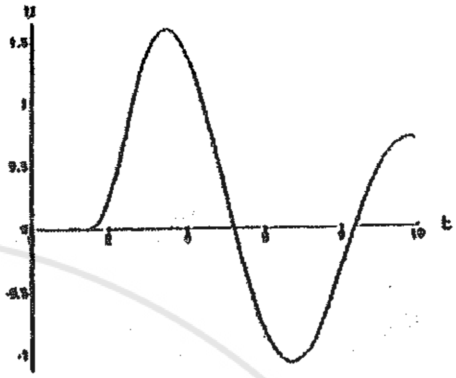
$H(s) = (\dfrac{1}{s} - \dfrac{s+1/4}{s^2+s/4+1})$, then, since

$s^2 + s/4+ 1 = (s+1/8)^2 + 63/64$,

$h(t) = 1 - e^{-t/8}(\cos\dfrac{3\sqrt{7}}{8}t + \dfrac{\sqrt{7}}{21}\sin\dfrac{3\sqrt{7}}{8}t)$ and

$u(t) = ku_{3/2}(t)h(t-3/2) - ku_{5/2}(t)h(t-5/2)$.

16c. In all cases the plot will be zero for $0 \le t < 3/2$. For $3/2 \le t < 5/2$ the plot will be the system response (damped sinusoid) to a step input of magnitude k. For $t \ge 5/2$, the plot will be the system response to the I.C. $u(5^-/2)$, $u'(5^-/2)$ with no forcing function. The graph shown is for $k = 2$. Varying k will just affect the amplitude. Note that the amplitude never reaches 2, which would be the steady state response for the step input $2u_{3/2}(t)$. Note also that the solution and its derivative are continuous at $t = 5/2$.



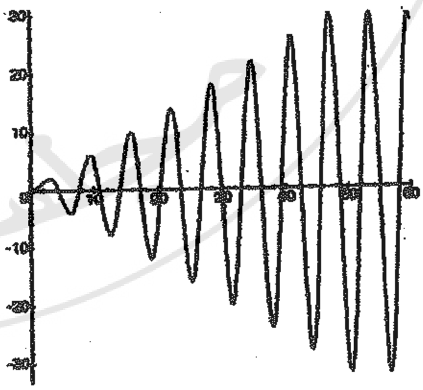
19a. The graph on $0 \le t < 6\pi$ will depend on how large n is. For instance, if $n = 2$ then

$$f(t) = \begin{cases} 1 & 0 \le t < \pi, \quad 2\pi \le t \\ -1 & \pi \le t < 2\pi \end{cases}.$$

For $n = 4$, $f(t) = \begin{cases} 1 & 0 \le t < \pi,\ 2\pi \le t < 3\pi,\ 4\pi \le t \\ -1 & \pi \le t < 2\pi,\ 3\pi \le t < 4\pi \end{cases}.$

19b. Taking the Laplace transform of the D.E. and using the I.C. we have $Y(s) = \dfrac{1}{s(s^2+1)}[1 + 2\sum_{k=1}^{n}(-1)^k e^{-\pi ks}]$, since $\mathcal{L}\{u_{\pi k}(t)\} = \dfrac{e^{-\pi ks}}{s}$. By partial fractons $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, so $y(t) = 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k u_{\pi k}(t)[1 - \cos(t-\pi k)]$, using line 13 in Table 6.2.1.

19c. For $0 \le t < \pi$, $y(t) = 1 - \cos t$, which peaks at $t = \pi$, just when the forcing function changes from +1 to -1. Thus the forcing function "reinforces" the natural motion, creating a "resonance". This occurs at each π interval until $t > 15\pi$, at which time the forcing function no longer changes and the solution continues oscillating about -1. If $n = 16$, the solution would continue to oscillate about +1 for $t > 16\pi$.



19d Since $\cos(t-\pi k) = (-1)^k \cos t$, the solution in part b can be written as

$$y(t) = 1 - \cos t + 2\sum_{k=1}^{n}(-1)^k u_{\pi k}(t) - 2\sum_{k=1}^{n}(-1)^{2k} \cos t$$

$$= 1 - \cos t - 2n\cos t + 2\sum_{k=1}^{n}(-1)^k u_{\pi k}(t)$$ which diverges for

$n\to\infty$, since the nth erm does not approach zero.

20. In this case

$$Y(s) = \frac{1}{s(s^2+.1s+1)}\left[1 + 2\sum_{k=1}^{n}(-1)^k e^{-\pi ks}\right].$$ Using partial fractions we have

$$H(s) = \frac{1}{s(s^2+.1s+1)} = \frac{1}{s} - \frac{s+.1}{s^2+.1s+1}$$

$$= \frac{1}{s} - \frac{s+.05}{(s+.05)^2+b^2} - \frac{.05}{(s+.05)^2+b^2},$$ where

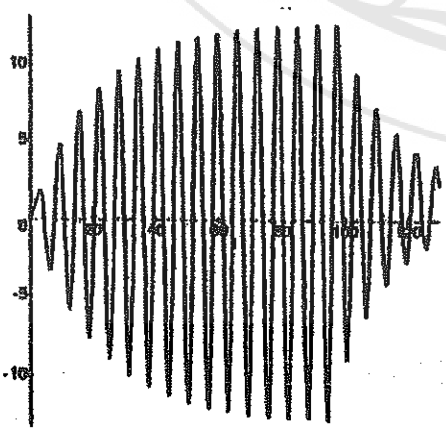
$b^2 = [1-(.05)^2] = .9975$. Now let

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 1 - e^{-.05t}\cos bt - \frac{.05}{b}e^{-.05t}\sin bt.$$ Hence,

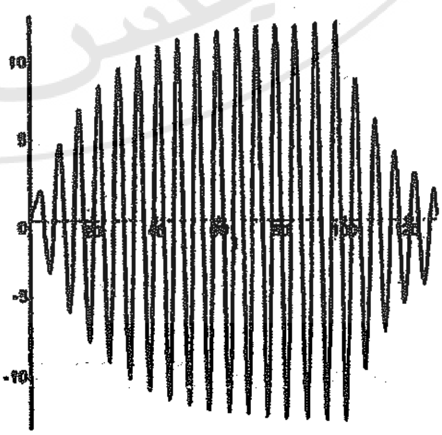
$$y(t) = h(t) + 2\sum_{k=1}^{n}(-1)^k u_{\pi k}(t)\, h(t-\pi k),$$ and thus, for $t > n\pi$ the solution will be approximated by

$\pm 1 - Ae^{-.05(t-n\pi)} \cos[b(t-n\pi) + \delta]$, and therefore converges as $t\to\infty$.

20a. y(t) for n = 30

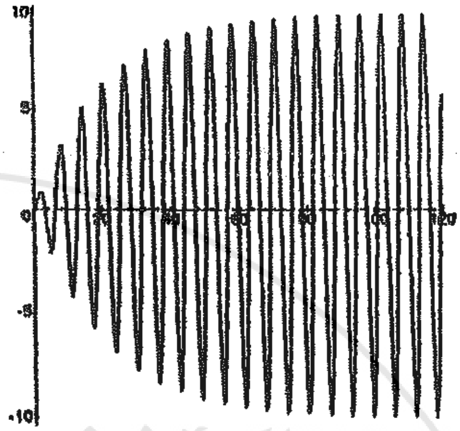


y(t) for n = 31



20b. From the graph of part a, A ≅ 12.5 and the frequency is 2π.

20c. From the graph (or analytically) A = 10 and the frequency is 2π.



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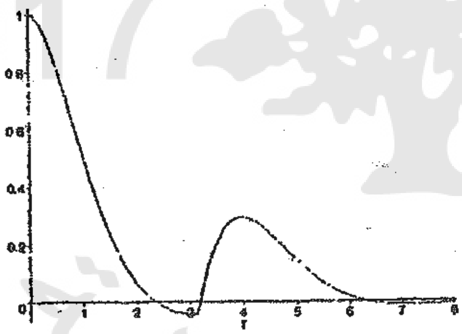
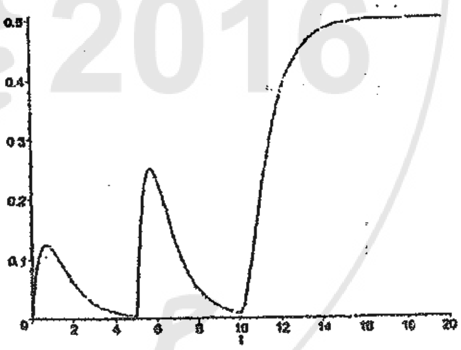
1a. Proceeding as in Ex. 1, we take the Laplace transform of the D.E. and apply the I.C.:
$(s^2 + 2s + 2)Y(s) = s + 2 + e^{-\pi s}$. Thus,

$Y(s) = (s+2)/[(s+1)^2 + 1] + e^{-\pi s}/[(s+1)^2 + 1]$. We write the first term as $(s+1)/[(s+1)^2 + 1] + 1/[(s+1)^2 + 1]$. Applying Theorem 6.3.1 and using Table 6.2.1, we obtain the solution, $y = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)e^{-(t-\pi)}\sin(t-\pi)$.

1b.



3b.



3a. Taking the Laplace transform and using the I.C. we have

$$(s^2+3s+2)Y(s) = \frac{1}{2} + e^{-5s} + \frac{e^{-10s}}{s}. \quad \text{Thus}$$

$$Y(s) = \frac{1/2}{s^2+3s+2} + \frac{e^{-5s}}{s^2+3s+2} + e^{-10s}\left(\frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}\right) \text{ and}$$

hence

$$y(t) = \frac{1}{2}h(t) + u_5(t)h(t-5) + u_{10}(t)\left[\frac{1}{2}+\frac{1}{2}e^{-2(t-10)}-e^{-(t-10)}\right]$$

where $h(t) = e^{-t} - e^{-2t}$.

5a. The Laplace transform of the D.E. is

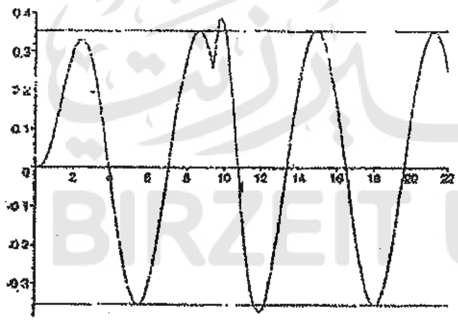
$(s^2+2s+3)Y(s) = \dfrac{1}{s^2+1} + e^{-3\pi s}$, so

$Y(s) = \dfrac{1}{(s^2+1)(s^2+2s+3)} + e^{-3\pi s}\left[\dfrac{1}{s^2+2s+3}\right]$. Using partial fractions or a computer algebra system we obtain

$$y(t) = \frac{1}{4}\sin t - \frac{1}{4}\cos t + \frac{1}{4}e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_{3\pi}(t)h(t-3\pi),$$

where $h(t) = e^{-t}\sin\sqrt{2}\,t$.

5b.

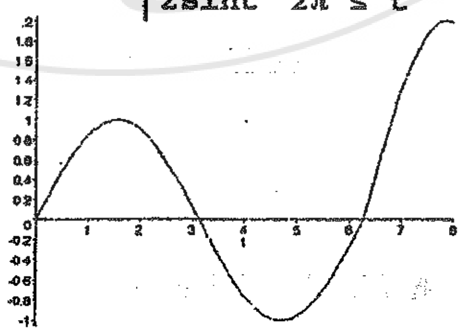


Notice the response to the impulse at $t = 3\pi$ and that the effect of the impulse is negligible after $t = 15$ or 16.

7a. Taking the Laplace transform of the D.E. yields $(s^2+1)Y(s) - y'(0) = \int_0^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$. Since $\delta(t-2\pi) = 0$ for $t \neq 2\pi$ the integral on the right is equal to $\int_{-\infty}^\infty e^{-st}\delta(t-2\pi)\cos t\,dt$ which equals $e^{-2\pi s}\cos 2\pi$ from Eq.(16). Substituting for $y'(0)$ and solving for $Y(s)$ gives $Y(s) = \dfrac{1}{s^2+1} + \dfrac{e^{-2\pi s}}{s^2+1}$ and hence

$$y(t) = \sin t + u_{2\pi}(t)\sin(t-2\pi) = \begin{cases} \sin t & 0 \le t < 2\pi \\ 2\sin t & 2\pi \le t \end{cases}.$$

7b. The effect of the impulse is barely seen at $t = 2\pi$.



10. Follow the same steps as in the solution for Prob. 7.

13a. From Eq. (22) y(t) will complete one cycle when

$\sqrt{15}(t-5)/4 = 2\pi$ or $T = t - 5 = 8\pi/\sqrt{15}$, which is consistent with the plot in Fig. 6.5.3. Since an impulse causes a discontinuity in the first derivative, we need to find the value of y' at $t = 5$ and $t = 5 + T$. From Eq. (22) we have, for $t \geq 5$,

$$y' = e^{-(t-5)/4}\left[\frac{-1}{2\sqrt{15}}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{1}{2}\cos\frac{\sqrt{15}}{4}(t-5)\right].$$ Thus

$y'(5) = \frac{1}{2}$ and $y'(5+T) = \frac{1}{2}e^{-T/4}$. Since the original impulse, $\delta(t-5)$, caused a discontinuity in y' of 1/2 at $t = 5$, we must choose the impulse at $t = 5 + T$ to be $-e^{-T/4}$, which is equal and opposite to y' at $5 + T$.

13b. Now consider $2y'' + y' + 2y = \delta(t-5) + k\delta(t-5-T)$ with $y(0) = 0$, $y'(0) = 0$. Using the results of Ex. 1 we have

$$y(t) = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{2k}{\sqrt{15}}u_{5+T}(t)e^{-(t-5-T)/4}\sin\frac{\sqrt{15}}{4}(t-5-T)$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t)\sin\frac{\sqrt{15}}{4}(t-5)+ku_{5+T}(t)e^{T/4}\sin\frac{\sqrt{15}}{4}(t-5-T)\right]$$

$$= \frac{2}{\sqrt{15}}e^{-(t-5)/4}\left[u_5(t)+ke^{T/4}u_{5+T}(t)\right]\sin\frac{\sqrt{15}}{4}(t-5),$$ since

$T = 8\pi/\sqrt{15}$. If $k = -e^{-T/4}$ then $1 + ke^{-T/4} = 0$ for $t > 5 + T$ and $y(t) \equiv 0$, which is the desired result.

17b. We have $(s^2+1)Y(s) = \sum_{k=1}^{20} e^{-k\pi s}$ so that $Y(s) = \sum_{k=1}^{20}\frac{e^{-ks}}{s^2+1}$ and

hence $y(t) = \sum_{k=1}^{20} u_{k\pi}(t)\sin(t-k\pi)$

$= u_\pi(t)\sin(t-\pi) + u_{2\pi}(t)\sin(t-2\pi) + \ldots + u_{20\pi}\sin(t-20\pi)$.

For $0 \leq t < \pi$, $y(t) = 0$.

For $\pi \leq t < 2\pi$, $y(t) = \sin(t-\pi) = -\sin t$. For $2\pi \leq t < 3\pi$, $y(t) = \sin(t-\pi)+\sin(t-2\pi) = -\sin t+\sin t = 0$. Due to the periodicity of $\sin t$, the solution will exhibit this behavior in alternate intervals for $0 \leq t < 20\pi$.

17c. After $t = 20\pi$ the solution remains at zero.

21b. Taking the transform and using the I.C. we have

$$(s^2+1)Y(s) = \sum_{k=1}^{15} e^{-(2k-1)\pi} \text{ so that } Y(s) = \sum_{k=1}^{15} \frac{e^{-(2k-1)\pi}}{s^2+1}.$$

Thus $y(t) = \sum_{k=1}^{15} u_{(2k-1)\pi}(t)\sin[t-(2k-1)\pi]$.

21c. For $t > 29\pi$, $y(t) = \sin(t-\pi) + \sin(t-3\pi) +\ldots+ \sin(t-29\pi)$
$= -\sin t - \sin t -\ldots-\sin t$
$= -15\sin t$.

25b. Substituting for f(t) in the integral of Part a, we have

$y = \int_0^t e^{-(t-\tau)}\delta(\tau-\pi)\sin(t-\tau)d\tau$. We know that the integration variable is always less than t (the upper limit) and thus for $t < \pi$ we have $\tau < \pi$ and thus

$\delta(\tau-\pi) = 0$. Hence $y = 0$ for $t < \pi$. For $t > \pi$ utilize Eq.(16).

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1c. Using the format of Eqs.(2) and (3) we have

$$f*(g*h) = \int_0^t f(t-\tau)(g*h)(\tau)d\tau$$
$$= \int_0^t f(t-\tau)[\int_0^\tau g(\tau-\eta)h(\eta)d\eta]d\tau$$
$$= \int_0^t [\int_\eta^t f(t-\tau)g(\tau-\eta)d\tau]h(\eta)(d\eta).$$

The last double integral is obtained from the previous line by interchanging the order of the η and τ integrations. Making the change of variable $\omega = \tau - \eta$ on the inside integral yields

$$f*(g*h) = \int_0^t [\int_0^{t-\eta} f(t-\eta-\omega)g(\omega)d\omega]h(\eta)d\eta$$
$$= \int_0^t [(f*g)(t-\eta)]h(\eta)d\eta = (f*g)*h.$$

3. Use the trigonometric identity $\sin A\sin B = (1/2)[\cos(A-B)-\cos(A+B)]$ with $A = t-\tau$ and $B = \tau$.

4. It is possible to determine f(t) explicitly by using integration by parts and then find its transform F(s). However, it is much more convenient to apply Theorem 6.6.1. Let us define $g(t) = t^2$ and $h(t) = \cos 2t$. Then, $f(t) = \int_0^t g(t-\tau)h(\tau)d\tau$. Using Table 6.2.1, we have

$G(s) = \mathcal{L}\{g(t)\} = 2/s^3$ and $H(s) = \mathcal{L}\{h(t)\} = s/(s^2+4)$. Hence, by Theorem 6.6.1,

$$\mathcal{L}\{f(t)\} = F(s) = G(s)H(s) = \frac{2}{s^2(s^2+4)}.$$

8. As was done in Ex. 1 think of F(s) as the product of s^{-4} and $(s^2+1)^{-1}$ which, according to Table 6.2.1, are the transforms of $t^3/6$ and $\sin t$, respectively. Hence, by Theorem 6.6.1, the inverse transform of F(s) is $f(t) = (1/6)\int_0^t (t-\tau)^3 \sin\tau d\tau$.

14. We take the Laplace transform of the D.E. and apply the I.C.: $(s^2 + 2s + 2)Y(s) = \alpha/(s^2 + \alpha^2)$. Solving for Y(s), we have $Y(s) = \frac{\alpha}{s^2+\alpha^2} \cdot \frac{1}{(s+1)^2+1}$, where the second factor has been written in a convenient way by completing the square. Thus Y(s) is seen to be the product of the transforms of $\sin\alpha t$ and $e^{-t}\sin t$ respectively. Hence, according to Theorem 6.6.1, $y = \int_0^t e^{-(t-\tau)}\sin(t-\tau)\sin\alpha\tau d\tau$.

16. Proceeding as in Prob.14 we obtain

$$(s^2+s+5/4)Y(s) - s = \frac{1-e^{-\pi s}}{s} \text{ or}$$

$$Y(s) = \frac{s}{s^2+s+5/4} + \frac{1-e^{-\pi s}}{s(s^2+s+5/4)}$$

$$= \frac{(s+1/2) - 1/2}{(s+1/2)^2+1} + \frac{1-e^{-\pi s}}{s} \cdot \frac{1}{(s+1/2)^2+1}$$

where the first term is obtained by completing the square in the denominator and the second term is written as the product of two terms whose inverse transforms are known, so that Theorem 6.6.1 can be used. Note that $\mathcal{L}^{-1}\{(1-e^{-\pi s})/s\} = 1 - u_\pi(t)$. Also note that a different form of the same solution would be obtained by writing the second term as $(1-e^{-\pi s})(\frac{a}{s} + \frac{bs + c}{(s+1/2)^2+1})$ and solving for a, b and c. In this case $\mathcal{L}^{-1}\{1-e^{-\pi s}\} = \delta(t) - \delta(t-\pi)$ from Sect.6.5.

18. Taking the Laplace transform, using the I.C. and solving, we have $Y(s) = \frac{s+3}{(s+1)(s+2)} + \frac{s}{(s^2+\alpha^2)(s+1)(s+2)}$. As in Prob. 16, there are several correct ways the second term can be treated in order to use the convolution integral. In order to obtain the desired answer, write the second term as

$\frac{s}{s^2+\alpha^2}(\frac{a}{s+1} + \frac{b}{s+2})$ and solve for a and b.

21. To find $\Phi(s)$ you must recognize the integral that appears in the equation as a convolution integral. Taking the transform of both sides then yields

$\Phi(s) + K(s)\Phi(s) = F(s)$, or $\Phi(s) = \frac{F(s)}{1+K(s)}$.

24a Again, we recognize $\int_0^t (t-\xi)\phi(\xi)d\xi$ as the convolution of t and $\phi(t)$. Thus, taking the transform of both sides, we get $\Phi(s) - \frac{1}{s^2}\Phi(s) = \frac{1}{s}$. Solving for Φ we get

$\Phi(s) = \frac{s}{s^2-1}$ and hence $\phi(t) = \cosh t$ from line 8 of Table 6.2.1.

24b. Taking the derivative of the given equation we get

$\phi'(t) - \frac{d}{dt}\int_0^t (t-\xi)\phi(\xi)d\xi = 0$, or

$\phi'(t) - \int_0^t \phi(\xi)d\xi - (t-\xi)\phi(\xi)|_{\xi=t} = 0$, using Leibnitz's Rule. Since the last term is zero, we have

$$\phi'(t) - \int_0^t \phi(\xi)d\xi = 0, \quad \text{(i)},$$

and then

$$\phi''(t) - \phi(t) = 0, \quad \text{(ii)}.$$

The original equation gives $\phi(0) = 1$ and Eq.(i) gives $\phi'(0) = 0$. Using Eq.(ii) the IVP is $\phi''(t) - \phi(t) = 0$ with $\phi(0) = 1$ and $\phi'(0) = 0$.

24c. The solution to $\phi''(t) - \phi(t) = 0$, $\phi(0) = 1$, $\phi'(0) = 0$ is also $\phi(t) = \dfrac{e^t+e^{-t}}{2} = \cosh t$.

26a. Taking the transform, we obtain

$$s\Phi(s) - \phi(0) + \frac{1}{s^2}\Phi(s) = \frac{1}{s^2},$$ and thus

$$\Phi(s) = \frac{1}{s^3+1} = \frac{1}{(s+1)(s^2-s+1)} = \frac{1/3}{s+1} - \frac{(s-2)/3}{s^2-s+1}.$$

Completing the square in the denominator of the last term and using lines 9 and 10 of Table 6.2.1 we obtain

$$\phi(t) = e^{-t}/3 - (1/3)e^{t/2}\cos(\sqrt{3}\,t/2) + (1/\sqrt{3})e^{t/2}\sin(\sqrt{3}\,t/2).$$

26b. From the given equation $\phi'(0) = 0$. Differentiating we get $\phi''(t) + \int_0^t \phi(\xi)d\xi = 1$ and $\phi''(0) = 1$. Another differentiation gives $\phi'''(t) + \phi(t) = 0$. Thus the IVP is $\phi'''(t) + \phi(t) = 0$, $\phi(0) = 0$, $\phi'(0) = 0$, and $\phi''(0) = 1$.

26c. The characteristic equation is $r^3 + 1 = 0$, so the roots are the cube roots of -1, which are -1 and $(1 \pm \sqrt{3})/2$.

CHAPTER 7

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2. As in Ex. 1, let $x_1 = u$ and $x_2 = u'$. Then $x_1' = x_2$ and $x_2' = u'' = 3\sin t - .5u' - 2u = -2x_1 - .5x_2 + 3\sin t$.

4. In this case let $x_1 = u$, $x_2 = u'$, $x_3 = u''$, and $x_4 = u'''$. The last equation gives $x_4' = x_1$.

6. Let $x_1 = u$ and $x_2 = u'$; then $x_1' = x_2$ is the first of the desired pair of equations. The second equation is obtained by substituting $u'' = x_2'$, $u' = x_2$, and $u = x_1$ in the given D.E. The I.C. become $x_1(0) = u_0$, $x_2(0) = u_0'$.

8a. Follow the steps outlined in Prob.7. Solve the first D.E. for x_2 to obtain $x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'$. Substitute this into the second D.E. to obtain $x_1'' - x_1' - 2x_1 = 0$.

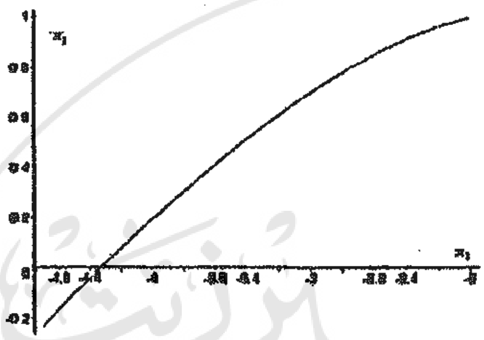
8b. The solution of the 2nd order ODE of Part a is $x_1 = c_1e^{2t} + c_2e^{-t}$. Differentiating this and substituting into the above equation for x_2 yields $x_2 = \frac{1}{2}c_1e^{2t} + 2c_2e^{-t}$. The I.C. then give
$c_1 + c_2 = 3$ and $\frac{1}{2}c_1 + 2c_2 = \frac{1}{2}$, which yield
$c_1 = \frac{11}{3}$, $c_2 = -\frac{2}{3}$. Thus $x_1 = \frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}$ and
$x_2 = \frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}$.

8c. Note that for large t, the second term in each solution of Part b vanishes and we have $x_1 \cong \frac{11}{3}e^{2t}$ and $x_2 \cong \frac{11}{6}e^{2t}$, so that $x_1 \cong 2x_2$. This says that the graph will be asymptotic to the line $x_1 = 2x_2$ for large t.

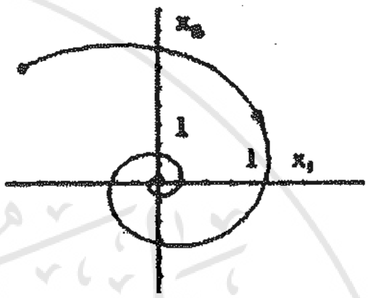
9a. Solving the first D.E. for x_2 gives $x_2 = \frac{4}{3}x_1' - \frac{5}{3}x_1$, which substituted into the second D.E. yields $x_1'' - 2.5x_1' + x_1 = 0$.

9b. From Part a, $x_1 = c_1e^{t/2} + c_2e^{2t}$ and $x_2 = -c_1e^{t/2} + c_2e^{2t}$. Using the I.C. yields $c_1 = -3/2$ and $c_2 = -1/2$. For large t, $x_1 \cong (-1/2)e^{2t}$ and $x_2 \cong (-1/2)e^{2t}$ and thus the graph is asymptotic to $x_1 = x_2$ in the third quadrant.

9c.



12c.



12a. Solving the first D.E. for x_2 gives $x_2 = \frac{1}{2}x_1' + \frac{1}{4}x_1$ and substitution into the second D.E. gives $x_1'' + x_1' + \frac{17}{4}x_1 = 0$.

12b. Solving the ODE of Part a we find $x_1 = e^{-t/2}(c_1\cos 2t + c_2\sin 2t)$ and substitution gives $x_2 = e^{-t/2}(c_2\cos 2t - c_1\sin 2t)$. The I.C. yields $c_1 = -2$ and $c_2 = 2$.

14. If $a_{12} \neq 0$, then solve the first equation for x_2, obtaining $x_2 = [x_1' - a_{11}x_1 - g_1(t)]/a_{12}$. Upon substituting this expression into the second equation, we have a second order linear O.D.E. for x_1. One I.C. is $x_1(0) = x_1^0$. The second I.C. is $x_2(0) = [x_1'(0) - a_{11}x_1(0) - g_1(0)]/a_{12} = x_2^0$. Solving for $x_1'(0)$ gives $x_1'(0) = a_{12}x_2^0 + a_{11}x_1^0 + g_1(0)$. If $a_{12} = 0$, then solve the second equation for x_1 and proceed as above. These results hold when $a_{11}, \ldots, a_{22}$ are functions of t as long as the derivatives exist and $a_{12}(t)$ and $a_{21}(t)$ are not both zero on the interval. The I.C. will involve $a_{11}(0)$ and $a_{12}(0)$.

20. Number the nodes 1,2, and 3 clockwise beginning with the top right node in Fig. 7.1.4. Also let I_1, I_2, I_3, and I_4 denote the currents through the resistor $R = 1$, the inductor $L = 1$, the capacitor $C = \frac{1}{2}$, and the resistor $R = 2$, respectively. Let V_1, V_2, V_3, and V_4 be the corresponding voltage drops. Kirchhoff's first law applied to nodes 1 and 2, respectively, gives (i) $I_1 - I_2 = 0$ and (ii) $I_2 - I_3 - I_4 = 0$. Kirchhoff's second law applied to each loop gives (iii) $V_1 + V_2 + V_3 = 0$ and (iv) $V_3 - V_4 = 0$. The current-voltage relation through each circuit element yields four more equations: (v) $V_1 = I_1$, (vi) $I_2' = V_2$, (vii) $(1/2)V_3' = I_3$ and (viii) $V_4 = 2I_4$. We thus have a system of eight equations in eight unknowns, and we wish to eliminate all of the variables except I_2 and V_3 from this system of equations. For example, we can use Eqs.(i) and (iv) to eliminate I_1 and V_4 in Eqs.(v) and (viii). Then use the new Eqs.(v) and (viii) to eliminate V_1 and I_4 in Eqs.(ii) and (iii). Finally, use the new Eqs. (ii) and (iii) in Eqs.(vi) and (vii) to obtain $I_2' = -I_2 - V_3$, $V_3' = 2I_2 - V_3$. These equations are identical (when subscripts on the remaining variables are dropped) to the equations given in the text.

22a. Note that the amount of water in each tank remains constant. Thus $Q_1(t)/30$ and $Q_2(t)/20$ represent oz./gal of salt in each tank. We assume the mixture in each tank is well stirred. Then, for the first tank we have $\frac{dQ_1}{dt} = 1.5 - 3\frac{Q_1(t)}{30} + 1.5\frac{Q_2(t)}{20}$, where the first term on the right represents the amount of salt per minute entering the mixture from an external source, the second term represents the loss of salt per minute going to Tank 2 and the third term represents the gain of salt per minute entering from Tank 2. Similarly, we have $\frac{dQ_2}{dt} = 3 + 3\frac{Q_1(t)}{30} - 4\frac{Q_2(t)}{20}$ for Tank 2. We are given that $Q_1(0) = 25$ oz. and $Q_2(0) = 15$ oz.

22b. Solve the second equation for $Q_1(t)$ to obtain $Q_1(t) = 10Q_2' + 2Q_2 - 30$. Substitution into the first equation then yields $10Q_2'' + 3Q_2' + \frac{1}{8}Q_2 = \frac{9}{2}$. Equilibrium is the steady state solution, which is $Q_2^E = 8(9/2) = 36$. Substituting this value into the equation for Q_1 yields $Q_1^E = 72 - 30 = 42$. It can be shown that $Q_1(t)$ satisfies the same second order DE as $Q_2(t)$ (except with the constant 21/4 on the right side) and thus the exponentials in the solutions for each are the same. Hence each tank approaches the equilibrium solution at then same rate.

22c. Substitute $Q_1 = x_1 + 42$ and $Q_2 = x_2 + 36$ into the equations found in Part a.

Section 7.2, Page 371

1a. $2\mathbf{A} = \begin{pmatrix} 2 & -4 & 0 \\ 6 & 4 & -2 \\ -4 & 2 & 6 \end{pmatrix}$ so that

$$2\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2+4 & -4-2 & 0+3 \\ 6-1 & 4+5 & -2+0 \\ -4+6 & 2+1 & 6+2 \end{pmatrix} = \begin{pmatrix} 6 & -6 & 3 \\ 5 & 9 & -2 \\ 2 & 3 & 8 \end{pmatrix}$$

1c. Using Eq.(9) and following Ex. 1 we have

$$\mathbf{AB} = \begin{pmatrix} 4+2+0 & -2-10+0 & 3+0+0 \\ 12-2-6 & -6+10-1 & 9+0-2 \\ -8-1+18 & 4+5+3 & -6+0+6 \end{pmatrix},$$

which yields the correct answer.

6a. $\mathbf{AB} = \begin{pmatrix} 6 & -5 & -7 \\ 1 & 9 & 1 \\ -1 & -2 & 8 \end{pmatrix}$ and $\mathbf{BC} = \begin{pmatrix} 5 & 3 & 3 \\ -1 & 7 & 3 \\ 2 & 3 & -2 \end{pmatrix}$ so that

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 7 & -11 & -3 \\ 11 & 20 & 17 \\ -4 & 3 & -12 \end{pmatrix}.$$

In Problems 10 through 19 the method of row reduction, as illustrated in Ex. 2, can be used to find the inverse matrix or else to show that none exists. We start with the original matrix augmented by the identity matrix, describe a suitable sequence of elementary row operations, and show the result of applying these operations.

10. Start with the given matrix augmented by the identity matrix.
$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right)$$

Add 2 times the first row to the second row.
$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 11 & 2 & 1 \end{array}\right)$$

Multiply the second row by (1/11).
$$\left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Add (-4) times the second row to the first row.
$$\left(\begin{array}{cc|cc} 1 & 0 & 3/11 & -4/11 \\ 0 & 1 & 2/11 & 1/11 \end{array}\right)$$

Since we have performed the same operation on the given matrix and the identity matrix, the 2 x 2 matrix appearing on the right side of this augmented matrix is the desired inverse matrix. The answer can be checked by multiplying it by the given matrix; the result should be the identity matrix.

12. The augmented matrix in this case is:
$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right)$$

Add (−2) times the first row to the second row and (−3) times the first row to the third row.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)$$

Multiply the second and third rows by (−1) and interchange them.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

Add (−3) times the third row to the first and second rows.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

Add (−2) times the second row to the first row.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

The desired answer appears on the right side of this augmented matrix.

14. Again, start with the given matrix augmented by the identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right)$$

Add (2) times the first row to the second row and add(−1) times the first row to the third row.

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 5 & 10 & \vdots & 2 & 1 & 0 \\ 0 & -4 & -8 & \vdots & -1 & 0 & 1 \end{pmatrix}$$

Add (4/5) times the second row to the third row.

$$\begin{pmatrix} 1 & 2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 5 & 10 & \vdots & 2 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 3/5 & 4/5 & 0 \end{pmatrix}$$

Since the third row of the left matrix is all zeros, no further reduction can be performed, and the given matrix is singular.

21c. Differentiate each element of **A**(t). For instance, $4e^{2t}$ is the derivative of $a_{33}(t)$ and this will then be the element in the 3rd row 3rd column of **A**′(t).

21d. Integrate each element of **A**(t) from t = 0 to t = 1. For instance, $\int_0^1 e^{2t}dt = (1/2)e^{2t}\Big|_0^1 = (1/2)(e^2-1) = (e+1)(e-1)/2$ is the integral of $a_{13}(t)$. Thus (e+1)(e−1)/2 will be the element in the 1st row 3rd column of $\int_0^1 \mathbf{A}(t)dt$

22. $\mathbf{x}' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} 2e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}$; and

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t} = \begin{pmatrix} 12-4 \\ 8-4 \end{pmatrix} e^{2t} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} e^{2t}.$$

25. $\Psi' = \begin{pmatrix} -3e^{-3t} & 2e^{2t} \\ 12e^{-3t} & 2e^{2t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}.$

<u>Section 7.3, Page 383</u>

1. As in Ex. 1, form the augmented matrix and use row reduction:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 3 & 1 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right)$$

Add (-3) times the first row to the second and add the first row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right)$$

Add (-1) times the second row to the third.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

The third row is equivalent to $-3x_3 = 1$ or $x_3 = -1/3$. Likewise the second row is equivalent to $x_2 + 4x_3 = 1$, so $x_2 = 7/3$. Finally, from the first row, $x_1 - x_3 = 0$, so $x_1 = -1/3$. The answer can be checked by substituting into the original equations.

2. Forming the augmented matrix and using row reduction will yield an augmented mstrix whose third row is equivalent to: $0x_1 + 0x_2 + 0x_3 = 1$, or $0 = 1$. Thus there is no solution.

3. Form the augmented matrix and use row reduction.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 \end{array}\right)$$

Add (-2) times the first row to the second and add (-1) times the first row to the third.

$$\begin{pmatrix} 1 & 2 & -1 & . & 2 \\ 0 & -3 & 3 & . & -3 \\ 0 & -3 & 3 & . & -3 \end{pmatrix}$$

Add (-1) times the second row to the third row and then multiply the second row by (-1/3).

$$\begin{pmatrix} 1 & 2 & -1 & . & 2 \\ 0 & 1 & -1 & . & 1 \\ 0 & 0 & 0 & . & 0 \end{pmatrix}$$

Since the last row has only zero entries, it may be dropped. The second row corresponds to the equation $x_2 - x_3 = 1$. We can assign an arbitrary value to either x_2 or x_3 and use this equation to solve for the other. For example, let $x_3 = c$, where c is arbitrary. Then $x_2 = 1 + c$. The first row corresponds to the equation $x_1 + 2x_2 - x_3 = 2$, so $x_1 = 2 - 2x_2 + x_3 = 2 - 2(1+c) + c = -c$. The solution can be written in vector form as $\mathbf{x} = \begin{pmatrix} -c \\ 1+c \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, where the first vector on the right is a solution of the given nonhomogeneous equation and the second vector is a solution of the related homogeneous equation.

7. To determine whether the given set of vectors is linearly independent we must solve the system $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$ for c_1, c_2, and c_3. Writing this in scalar form, we have
$$\begin{aligned} c_1 \quad\quad + c_3 &= 0 \\ c_1 + c_2 \quad\quad &= 0, \\ c_2 + c_3 &= 0 \end{aligned}$$
so the augmented matrix is $\begin{pmatrix} 1 & 0 & 1 & . & 0 \\ 1 & 1 & 0 & . & 0 \\ 0 & 1 & 1 & . & 0 \end{pmatrix}$

Row reduction yields $\begin{pmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & \vdots & 0 \end{pmatrix}$.

From the third row we have $c_3 = 0$. Then from the second row, $c_2 - c_3 = 0$, so $c_2 = 0$. Finally from the first row $c_1 + c_3 = 0$, so $c_1 = 0$. Since $c_1 = c_2 = c_3 = 0$, we conclude that the given vectors are linearly independent.

9. As in Prob.7 we wish to solve the system $c_1x^{(1)} + c_2x^{(2)} + c_3x^{(3)} + c_4x^{(4)} = 0$ for c_1, c_2, c_3, and c_4. Form the augmented matrix and use row reduction:

$$\begin{pmatrix} 1 & -1 & -2 & -3 & \vdots & 0 \\ 2 & 0 & -1 & 0 & \vdots & 0 \\ 2 & 3 & 1 & -1 & \vdots & 0 \\ 3 & 1 & 0 & 3 & \vdots & 0 \end{pmatrix}$$

Add (-2) times the first row to the second, add (-2) times the first row to the third, and add (-3) times the first row to the fourth.

$$\begin{pmatrix} 1 & -1 & -2 & -3 & \vdots & 0 \\ 0 & 2 & 3 & 6 & \vdots & 0 \\ 0 & 5 & 5 & 5 & \vdots & 0 \\ 0 & 4 & 6 & 12 & \vdots & 0 \end{pmatrix}$$

Multiply the second row by (1/2) and then add (-5) times the second row to the third and add (-4) times the second row to the fourth.

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -3 & 0 \\ 0 & 1 & 3/2 & 3 & 0 \\ 0 & 0 & -5/2 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The third row is equivalent to the equation $c_3 + 4c_4 = 0$. One way to satisfy this equation is by choosing $c_4 = -1$; then $c_3 = 4$. From the second row we then have $c_2 = -(3/2)c_3 - 3c_4 = -6 + 3 = -3$. Then, from the first row, $c_1 = c_2 + 2c_3 + 3c_4 = -3 + 8 - 3 = 2$. Hence the given vectors are linearly dependent, and satisfy $2x^{(1)} - 3x^{(2)} + 4x^{(3)} - x^{(4)} = 0$.

14. Consider $c_1x^{(1)}(t) + c_2x^{(2)}(t) = 0$, which has the augmented matrix $\left(\begin{array}{cc|c} 2\sin t & \sin t & 0 \\ \sin t & 2\sin t & 0 \end{array}\right)$. Multiplying the first row by $-1/2$ and adding to the second row yields $\left(\begin{array}{cc|c} 2\sin t & \sin t & 0 \\ 0 & (3/2)\sin t & 0 \end{array}\right)$. Since sint is not identically zero, we conclude, from the last row, that $c_2 = 0$. Using this in the first row gives $c_1 = 0$ and thus $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent for $-\infty < t < \infty$.

15. Let $t = t_0$ be a fixed value of t in the interval $0 \le t \le 1$. To determine whether $x^{(1)}(t_0)$ and $x^{(2)}(t_0)$ are linearly dependent we must solve $c_1x^{(1)}(t_0)+c_2x^{(2)}(t_0)=0$. We have the augmented matrix

$$\left(\begin{array}{cc|c} e^{t_0} & 1 & 0 \\ t_0e^{t_0} & t_0 & 0 \end{array}\right).$$

Multiply the first row by $(-t_0)$ and add to the second row to obtain $\begin{pmatrix} e^{t_0} & 1 & . & 0 \\ 0 & 0 & . & 0 \end{pmatrix}$.

Thus, for example, we can choose $c_1 = 1$ and $c_2 = -e^{t_0}$, and hence the given vectors are linearly dependent at t_0. Since t_0 is arbitrary the vectors are linearly dependent at each point in the interval. However, there is no linear relation between $x^{(1)}$ and $x^{(2)}$ that is valid throughout the interval $0 \le t \le 1$. For example, if $t_1 \ne t_0$, and if c_1 and c_2 are chosen as above, then

$$c_1x^{(1)}(t_1) + c_2x^{(2)}(t_1)$$
$$= \begin{pmatrix} e^{t_1} \\ t_1e^{t_1} \end{pmatrix} + -e^{t_0}\begin{pmatrix} 1 \\ t_1 \end{pmatrix} = \begin{pmatrix} e^{t_1} - e^{t_0} \\ t_1e^{t_1} - t_1e^{t_0} \end{pmatrix} \ne \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence the given vectors must be linearly independent on $0 \le t \le 1$. In fact, the same argument applies to any interval.

16. To find the eigenvalues and eigenvectors of the given matrix we must solve $\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(5-\lambda)(1-\lambda) - (-1)(3) = 0$, or $\lambda^2 - 6\lambda + 8 = 0$. Hence $\lambda_1 = 2$ and $\lambda_2 = 4$ are the eigenvalues. The eigenvector corresponding to λ_1 must satisfy $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $3x_1 - x_2 = 0$. If we let $x_1 = 1$, then $x_2 = 3$ and the eigenvector is $x^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, or any constant multiple of this vector. Similarly, the eigenvector corresponding to λ_2 must satisfy $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 - x_2 = 0$. Hence $x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, or a multiple thereof.

19. Since $\bar{a}_{12} = a_{21}$, the given matrix is Hermitian and we know in advance that its eigenvalues are real. To find the eigenvalues and eigenvectors we must solve

$\begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)^2 - i(-i) = \lambda^2 - 2\lambda$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$; observe that they are indeed real even though the given matrix has imaginary entries. The eigenvector corresponding to λ_1 must satisfy

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $x_1 + ix_2 = 0$. Note that the second equation $-ix_1 + x_2 = 0$ is a multiple of the first. If $x_1 = 1$, then $x_2 = i$, and the eigenvector is $x^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. In a similar way we find that the eigenvector associated with λ_2 is $x^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

22. The eigenvalues and eigenvectors satisfy

$\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(1-\lambda)[(1-\lambda)^2 + 4] = 0$, which has roots $\lambda = 1, 1 \pm 2i$. For $\lambda = 1$, we then have $2x_1 - 2x_3 = 0$ and $3x_1 + 2x_2 = 0$. Choosing $x_1 = 2$ then yields $\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$ as the eigenvector corresponding to $\lambda = 1$. For $\lambda = 1 + 2i$ we have $-2ix_1 = 0$, $2x_1 - 2ix_2 - 2x_3 = 0$ and $3x_1 + 2x_2 - 2ix_3 = 0$, yielding $x_1 = 0$ and $x_3 = -ix_2$. Thus $\begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 + 2i$. A similar calculation shows that $\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 1 - 2i$.

25. Since the given matrix is real and symmetric, we know that the eigenvalues are real. Further, even if there are repeated eigenvalues, there will be a full set of three linearly independent eigenvectors. To find the eigenvalues and eigenvectors we must solve

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ The determinant of

coefficients is $(3-\lambda)[-\lambda(3-\lambda)-4] - 2[2(3-\lambda) -8] + 4[4+4\lambda]$ $= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$. Setting this equal to zero and solving we find $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 8$. The eigenvectors corresponding to λ_1 and λ_2 must satisfy

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$ hence there is only the single

relation $2x_1 + x_2 + 2x_3 = 0$ to be satisfied. Consequently, two of the variables can be selected arbitrarily and the third is then determined by this equation. For example, if $x_1 = 1$ and $x_3 = 1$, then $x_2 = -4$,

and we obtain the eigenvector $\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$. Similarly, if

$x_1 = 1$ and $x_2 = 0$, then $x_3 = -1$, and we have the

eigenvector $\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, which is linearly independent of

$\mathbf{x}^{(1)}$. There are many other choices that could have been made; however, by Eq.(38) there can be no more than two linearly independent eigenvectors corresponding to the eigenvalue -1. To find the eigenvector corresponding to λ_3 we must solve

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ Interchange the first and

second rows and use row reduction to obtain the equivalent system $x_1 - 4x_2 + x_3 = 0$, $2x_2 - x_3 = 0$. Since there are two equations to satisfy only one variable can be assigned an arbitrary value. If we let $x_2 = 1$, then

$x_3 = 2$ and $x_1 = 2$, so we find that $x^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

28. We are given that $Ax = b$ has solutions and thus we have $(Ax,y) = (b,y)$. Using $A^*y= 0$ and the result of Prob.26b, we have $(Ax,y) = (x,A^*y) = 0$. Thus $(b,y) = 0$. For Ex. 2, since A is real,

$A^* = A^T = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 3 & -2 & 3 \end{pmatrix}$ and, using row reduction, the

augmeted matrix for $A^*y = 0$ becomes $\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Thus $y = c\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and hence $(b,y) = b_1 + 3b_2 + b_3 = 0$.

<u>Section 7.4, Page 389</u>

1. Use Mathematical Induction. It has already been proven that if $x^{(1)}$ and $x^{(2)}$ are solutions, then so is $c_1x^{(1)} + c_2x^{(2)}$. Assume that if $x^{(1)}$, $x^{(2)}$, ..., $x^{(k)}$ are solutions, then $x = c_1x^{(1)} + \cdots + c_kx^{(k)}$ is a solution. Then use Theorem 7.4.1 to conclude that $x + c_{k+1}x^{(k+1)}$ is also a solution and thus $c_1x^{(1)} + \cdots + c_{k+1}x^{(k+1)}$ is a solution if $x^{(1)}$, ..., $x^{(k+1)}$ are solutions.

2a. From Eq.(10) we have

$W = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = x_1^{(1)}\, x_2^{(2)} - x_2^{(1)}\, x_1^{(2)}$. Taking the derivative of these two products yields four terms which may be written as

$$\frac{dW}{dt} = \left[\frac{dx_1^{(1)}}{dt}x_2^{(2)} - x_2^{(1)}\,\frac{dx_1^{(2)}}{dt}\right] + \left[x_1^{(1)}\,\frac{dx_2^{(2)}}{dt} - \frac{dx_2^{(1)}}{dt}x_1^{(2)}\right].$$

The terms in the square brackets can now be recognized as the respective determinants appearing in the desired result. A similar result was obtained in Prob. 20 of Sect. 4.1.

2b. If $x^{(1)}$ is substituted into Eq.(3) we have

$$\frac{dx_1^{(1)}}{dt} = p_{11}\, x_1^{(1)} + p_{12}\, x_2^{(1)}$$

$$\frac{dx_2^{(1)}}{dt} = p_{21}\, x_1^{(1)} + p_{22}\, x_2^{(1)}.$$

Substituting the first equation above and its counterpart for $x^{(2)}$ into the first determinant appearing in dW/dt and evaluating the result yields $p_{11} \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} = p_{11}W$. Similarly, the second determinant in dW/dt is evaluated as $p_{22}W$, yielding the desired result.

2c. From Part(b) we have $\frac{dW}{W} = [p_{11}(t) + p_{22}(t)]dt$ which gives $W(t) = c\exp\int[p_{11}(t) + p_{22}(t)]dt$.

6a. $W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$.

6b. Pick $t = t_0$, then $c_1x^{(1)}(t_0) + c_2x^{(2)}(t_0) = 0$ implies $c_1\begin{pmatrix} t_0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t_0^2 \\ 2t_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has a non-zero solution for c_1 and c_2 if and only if $\begin{vmatrix} t_0 & t_0^2 \\ 1 & 2t_0 \end{vmatrix} = 2t_0^2 - t_0^2 = t_0^2 = 0$. Thus $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent at each point except $t = 0$. Thus they are linearly independent on every interval.

6c. From Part(a) we see that the Wronskian vanishes at $t = 0$, but not at any other point. By Theorem 7.4.3, if $P(t)$, from Eq.(3), is continuous, then the Wronskian is either identically zero or else never vanishes. Hence, we conclude that the D.E. satisfied by $x^{(1)}(t)$ and $x^{(2)}(t)$ must have at least one discontinuous coefficient at $t = 0$.

6d. To obtain the system satisfied by $x^{(1)}$ and $x^{(2)}$ we consider

$x = c_1x^{(1)} + c_2x^{(2)}$, or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} t \\ 1 \end{pmatrix} + c_2\begin{pmatrix} t^2 \\ 2t \end{pmatrix}$.

Taking the derivative we obtain $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2t \\ 2 \end{pmatrix}$.

Solving this last system for c_1 and c_2 we find $c_1 = x_1' - tx_2'$ and $c_2 = x_2'/2$. Thus $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1' - tx_2') \begin{pmatrix} t \\ 1 \end{pmatrix} + \frac{x_2'}{2} \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$, which yields

$x_1 = tx_1' - \frac{t^2}{2}\, x_2'$ and $x_2 = x_1'$. Writing this system in

matrix form we have $x = \begin{pmatrix} t & -t^2/2 \\ 1 & 0 \end{pmatrix} x'$. Finding the

inverse of the matrix multiplying x' yields the desired solution. Note that at $t = 0$ two of the elements in $P(t)$ are discontinuous.

<u>Section 7.5, Page 398</u>

1. Assuming that there are solutions of the form $x = \xi e^{rt}$, we substitute into the D.E. to find

$r\xi e^{rt} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \xi e^{rt}$. Since $\xi = I\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi$, we can

write this equation as $\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \xi - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi = 0$ and

thus we must solve $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for r, ξ_1, ξ_2.

The determinant of the coefficients is

$(3-r)(-2-r) + 4 = r^2 - r - 2$, so the eigenvalues are

$r = -1, 2$. The eigenvector corresponding to $r = -1$

satisfies $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which yields $2\xi_1 - \xi_2 = 0$.

Thus $x^{(1)}(t) = \xi^{(1)}e^{-t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$, where we have set $\xi_1 = 1$.

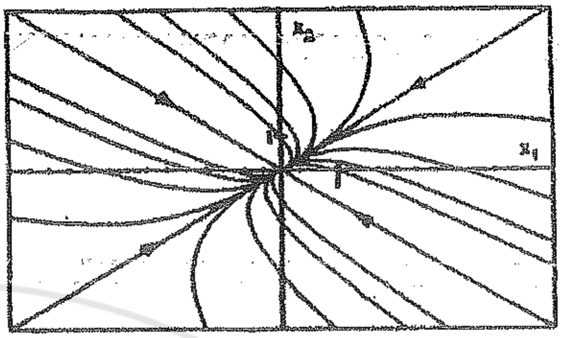
(Any other non zero choice would also work). In a similar fashion, for $r = 2$, we have $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $\xi_1 - 2\xi_2 = 0$. Hence $x^{(2)}(t) = \xi^{(2)}e^{2t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$ by setting $\xi_2 = 1$. The general solution is then

$x = c_1x^{(1)}(t) + c_2x^{(2)}(t)$. To sketch the trajectories we follow the steps illustrated in Exs. 1 and 2. Setting $c_2 = 0$ we have $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$ or $x_1 = c_1e^{-t}$ and

$x_2 = 2c_1e^{-t}$ and thus one asymptote is given by $x_2 = 2x_1$. In a similar fashion $c_1 = 0$ gives $x_2 = (1/2)x_1$ as a second asymptote. Since the roots differ in sign, the trajectories for this problem are similar in nature to those in Ex. 1. For $c_2 \neq 0$, all solutions will be asymptotic to $x_2 = (1/2)x_1$ as $t \to \infty$. For $c_2 = 0$, the solution approaches the origin along the line $x_2 = 2x_1$.

5. Proceeding as in Prob. 1 we assume a solution of the form $x = \xi e^{rt}$, where r, ξ_1, ξ_2 must now satisfy $\begin{pmatrix} -2-r & 1 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Evaluating the determinant of the coefficients set equal to zero yields $r = -1, -3$ as the eigenvalues. For $r = -1$ we find $\xi_1 = \xi_2$ and thus $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $r = -3$ we find $\xi_2 = -\xi_1$ and hence $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solution is then

$x = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$. Since there are two negative eigenvalues, we would expect the trajectories to be similar to those of Ex. 2. Setting $c_2 = 0$ and eliminating t (as in Prob. 1) we find



that $\begin{pmatrix}1\\1\end{pmatrix} e^{-t}$ approaches the origin along the line $x_2 = x_1$. Similarly $\begin{pmatrix}1\\-1\end{pmatrix} e^{-3t}$ approaches the origin along the line $x_2 = -x_1$. As long as $c_1 \neq 0$ (since e^{-t} is the dominnt term as $t \to 0$). all trajectories approach the origin asymptotic to $x_2 = x_1$. For $c_1 = 0$, the trajectory approaches the origin along $x_2 = -x_1$, as shown in the graph.

6. The characteristic equation is $(5/4 - r)^2 - 9/16 = 0$, so $r = 2, 1/2$. Since the roots are of the same sign, the behavior of the solutions is similar to Prob. 5, except the trajectories are reversed, since the roots are positive.

7. Again assuming $\mathbf{x} = \xi e^{rt}$ we find that r, ξ_1, ξ_2 must satisfy $\begin{pmatrix}4-r & -3\\ 8 & -6-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. The determinant of the coefficients set equal to zero yields $r = 0, -2$. For $r = 0$ we find $4\xi_1 = 3\xi_2$. Choosing $\xi_2 = 4$ we find $\xi_1 = 3$ and thus $\xi^{(1)} = \begin{pmatrix}3\\4\end{pmatrix}$. Similarly for $r = -2$ we have $\xi^{(2)} = \begin{pmatrix}1\\2\end{pmatrix}$ and thus $\mathbf{x} = c_1\begin{pmatrix}3\\4\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix} e^{-2t}$. To sketch the trajectories, note that the general solution is equivalent to the simultaneous equations $x_1 = 3c_1 + c_2e^{-2t}$ and $x_2 = 4c_1 + 2c_2e^{-2t}$. Solving the first equation for c_2e^{-2t} (assuming $c_2 \neq 0$) and substituting into the second yields $x_2 = 2x_1 - 2c_1$ and thus the trajectories are parallel straight lines. If $c_2 = 0$, the solution is fixed at a point.

9. The eigvalues are given by
$\begin{vmatrix}1-r & i\\ -i & 1-r\end{vmatrix} = (1-r)^2 + i^2 = r(r-2) = 0$. For $r = 0$ we have

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$ or $-i\xi_1 + \xi_2 = 0$ and thus $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is one eigenvector. Similarly $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigenvector for $r = 2$.

14. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant of coefficients set equal to zero reduces to $r^3 - 2r^2 - 5r + 6 = 0$, so the eigenvalues are $r_1 = 1$, $r_2 = -2$, and $r_3 = 3$. The eigenvector corresponding to r_1 must satisfy $\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Using row reduction we obtain the equivalent system $\xi_1 + \xi_3 = 0$, $\xi_2 - 4\xi_3 = 0$. Letting $\xi_1 = 1$, it follows that $\xi_3 = -1$ and $\xi_2 = -4$, so $\xi^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$. In a similar way the eigenvectors corresponding to r_2 and r_3 are found to be $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ and $\xi^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, respectively. Thus the general solution of the given D.E. is

$$x = c_1\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t} + c_3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}.$$

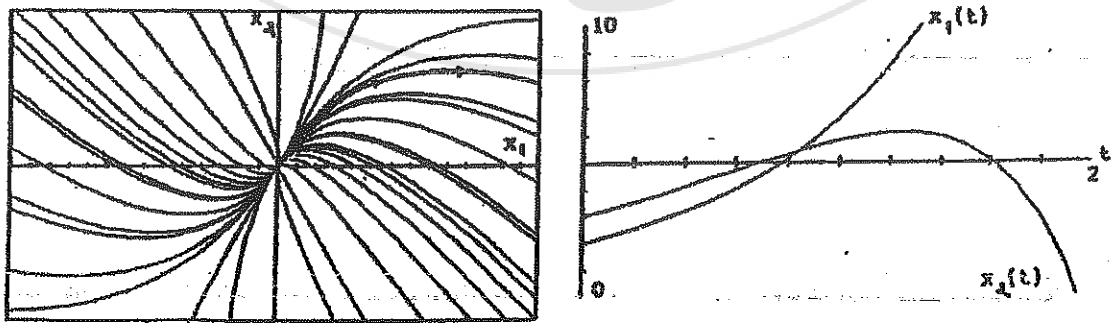
Notice that the "trajectories" of this solution would lie in the $x_1 x_2 x_3$ three dimensional space.

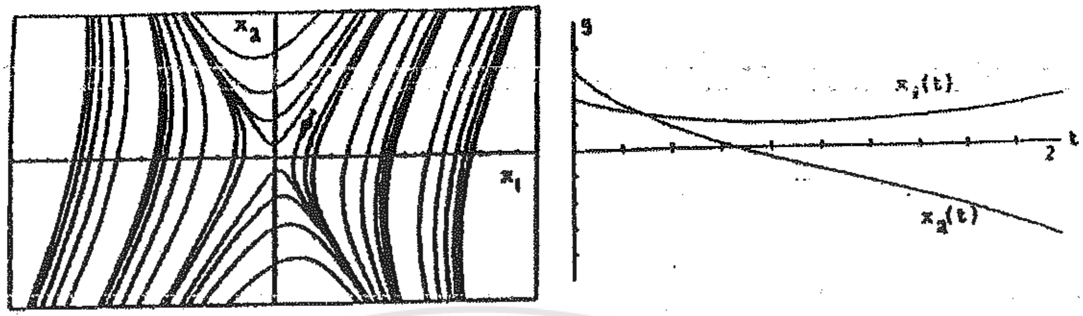
16. The eigenvalues and eigenvectors of the coefficient matrix are found to be $r_1 = -1$, $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = 3$,

$\xi^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$. Thus the general solution of the given D.E. is $x = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$. The I.C. yields the system of equations $c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. The augmented matrix of this system is $\begin{pmatrix} 1 & 1 & . & 1 \\ & & . & \\ 1 & 5 & . & 3 \end{pmatrix}$ and by row reduction we obtain $\begin{pmatrix} 1 & 1 & . & 1 \\ & & . & \\ 0 & 1 & . & 1/2 \end{pmatrix}$. Thus $c_2 = 1/2$ and $c_1 = 1/2$. Substituting these values in the general solution gives the solution of the I.V.P. As $t \to \infty$, the solution becomes asymptotic to $x = \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$, or $x_2 = 5x_1$.

20. Substituting $x = \xi t^r$ into the D.E. we obtain $r\xi t^r = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \xi t^r$. For $t \neq 0$ this equation can be written as $\begin{pmatrix} 2-r & -1 \\ 3 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The eigenvalues and eigenvectors are $r_1 = 1$, $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = -1$, $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Substituting these in the assumed form we obtain the general solution $x = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} t^{-1}$.

25.



27. 

31c. The eigenvalues are given by

$$\begin{vmatrix} -1-r & -1 \\ -\alpha & -1-r \end{vmatrix} = r^2 + 2r + 1 - \alpha = 0. \quad \text{Thus } r_{1,2} = -1\pm\sqrt{\alpha}.$$

Note that in Part (a) the eigenvalues are both negative while in Part (b) they differ in sign. Thus, in this part, if we choose $\alpha = 1$, then one eigenvalue is zero, which is the transition of the one root from negative to positive. This is the desired bifurcation point.

Section 7.6, Page 409

1a. We assume a solution of the form $\mathbf{x} = \boldsymbol{\xi}e^{rt}$ thus r and $\boldsymbol{\xi}$ are solutions of $\begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(r^2-2r-3) + 8 = r^2 - 2r + 5$, so the eigenvalues are $r = 1 \pm 2i$. The eigenvector corresponding to $1 + 2i$ satisfies $\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $(2-2i)\xi_1 - 2\xi_2 = 0$. If $\xi_1 = 1$, then $\xi_2 = 1-i$ and $\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$ and thus one complex-valued solution of the D.E. is $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+2i)t}$. To find real-valued solutions (see Eqs.10 and 11) we take the real and imaginary parts, respectively of $\mathbf{x}^{(1)}(t)$.

Thus $\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{t}(\cos 2t + i\sin 2t)$

$$= e^{t}\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + \sin 2t + i(\sin 2t - \cos 2t) \end{pmatrix}$$

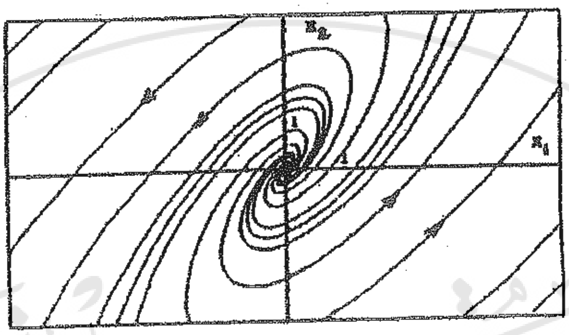
$$= e^{t}\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + ie^{t}\begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}.$$

Hence the general solution of the D.E. is

$$x = c_1e^t\begin{pmatrix}\cos 2t\\ \cos 2t + \sin 2t\end{pmatrix} + c_2e^t\begin{pmatrix}\sin 2t\\ \sin 2t - \cos 2t\end{pmatrix}.$$

The solutions spiral to ∞ as $t \to \infty$ due to the e^t terms.

1b.



7. The eigenvalues and eigenvectors of the coefficient matrix satisfy

$$\begin{pmatrix}1-r & 0 & 0\\ 2 & 1-r & -2\\ 3 & 2 & 1-r\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

The determinant of coefficients reduces to $(1-r)(r^2 - 2r + 5)$ so the eigenvalues are $r_1 = 1$, $r_2 = 1 + 2i$, and $r_3 = 1 - 2i$. The eigenvector corresponding to r_1 satisfies

$$\begin{pmatrix}0 & 0 & 0\\ 2 & 0 & -2\\ 3 & 2 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix};$$

hence $\xi_1 - \xi_3 = 0$ and $3\xi_1 + 2\xi_2 = 0$. If we let $\xi_2 = -3$ then $\xi_1 = 2$ and $\xi_3 = 2$, so one solution of the D.E. is $\begin{pmatrix}2\\-3\\2\end{pmatrix}e^t$. The eigenvector corresponding to r_2 satisfies

$$\begin{pmatrix}-2i & 0 & 0\\ 2 & -2i & -2\\ 3 & 2 & -2i\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence $\xi_1 = 0$ and $i\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_3 = -i$. Thus a complex-valued solution is

$$\begin{pmatrix}0\\1\\-i\end{pmatrix}e^t(\cos 2t + i\sin 2t).$$

Taking the real and imaginary

parts, see Prob. 1, we obtain $\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} e^t$ and $\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix} e^t$, respectively. Thus the general solution is

$$x = c_1\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}e^t + c_2e^t\begin{pmatrix} 0 \\ \cos 2t \\ \sin 2t \end{pmatrix} + c_3e^t\begin{pmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{pmatrix},$$

which spirals to ∞ about the x_1 axis in the $x_1x_2x_3$ space as $t \to \infty$.

9. The eigenvalues and eigenvectors of the coefficient matrix satisfy $\begin{pmatrix} 1-r & -5 \\ 1 & -3-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $r^2 + 2r + 2$ so that the eigenvalues are $r = -1 \pm i$. The eigenvector corresponding to $r = -1 + i$ is given by $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$ so that $\xi_1 = (2+i)\xi_2$ and thus one complex-valued solution is

$x^{(1)}(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{(-1+i)t}$. Finding the real and complex parts of $x^{(1)}$, as in Prob.1, leads to the general solution $x = c_1e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$.

Setting $t = 0$ we find $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which is equivalent to the system $\begin{matrix} 2c_1 + c_2 = 1 \\ c_1 + 0 = 1 \end{matrix}$. Thus $c_1 = 1$,

$$c_2 = -1 \text{ and } x(t) = e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} - e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$$

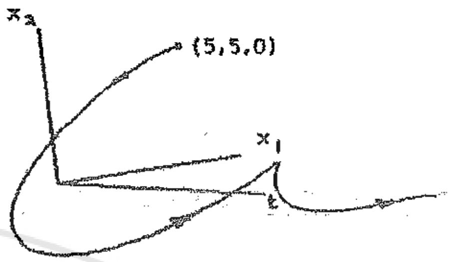
$$= e^{-t}\begin{pmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{pmatrix},$$

which spirals to zero as $t \to \infty$, due to the e^{-t} term.

11a. The eigenvalues are given by

$$\begin{vmatrix} 3/4-r & -2 \\ 1 & -5/4-r \end{vmatrix} = r^2 + r/2 + 17/16 = 0, \text{ so } r = -1/4 \pm i.$$

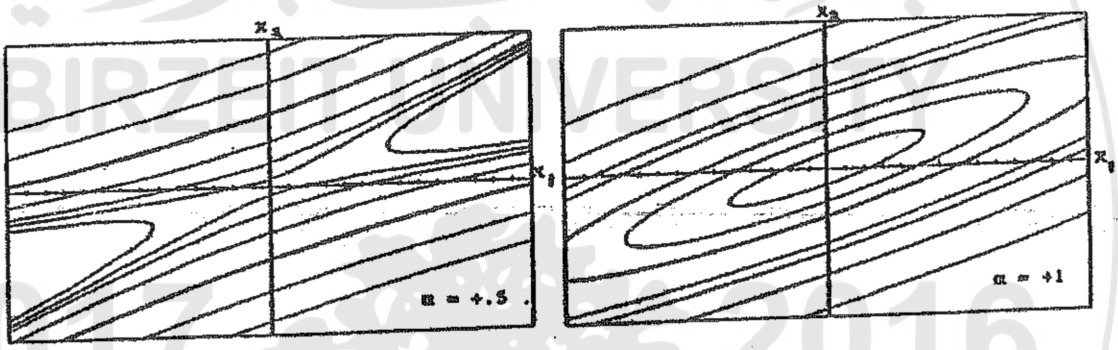
11d. Choose $x(0) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$, then the trajectory starts at (5,5) in the x_1x_2 plane and spirals around the t-axis and converges to the t axis as $t \to \infty$.



15a. The eigenvalues satisfy $\begin{vmatrix} 2-r & -5 \\ \alpha & -2-r \end{vmatrix} = r^2 - 4 + 5\alpha = 0$, so $r_1, r_2 = \pm\sqrt{4-5\alpha}$.

15b. The qualitative nature of the phase portrait changes when r goes from real to complex. Thus $\alpha = 4/5$ is the critical value and $r_1 = r_2 = 0$.

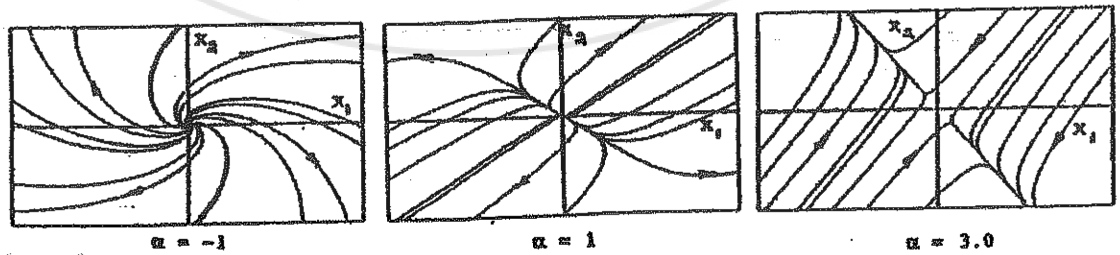
15c.



16a. $\begin{vmatrix} 5/4-r & 3/4 \\ \alpha & 5/4-r \end{vmatrix} = r^2 - 5r/2 + (25/16 - 3\alpha/4) = 0$, so $r_{1,2} = 5/4 \pm \sqrt{3\alpha}/2$.

16b. There are two critical values of α. For $\alpha < 0$ the eigenvalues are complex, while for $\alpha > 0$ they are real. There will be a second critical value of α when $r_2 = 0$, or $\alpha = 25/12$. In this case the second real eigenvalue goes from positive to negative.

16c.



18a. We have $\begin{vmatrix} 3-r & \alpha \\ -6 & -4-r \end{vmatrix} = r^2 + r - 12 + 6\alpha = 0$, so

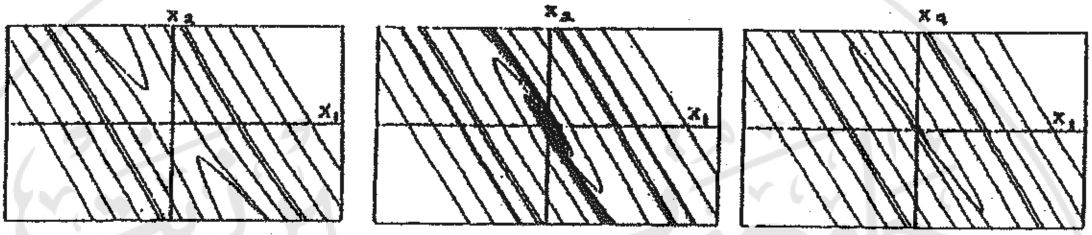
$r_1, r_2 = -1/2 \pm\sqrt{49-24\alpha}\,/2$.

18b. The critical values occur when $49 - 24\alpha = 1$ (in which case

$r_2 = 0$) and when $49 - 24\alpha = 0$, in which case $r_1 = r_2 = -1/2$.

Thus $\alpha = 2$ and $\alpha = 49/24 \cong 2.04$.

18c.



$\alpha = 1.8$ $\alpha = 2.1$ $\alpha = 2.03$

21. If we seek solutions of the form $x = \xi t^r$, then

$\xi r t^r = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}\xi t^r$. Thus r must be an eigenvalue and ξ a corresponding eigenvector of the coefficient matrix.

Thus r and ξ satisfy $\begin{pmatrix} -1-r & -1 \\ 2 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The

determinant of coefficients is $(-1-r)^2 + 2 = r^2 + 2r + 3$, so the eigenvalues are $r = -1 \pm \sqrt{2}\,i$. The eigenvector corresponding to $-1 + \sqrt{2}\,i$ satisfies

$\begin{pmatrix} -\sqrt{2}\,i & -1 \\ 2 & -\sqrt{2}\,i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\sqrt{2}\,i\xi_1 + \xi_2 = 0$. If we let

$\xi_1 = 1$, then $\xi_2 = -\sqrt{2}\,i$, and $\xi^{(1)} = \begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix}$. Thus a complex-valued solution of the given D.E. is

$\begin{pmatrix} 1 \\ -\sqrt{2}\,i \end{pmatrix} t^{-1+\sqrt{2}\,i}$. From Eq.(16) of Sect. 5.4

we have (since $t^{\sqrt{2}\,i} = e^{\ln t^{\sqrt{2}\,i}} = e^{\sqrt{2}\,i\ln t}$)

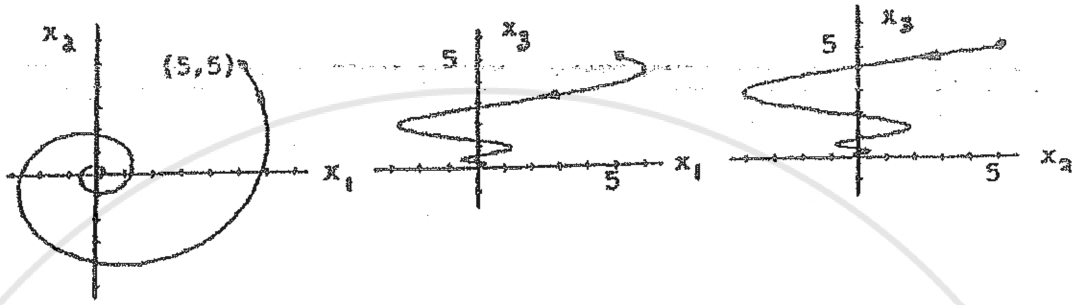
$t^{-1+\sqrt{2}\,i} = t^{-1}[\cos(\sqrt{2}\ln t) + i\sin(\sqrt{2}\ln t)]$ for $t > 0$.

Separating the complex valued solution into real and imaginary parts, we obtain the two real-valued solutions

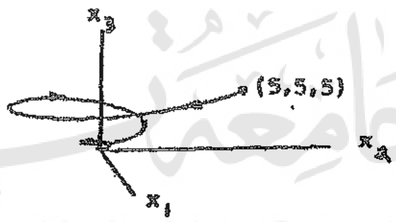
$$u = t^{-1}\begin{pmatrix} \cos(\sqrt{2}\ln t) \\ \sqrt{2}\sin(\sqrt{2}\ln t) \end{pmatrix} \text{ and } v = t^{-1}\begin{pmatrix} \sin(\sqrt{2}\ln t) \\ -\sqrt{2}\cos(\sqrt{2}\ln t) \end{pmatrix}.$$

23a. The eigenvalues are given by $(r+1/4)[(r+1/4)^2 + 1] = 0$.

23b.



23c. Graph starts in the first octant and spirals around the x_3 axis, converging to zero.



30a. Following the steps leading to Eq.(24), and using the given values for the m's and k's, we obtain $y_1' = y_3$, $y_2' = y_4$, $y_3' = -4y_1 + 3y_2$, and $y_4' = (9/4)y_1 - (13/4)y_2$.

Thus $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 3 & 0 & 0 \\ 9/4 & -13/4 & 0 & 0 \end{pmatrix}$.

30b. The eigenvalues of $\mathbf{A}$ are given by $\det(\mathbf{A} - r\mathbf{I}) = r^4 + (29/4)r^2 + 25/4 = 0$, and thus $r_{1,2} = \pm i$ and $r_{3,4} = \pm(5/2)i$. The eigenvector corresponding to $r_1 = i$ satisfies

$$\begin{pmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \\ -4 & 3 & -i & 0 \\ 9/4 & -13/4 & 0 & -i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$ or $\xi_3 = i\xi_1$, $\xi_4 = i\xi_2$,

$-4\xi_1 + 3\xi_2 = i\xi_3$, and $(9/4)\xi_1 - (13/4)\xi_2 = i\xi_4$. Setting

$\xi_1 = 1$, the first three equations yield $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix}$ and thus $\xi^{(2)} = \begin{pmatrix} 1 \\ 1 \\ -i \\ -i \end{pmatrix}$ by Eq.(13). It should be noted that the fourth equation is also satisfied by this choice.

Similarly, $\xi^{(3)}$ satisfies $\begin{pmatrix} -5i/2 & 0 & 1 & 0 \\ 0 & -5i/2 & 0 & 1 \\ -4 & 3 & -5i/2 & 0 \\ 9/4 & -13/4 & 0 & -5i/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

which yields $\xi^{(3)} = \begin{pmatrix} 4 \\ -3 \\ 10i \\ -15i/2 \end{pmatrix}$ and $\xi^{(4)} = \begin{pmatrix} 4 \\ -3 \\ -10i \\ 15i/2 \end{pmatrix}$.

30c. Taking the real and imaginary parts of $e^{it}\begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix}$ yields

$w_1(t) = (\text{cost},\text{cost},-\text{sint},-\text{sint})^T$ and
$w_2(t) = (\text{sint},\text{sint},\text{cost},\text{cost})^T$ as the corresponding two real valued solutions. Similarly, $e^{(5/2)it}\begin{pmatrix} 4 \\ -3 \\ 10i \\ -15i/2 \end{pmatrix}$ yields the other two real valued solutions, denoted as $w_3(t)$ and $w_4(t)$. The general solution of the system is then

$y(t) = c_1w_1(t) + c_2w_2(t) + c_3w_3(t) + c_4w_4(t)$.

30d. There are two fundamental modes, one represented by $\cos(t - \delta_1)$, of frequency 1, and the other represented by $\cos(5t/2 - \delta_2)$, of frequency 5/2 (see Sect. 3.7).

30e. From Part(c),

$$y(0) = c_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}4\\-3\\0\\0\end{pmatrix} + c_4\begin{pmatrix}0\\0\\10\\-15/2\end{pmatrix} = \begin{pmatrix}2\\1\\0\\0\end{pmatrix},$$ which

yields $c_2 = c_4 = 0$, $c_1 = 10/7$, and $c_3 = 1/7$.

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Each of the Problems 1 through 10, except 2 and 8, has been solved in one of the previous sections. Thus a fundamental matrix for the given systems can be readily written down. The fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ can then be found, as shown in the following problems.

2a. The characteristic equation is given by

$$\begin{vmatrix}-3/4-r & 1/2\\ 1/8 & -3/4-r\end{vmatrix} = r^2 + 3r/2 + 1/2 = 0,$$ so $r = -1, -1/2$. For

$r = -1$ we have $\begin{pmatrix}1/4 & 1/2\\ 1/8 & 1/4\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$, and $\xi^{(1)} = \begin{pmatrix}-2\\1\end{pmatrix}$.

Likewise $\xi^{(2)} = \begin{pmatrix}2\\1\end{pmatrix}$. Thus $x^{(1)}(t) = \begin{pmatrix}-2\\1\end{pmatrix}e^{-t}$ and

$x^{(2)}(t) = \begin{pmatrix}2\\1\end{pmatrix}e^{-t/2}$ so a fundamental matrix Ψ is $\begin{pmatrix}-2e^{-t} & 2e^{-t/2}\\ e^{-t} & e^{-t/2}\end{pmatrix}$.

2b. To find the first column of Φ we choose c_1 and c_2 so that

$c_1x^{(1)}(0) + c_2x^{(2)}(0) = \begin{pmatrix}1\\0\end{pmatrix}$, which yields $-2c_1 + 2c_2 = 1$

and $c_1 + c_2 = 0$. Thus $c_1 = -1/4$ and $c_2 = 1/4$ and the first

column of Φ is $\begin{pmatrix}e^{-t}/2 + e^{-t/2}/2\\ -e^{-t}/4 + e^{-t/2}/4\end{pmatrix}$. The second colunm of Φ

is determined by $d_1x^{(1)}(0) + d_2x^{(2)}(0) = \begin{pmatrix}0\\1\end{pmatrix}$ which yields

$d_1 = d_2 = 1/2$ and thus the second column of Φ is

$\begin{pmatrix}-e^{-t} + e^{-t/2}\\ e^{-t}/2 + e^{-t/2}/2\end{pmatrix}$.

4a. From Prob. 4 of Sect. 7.5 we have the two linearly independent solutions $x^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$ and $x^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$. Hence a fundamental matrix Ψ is given by $\Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix}$.

4b. To find the fundamental matrix $\Phi(t)$ satisfying the I.C. $\Phi(0) = I$ we can proceed in either of two ways. One way is to find $\Psi(0)$, invert it to obtain $\Psi^{-1}(0)$, and then to form the product $\Psi(t)\Psi^{-1}(0)$, which is $\Phi(t)$. Alternatively, we can find the first column of Φ by determining the linear combination $c_1x^{(1)}(t) + c_2x^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This requires that $c_1 + c_2 = 1$, $-4c_1 + c_2 = 0$, so we obtain $c_1 = 1/5$ and $c_2 = 4/5$. Thus the first column of $\Phi(t)$ is $\begin{pmatrix} (1/5)e^{-3t} + (4/5)e^{2t} \\ -(4/5)e^{-3t} + (4/5)e^{2t} \end{pmatrix}$. Similarly, the second column of Φ is that linear combination of $x^{(1)}(t)$ and $x^{(2)}(t)$ that satisfies the I.C. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus we must have $c_1 + c_2 = 0$, $-4c_1 + c_2 = 1$; therefore $c_1 = -1/5$ and $c_2 = 1/5$. Hence the second column of $\Phi(t)$ is $\begin{pmatrix} -(1/5)e^{-3t} + (1/5)e^{2t} \\ (4/5)e^{-3t} + (1/5)e^{2t} \end{pmatrix}$.

6a. Two linearly independent real-valued solutions of the given D.E. were found in Prob. 2 of Sect. 7.6. Using the result of that problem, we have

$$\Psi(t) = \begin{pmatrix} -2e^{-t}\sin 2t & 2e^{-t}\cos 2t \\ e^{-t}\cos 2t & e^{-t}\sin 2t \end{pmatrix}.$$

6b. To find $\Phi(t)$ we determine the linear combinations of the columns of $\Psi(t)$ that satisfy the I.C. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. In the first case c_1 and c_2 satisfy $0c_1 + 2c_2 = 1$ and $c_1 + 0c_2 = 0$. Thus $c_1 = 0$ and $c_2 = 1/2$. In the second case we have $0c_1 + 2c_2 = 0$ and $c_1 + 0c_2 = 1$, so $c_1 = 1$ and $c_2 = 0$. Using these values of c_1 and c_2 to form the first and second columns of $\Phi(t)$ respectively, we obtain $\Phi(t) = \begin{pmatrix} e^{-t}\cos 2t & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t/2 & e^{-t}\cos 2t \end{pmatrix}$.

10b. From Prob. 14 Sect. 7.5 we have $x^{(1)} = \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix} e^t$,

$x^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-2t}$ and $x^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} e^{3t}$. For the first column of Φ we want to choose c_1, c_2, c_3 such that

$c_1 x^{(1)}(0) + c_2 x^{(2)}(0) + c_3 x^{(3)}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Thus

$c_1 + c_2 + c_3 = 1$, $-4c_1 - c_2 + 2c_3 = 0$ and $-c_1 - c_2 + c_3 = 0$, which yield $c_1 = 1/6$, $c_2 = 1/3$ and $c_3 = 1/2$. The first column of Φ is then

$(e^t/6 + e^{-2t}/3 + e^{3t}/2,\ -2e^t/3 - e^{-2t}/3 + e^{3t},\ -e^t/6 - e^{-2t}/3 + e^{3t}/2)^T$.

Likewise, for the second column we have

$d_1 x^{(1)}(0) + d_2 x^{(2)}(0) + d_3 x^{(3)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, which yields

$d_1 = -1/3$, $d_2 = 1/3$ and $d_3 = 0$ and thus $(-e^t/3 + e^{-2t}/3,\ 4e^t/3 - e^{-2t}/3,\ e^t/3 - e^{-2t}/3)^T$ is the second column of $\Phi(t)$. Finally, for the third column we

have $e_1 x^{(1)}(0) + e_2 x^{(2)}(0) + e_3 x^{(3)}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which gives

$e_1 = 1/2$, $e_2 = -1$ and $e_3 = 1/2$ and hence $(e^t/2 - e^{-2t} + e^{3t}/2,\ -2e^t + e^{-2t} + e^{3t},\ -e^t/2 + e^{-2t} + e^{3t}/2)^T$ is the third column of $\Phi(t)$.

11. From Eq. (14) the solution is given by $\Phi(t)x^0$. Thus

$$x = \begin{pmatrix} 3e^t/2 - e^{-t}/2 & -e^t/2 + e^{-t}/2 \\ 3e^t/2 - 3e^{-t}/2 & -e^t/2 + 3e^{-t}/2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 7e^t/2 - 3e^{-t}/2 \\ 7e^t/2 - 9e^{-t}/2 \end{pmatrix} = \frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t - \frac{3}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}.$$

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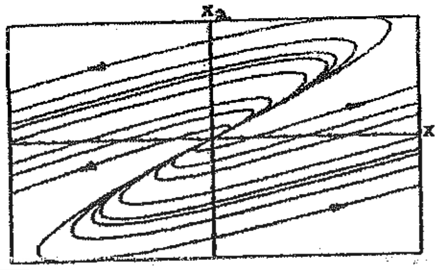
1a.



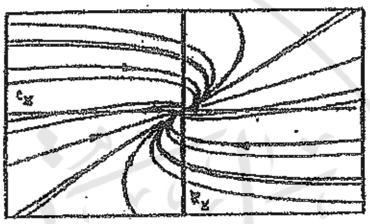
1b. From the general solution we have $\dfrac{x_2}{x_1} = \dfrac{c_1 + c_2t}{2c_1 + 2c_2t + c_2}$, so that $\lim\limits_{t\to\infty} \dfrac{x_2}{x_1} = \dfrac{1}{2}$. Thus all solutions diverge to infinity along lines of slope $\frac{1}{2}$ which can be seen in the trajectories shown in Part(a).

1c. The eigenvalues and eigenvectors of the given coefficient matrix satisfy $\begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The determinant of coefficients is $(3-r)(-1-r) + 4 = r^2 - 2r + 1 = (r-1)^2$ so $r_1 = 1$ and $r_2 = 1$. The eigenvectors corresponding to this double eigenvalue satisfy $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or $\xi_1 - 2\xi_2 = 0$. Thus the only eigenvectors are multiples of $\xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. One solution of the given D.E. is $x^{(1)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t$, but there is no second solution of this form. To find a second solution we assume, as in Eq. (13), that $x = \xi te^t + \eta e^t$ and substitute this expression into the D.E. As in Ex. 2 we find that ξ is an eigenvector, so we choose $\xi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then η must satisfy Eq.(24): $(A - rI)\eta = \xi$, or $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for this problem. Solving these equations yields $\eta_1 - 2\eta_2 = 1$. If $\eta_2 = k$, where k is an arbitrary constant, then $\eta_1 = 1 + 2k$. Hence the second solution that we obtain is

$$x^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 + 2k \\ k \end{pmatrix} e^t = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + k\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t.$$

The last term is a multiple of the first solution $x^{(1)}(t)$ and may be neglected, that is, we may set $k = 0$. Thus $x^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ and the general solution is $x = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$.

3b. The origin is attracting. That is, as $t \to \infty$ the solution approaches the origin tangent to the line $x_2 = x_1/2$, which is obtained by taking the $\lim_{t \to 0} \frac{x_2}{x_1}$ similar to Prob.1.



5. Substituting $x = \xi e^{rt}$ into the given system, we find that the eigenvalues and eigenvectors satisfy

$$\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of coefficients is $-r^3 + 3r^2 - 4$ and thus $r_1 = -1$, $r_2 = 2$ and $r_3 = 2$. The eigenvector corresponding to r_1 satisfies

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which yields $\xi^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and

$$x^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}.$$

The eigenvectors corresponding to the double eigenvalue must satsify

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields the single eigenvector $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and hence

$$x^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}.$$

The second solution corresponding to the double eigenvalue will have the form specified by

Eq.(13), which yields $x^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \eta e^{2t}$.

Substituting this into the given system, or using Eq.(24), we find that η satisfies $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

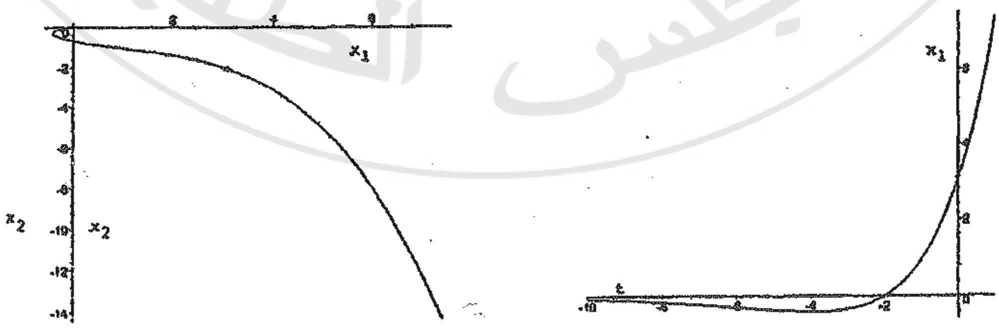
Using row reduction we find that $\eta_1 = 1$ and $\eta_2 + \eta_3 = 1$, where either η_2 or η_3 is arbitrary. If we choose $\eta_2 = 0$, then $\eta = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and thus $x^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$. The general solution is then $x = c_1x^{(1)} + c_2x^{(2)} + c_3x^{(3)}$.

9a. We have $\begin{vmatrix} 2-r & 3/2 \\ -3/2 & -1-r \end{vmatrix} = (r-1/2)^2 = 0$. For $r = 1/2$, the eigenvector is given by $\begin{pmatrix} 3/2 & 3/2 \\ -3/2 & -3/2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$, so $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t/2}$ is one solution. For the second solution we have $x = \xi te^{t/2} + \eta e^{t/2}$, where $(A - \frac{1}{2}I)\eta = \xi$, A being the coefficient matrix for this problem. This last equation reduces to $3\eta_1/2 + 3\eta_2/2 = 1$ and $-3\eta_1/2 - 3\eta_2/2 = -1$. Choosing $\eta_2 = 0$ yields $\eta_1 = 2/3$ and hence the general solution is

$$x = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t/2} + c_2\begin{pmatrix} 2/3 \\ 0 \end{pmatrix}e^{t/2} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}te^{t/2}.$$

$x(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ gives $c_1 + 2c_2/3 = 3$ and $-c_1 = -2$, and hence $c_1 = 2$, $c_2 = 3/2$. The graphs are shown for $-10 \le t \le 1$.

9b.



11a. Since the coefficient matrix is lower triangular, the eigenvalues are easily found to be r = 1,1,2. For r = 2, we have $\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields $\xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, so one solution is $x^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t}$. For r = 1, we have $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, which yields the second solution

$x^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}e^{t}$. The third solution is of the form

$x^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t} + \eta e^{t}$, where $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix}\eta = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$ and thus $\eta_1 = -1/4$ and $6\eta_2 + \eta_3 = -21/4$. Choosing $\eta_2 = 0$ gives $\eta_3 = -21/4$ and hence

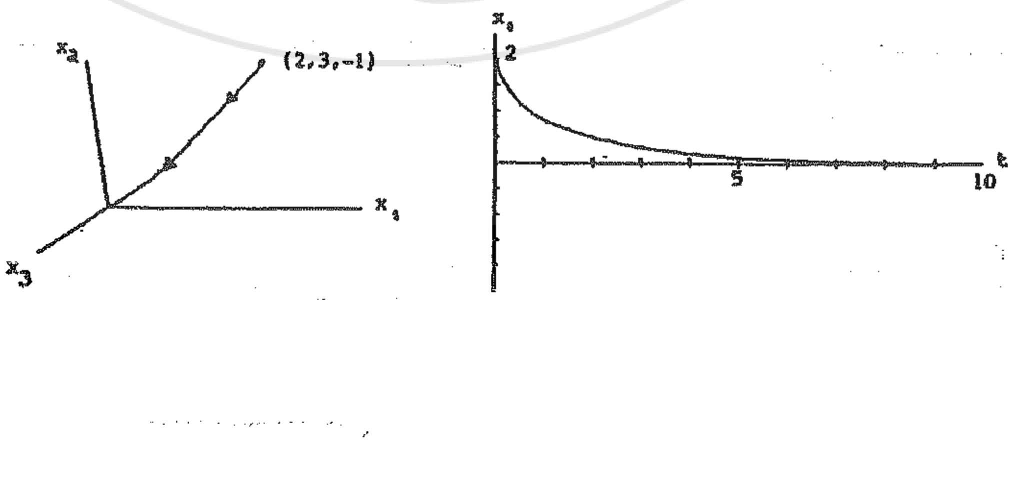
$$x(t) = c_1\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}e^{t} + c_3\left[\begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix}e^{t} + \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t}\right].$$

The I.C. then yield $c_1 = 2$, $c_2 = 4$ and $c_3 = 3$ and hence

$$x = \begin{pmatrix} -1 \\ 2 \\ -33 \end{pmatrix}e^{t} + 4\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}te^{t} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{2t},$$

which becomes unbounded as $t \to \infty$.

12b.



14. Assuming $x = \xi t^r$ and substituting into the given system, we find r and ξ must satisfy $\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has the double eigenvalue $r = -3$ and single eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence one solution of the given D.E. is

$x^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3}$. By analogy with the scalar case considered in Sect. 5.4 and Ex. 2 of this section, we seek a second solution of the form $x = \eta t^{-3}\ln t + \zeta t^{-3}$. Substituting this expression into the D.E. we find that η and ζ satisfy the equations $(A + 3I)\eta = 0$ and

$(A + 3I)\zeta = \eta$, where $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ and I is the identity

matrix. Thus $\eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, from above, and ζ then satisfies

$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Choosing $\zeta_1 = 0$ we obtain $\zeta_2 = -1/4$ and

hence a second solution is $x^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^{-3}\ln t + \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} t^{-3}$.

15. The eigenvalues are given by $r^2 - (a+d)r + ad-bc = 0$. Use the quadratic formula to find the roots. Then show that the roots are either real and negative or else are complex with negative real part when $a+d < 0$ and $ad-bc>0$. In both these cases the solution approaches zero as $t \to \infty$.

17a. The eigenvalues and eigenvectors of the coefficient

matrix satisfy $\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The determinant

of coefficients is $8 - 12r + 6r^2 - r^3 = (2-r)^3$, so the eigenvalues are $r_1 = r_2 = r_3 = 2$. The eigenvectors corresponding to this triple eigenvalue satisfy

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Using row reduction we can reduce this to the equivalent system $\xi_1 - \xi_2 - \xi_3 = 0$, and $\xi_2 + \xi_3 = 0$. If we let $\xi_2 = 1$, then $\xi_1 = 0$ and $\xi_3 = -1$, so the only eigenvectors are multiples of $\xi = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

17b. From Part(a), one solution of the given D.E. is $x^{(1)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$, but there are no other linearly independent solutions of this form.

17c. We now seek a second solution of the form $x = \xi te^{2t} + \eta e^{2t}$. Thus $Ax = A\xi te^{2t} + A\eta e^{2t}$ and $x' = 2\xi te^{2t} + \xi e^{2t} + 2\eta e^{2t}$. Equating like terms, we then have $(A-2I)\xi = 0$ and $(A-2I)\eta = \xi$. Thus ξ is the same as in Part(a) and the second equation yields $\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. By row reduction this is equivalent to the system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. If we choose $\eta_3 = 0$, then $\eta_2 = 1$ and $\eta_1 = 1$, so $\eta = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence a second solution of the D.E. is

$$x^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$$

17d. Assuming $x = \xi(t^2/2)e^{2t} + \eta te^{2t} + \zeta e^{2t}$, we have $Ax = A\xi(t^2/2)e^{2t} + A\eta te^{2t} + A\xi e^{2t}$ and $x' = \xi te^{2t} + 2\xi(t^2/2)e^{2t} + \eta e^{2t} + 2\eta te^{2t} + 2\zeta e^{2t}$ and thus $(A-2I)\xi = 0$, $(A-2I)\eta = \xi$ and $(A-2I)\zeta = \eta$. Again, ξ and η are as found previously and the last equation is equivalent to

$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. By row reduction we find the equivalent system $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$. If we let $\zeta_2 = 0$, then $\zeta_3 = 3$ and $\zeta_1 = 2$, so $\zeta = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and

$$x^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}(t^2/2)e^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}te^{2t} + \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}e^{2t}.$$

17e. Ψ is the matrix with $x^{(1)}$ as the first column, $x^{(2)}$ as the second column and $x^{(3)}$ as the third column.

17f. $T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and using row operations on T and I, or a computer algebra system, $T^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$ and thus

$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} = J$, which is equivalent to Eq.(29) for this problem.

19a. $J^2 = JJ = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$

$J^3 = JJ^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$

19b. Based upon the results of Part(a), assume

$J^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$, then

$$J^{n+1} = JJ^n = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$= \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$, which is the same as J^n with n replaced by n+1. Thus, by mathematical induction, J^n has the desired form.

19c. From Eq.(23), Sect. 7.7, we have

$$\exp(Jt) = I + \sum_{n=1}^{\infty} \frac{J^n t^n}{n!}$$

$$= I + \sum_{n=1}^{\infty} \begin{pmatrix} \dfrac{\lambda^n t^n}{n!} & \dfrac{n\lambda^{n-1}t^n}{n!} \\ 0 & \dfrac{\lambda^n t^n}{n!} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} & \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} \\ 0 & 1 + \displaystyle\sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!} \end{pmatrix}$$

$= \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$, since

$$\sum_{n=1}^{\infty} \frac{\lambda^{n-1}t^n}{(n-1)!} = t(1 + \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{n!}) = te^{\lambda t}.$$

19d. From Eq.(28), Sect. 7.7, we have

$$x = \exp(Jt)x^0 = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} = \begin{pmatrix} x_1^0 e^{\lambda t} + x_2^0 te^{\lambda t} \\ x_2^0 e^{\lambda t} \end{pmatrix}$$

$$= \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix} e^{\lambda t} + \begin{pmatrix} x_2^0 \\ 0 \end{pmatrix} te^{\lambda t}.$$

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1. From Sect. 7.5 Prob. 3 we have

$x^{(c)} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$. Note that

$g(t) = \begin{pmatrix}1\\0\end{pmatrix} e^t + \begin{pmatrix}0\\1\end{pmatrix} t$ and that $r = 1$ is an eigenvalue of the coefficient matrix. Thus if the method of undetermined coefficients is used, the assumed form is given by Eq.(18). Following the steps of Ex.2 then yields the desired solution.

2. Using methods of previous sections, we find that the eigenvalues are $r_1 = 2$ and $r_2 = -2$, with corresponding eigenvectors $\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix}$. Thus

$x^{(c)} = c_1\begin{pmatrix}\sqrt{3}\\1\end{pmatrix} e^{2t} + c_2\begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} e^{-2t}$. Writing the nonhomogeneous term as $\begin{pmatrix}1\\0\end{pmatrix} e^t + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} e^{-t}$ we see that we can assume $v(t) = ae^t + be^{-t}$ as the particlar solution. Substituting this in the D.E., we obtain

$ae^t - be^{-t} = Aae^t + Abe^{-t} + \begin{pmatrix}1\\0\end{pmatrix} e^t + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} e^{-t}$, where A is the given coefficient matrix. All the terms involving e^t must add to zero and thus we have $Aa - a + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. This is equivalent to the system

$\sqrt{3}a_2 = -1$ and $\sqrt{3}a_1 - 2a_2 = 0$, or $a_1 = -2/3$ and $a_2 = -1/\sqrt{3}$. Likewise the terms involving e^{-t} must add to zero, which yields $Ab + b + \begin{pmatrix}0\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. This is equilvalent to $2b_1 + \sqrt{3}b_2 = 0$ and $\sqrt{3}b_1 = -\sqrt{3}$ and thus $b_1 = -1$ and $b_2 = 2/\sqrt{3}$. Substituting these values for a and b into v(t) and adding v(t) to $x^{(c)}$ yields the desired solution.

3. From Prob.3 of Sect.7.6 we find that a fundamental matrix is

$\Psi(t) = \begin{pmatrix}5\cos t & 5\sin t\\2\cos t + \sin t & -\cos t + 2\sin t\end{pmatrix}$. The inverse matrix is

$$\Psi^{-1}(t) = \begin{pmatrix} \frac{\cos t - 2\sin t}{5} & \sin t \\ \frac{2\cos t + \sin t}{5} & -\cos t \end{pmatrix},$$ which may be found as in Sect. 7.2 or, more efficiently, by using a computer algebra system. Thus we may use the method of variation of parameters where $x = \Psi(t)u(t)$ and $u(t)$ is given by $u'(t) = \Psi^{-1}(t)g(t)$ from Eq.(27). For this problem

$g(t) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$ and thus

$$u'(t) = \begin{pmatrix} \frac{\cos t - 2\sin t}{5} & \sin t \\ \frac{2\cos t + \sin t}{5} & -\cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 2 - 3\cos 2t + \sin 2t \\ -1 - \cos 2t - 3\sin 2t \end{pmatrix},$$

after multiplying and using appropriate trigonometric identities. Integration and multiplication by Ψ yields the desired solution, using trigonometric identities.

This problem may also be solved using undetermined coefficients. However it is not straight forward, since the assumed form of $v(t) = a\cos t + b\sin t$ leads to singular equations for a and b. The assumed form requires that $t\cos t$ and $t\sin t$ be used, as illustrated in Eq.(18) and the subsequent discussion.

4. In this problem we use the method illustrated in Ex.1. From Prob 4 of Sect.7.5 we have the transformation matrix $T = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. Inverting T we find that $T^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$. If we let $x = Ty$ and substitute into the D.E., we obtain

$$y' = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} y + \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} y + \frac{1}{5}\begin{pmatrix} e^{-2t} + 2e^{t} \\ 4e^{-2t} - 2e^{t} \end{pmatrix}.$$ This corresponds to the two scalar equations

$$y_1' + 3y_1 = (1/5)e^{-2t} + (2/5)e^{t},$$
$$y_2' - 2y_2 = (4/5)e^{-2t} - (2/5)e^{t},$$

which may be solved by the methods of Sect. 2.1. For the first equation the integrating factor is e^{3t} and we obtain $(e^{3t}y_1)' = (1/5)e^t + (2/5)e^{4t}$, so $e^{3t}y_1 = (1/5)e^t + (1/10)e^{4t} + c_1$. For the second equation the integrating factor is e^{-2t}, so $(e^{-2t}y_2)' = (4/5)e^{-4t} - (2/5)e^{-t}$. Hence $e^{-2t}y_2 = -(1/5)e^{-4t} + (2/5)e^{-t} + c_2$. Thus

$$\mathbf{y} = \begin{pmatrix} 1/5 \\ -1/5 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/10 \\ 2/5 \end{pmatrix} e^t + \begin{pmatrix} c_1e^{-3t} \\ c_2e^{2t} \end{pmatrix}.$$ Finally,

multiplying by $\mathbf{T}$, we obtain

$$\mathbf{x} = \mathbf{T}\mathbf{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^t + c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$

The last two terms are the general solution of the corresponding homogeneous system, while the first two terms constitute a particular solution of the nonhomogeneous system.

8. For this problem we illustrate the use of Laplace Transforms. As in Eq.(43), $(s\mathbf{I} - \mathbf{A})\mathbf{X} = \begin{pmatrix} \frac{1}{s-1} \\ \frac{-1}{s-1} \end{pmatrix}$, where

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ (and we have assumed zero I.C. in order to find a particular solution), thus $\mathbf{X} = (s\mathbf{I} - \mathbf{A})^{-1}\begin{pmatrix} \frac{1}{s-1} \\ \frac{-1}{s-1} \end{pmatrix}$.

$(s\mathbf{I} - \mathbf{A})^{-1}$ is found to be $\frac{1}{s^2-1}\begin{pmatrix} s+2 & -1 \\ 3 & s-2 \end{pmatrix}$ and hence

$$\mathbf{X}(s) = \frac{1}{(s^2-1)(s-1)}\begin{pmatrix} s+3 \\ 5-s \end{pmatrix} = \begin{pmatrix} \frac{2}{(s-1)^2} - \frac{1/2}{s-1} + \frac{1/2}{s+1} \\ \frac{2}{(s-1)^2} - \frac{3/2}{s-1} + \frac{3/2}{s+1} \end{pmatrix},$$ using

partial fractions. The inverse transform gives

$\mathbf{x}(t) = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}te^t - \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t + \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}$. Note that this particular solution differs from the one shown in the text by a multiple of the homogeneous solution.

12. Since the coefficient matrix is the same as that of Prob.3, use the same procedure as done in that problem, including the Ψ^{-1} found there. In the interval $\pi/2 < t < \pi$ $\sin t > 0$ and $\cos t < 0$; hence $|\sin t| = \sin t$, but $|\cos t| = -\cos t$.

14. To verify that $x^{(c)}$ is the general solution of the corresponding homogeneous system it is sufficient to substitute $x_1(t) = \begin{pmatrix}1\\1\end{pmatrix}t$ and $x_2(t) = \begin{pmatrix}1\\3\end{pmatrix}t^{-1}$ individually into $tx' = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}x$, since x_1 and x_2 are linearly independent. For the nonhomogeous solution, substitute $x = \Psi(t)u(t)$, where $\Psi(t) = \begin{pmatrix}t & 1/t\\t & 3/t\end{pmatrix}$, into the given D.E. to obtain $t\Psi'u + t\Psi u' = A\Psi u + g(t)$. Here A is the coefficeint matrix and $g(t) = \begin{pmatrix}1-t^2\\2t\end{pmatrix}$. Since $t\Psi' = A\Psi$, we then have $u' = (1/t)\Psi^{-1}(t)g(t)$. Using a computer algebra system or row operations on Ψ and I, we find that $\Psi^{-1} = \begin{pmatrix}3/2t & -1/2t\\-t/2 & t/2\end{pmatrix}$ and hence

$u_1' = \frac{3}{2t^2} - \frac{3}{2} - \frac{1}{t}$ and $u_2' = \frac{-1}{2} + \frac{t^2}{2} + t$, which yields

$u_1 = \frac{-3}{2t} - \frac{3t}{2} - \ln t + c_1$ and $u_2 = -\frac{1}{2}t + \frac{t^3}{6} + \frac{t^2}{2} + c_2$.

Multiplication of u by $\Psi(t)$ yields the desired solution.