Let $x=e^t$ and $t=\ln x$. Then

$$\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2}=\frac{1}{x^2}\left[\frac{d^2y}{dt^2}-\frac{dy}{dt}\right]$$

The new ODE with the independent variable t is

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0 \tag{1.15}$$

The characteristic equation

$$m^2+6m+9=0$$

has a double root -3. Equation (1.15) has a general solution

$$y(t)=(c_1+c_2t)e^{-3t}$$

Using the transformation again, one obtains

$$y(x)=(c_1+c_2\ln x)x^{-3}$$

Euler equations appear in solutions of BVPs involving spherical geometry.

Exercises 1.2

1. Determine the general solution for the equation $y''-4y'+4y=0$.
2. Solve the differential equation $y''+2y'+2y=0$.
3. Find a general solution for $y'''-2y'-4y=0$.
   Hint: Show first that the characteristic equation has a root 2.
4. Solve the boundary value problem $y''-y=0$, $y(0)=0$, $y'(\pi)=1$.
5. Find a general solution for $y^{(4)}-y=0$.
6. Solve the differential equation $y'''-5y''+6y'=0$.
7. Determine a general solution for the equation $x^2y''-3xy'+3y=0$.
8. Solve the BVP $x^2y''-3xy'+4y=0$, $y(1)=0$, $y(e)=e^2$.
9. Find a general solution for $x^2y''-xy'+5y=0$.
10. Find a solution for the BVP $x^2y''+xy'+y=0$, $y(0)=1$, $y(\pi/2)=2$.

1.5. LINEAR PDEs

A PDE is called *linear* if L is a linear partial differential operator so that

$$Lu=f \tag{1.16}$$

The variable u is dependent and f is a function of the independent variables alone. If the equation is not linear it is described as *nonlinear*. Equation (1.16) is *homogeneous* if $f\equiv 0$; otherwise it is referred to as *nonhomogeneous*. A *solution* for the equation is a function of independent variables which satisfies (1.16). The order of a PDE is the order of its highest order derivative. The following are examples of PDEs.

$$Lu=u_x+u_y=x(x+2y) \tag{1.17}$$

$$Lu=u_{xy}+u_{yy}=0 \tag{1.18}$$

$$Lu=u_yu_{yy}+uu_x=0 \tag{1.19}$$

Equation (1.17) is linear, nonhomogeneous of order 1 with a solution $u=x^2y$. The second equation (1.18) is linear, homogeneous of order 2. One can verify that $u=\sin x$, $u=e^{y-x}$, $u=g(x)$ and $u=h(y-x)$ are all solutions of (1.18). The functions g and h are arbitrary. The last equation (1.19) is nonlinear, homogeneous of order 2. It has a solution $u=\sin(x+y)$.

For ODEs of nth order, general solutions are families of functions with n arbitrary constants. Instead of arbitrary constants, general solutions for PDEs are arbitrary functions of definite functions. The last two solutions mentioned for (1.18) were arbitrary functions $g(x)$ and $h(y-x)$. This implies that functions e^x, $\cos x$, $\sin(y-x)$, $(y-x)^2$, $\ln(y-x)$, and all others that are appropriately differentiable functions of x alone or $y-x$ are solutions of (1.18). Finding a particular solution from a general solution satisfying a constraint may be a difficult task. It may be preferable to find a particular solution satisfying specified conditions directly.

1.6. CLASSIFICATION OF A LINEAR PDE OF SECOND ORDER

A second order linear PDE with two independent variables has the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \tag{1.20}$$

where coefficients $A,\ldots,G$ are functions of x and y alone. The equation is *hyperbolic*, *elliptic*, or *parabolic* at a specific point in a domain as

$$B^2-4AC \tag{1.21}$$

is positive, negative, or zero. The classification is analogous to the geometry classification of conic sections. It can be shown by proper coo...

transformation that the nature of (1.20) is invariant and the sign of (1.21) is unaltered. Equation (1.20) can be classified different at different points. Should the coefficients $A, \ldots, G$ be constants, then the equation is a single type for all points of the domain. For details of the classification, and information on canonical forms and characteristic equations, the reader may refer to Sommerfeld [31, pp. 36–43]. Illustrations of the classification follow:

(a) $u_{xx} - u_{yy} = 0$ is hyperbolic with $B^2 - 4AC = 4$.
(b) $u_{xx} + u_{yy} + u = xy$ is elliptic with $B^2 - 4AC = -4$.
(c) $u_{xx} + u_x - u_y + u = 0$ is parabolic with $B^2 - 4AC = 0$.
(d) $u_{xx} + xu_{yy} = 0$ is elliptic, parabolic, or hyperbolic as $x > 0$, $x = 0$, or $x < 0$ since $B^2 - 4AC = -4x$.

1.7. BOUNDARY VALUE PROBLEMS WITH PDEs

A mathematical problem composed of a PDE and certain constraints on the boundary of the domain is called *a boundary value problem*. If u is the dependent variable of the PDE it must satisfy the PDE in a domain of its independent variables and also constraint equations involving u and appropriate partial derivatives of u.

Problems involving time t as one of the independent variables of the PDE may have a condition given at one specified time, frequently when $t = 0$. Such a constraint is referred to as an initial condition. If all the supplementary conditions are initial conditions then the problem is an *initial value problem*. A problem that has both initial and boundary conditions is properly called an *initial-boundary value problem*. In the literature one often finds the use of the terminology *boundary value problem* to include the initial-boundary value problem or mixed problem. In the problem

$$u_t(x,t) = a^2 u_{xx}(x,t), \qquad (0 < x < 1,\ t > 0) \tag{1.22}$$

$$u(0,t) = u(1,t) = 0, \qquad (t \geq 0) \tag{1.23}$$

$$u(x,0) = f(x), \qquad (0 \leq x \leq 1) \tag{1.24}$$

the condition (1.24) is an initial condition, while (1.23) are boundary conditions. The problem (1.22)–(1.24) is an initial-boundary value problem or simply a boundary value problem depending on one's preference.

Existence and uniqueness are important topics for boundary or initial value problems of PDEs. At this time we indicate only a Cauchy-Kovalevsky theorem for the second order PDE with initial conditions. For details see Zachmanoglou and Thoe [39, pp. 100–109].

Theorem.* Let

$$u_{tt} = F(t, x, u_t, u_x, u_{tx} u_{xx}) \tag{1.25}$$

be the PDE with initial conditions

$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x) \tag{1.26}$$

Functions $f(x)$ and $g(x)$ are defined on an interval of the x axis containing the origin. Assume that $f(x)$ and $g(x)$ are analytic in a neighborhood of the origin and F is analytic in a neighborhood of the point $(0, 0, f(0), g(0), f'(0), g'(0), f''(0))$. Then the problem (1.25), (1.26) has a unique analytic solution $u(x,t)$ in a neighborhood of the origin.

The Cauchy-Kovalevsky theorem serves as an example of an existence-uniqueness theorem for an IVP with a PDE. At a later time we will investigate properties of existence and uniqueness for a few problems of mathematical physics.

A mathematical problem is *well posed* if it has a unique solution that depends continuously on initial or boundary data. The last requirement implied above is sometimes referred to as *stability*. For a mathematical model to describe a specified phenomenon, a small modification in the original data should result only in a small variation of the solution. Even though most of our problems are well posed, it is important to know that there are problems that fail to meet these conditions. From a family of examples attributed to Hadamard [16, p. 33–34] the elliptic equation

$$u_{xx} + u_{yy} = 0, \qquad -\infty < x < \infty, \qquad y > 0$$

with the initial conditions on the x axis

$$u(x,0) = 0, \qquad -\infty < x < \infty$$

$$u_y(x,0) = e^{-\sqrt{n}} \sin nx, \qquad -\infty < x < \infty$$

has the solution

$$u(x,y) = \frac{e^{-\sqrt{n}}}{n} \sin nx \sinh ny \tag{1.27}$$

As $n \to \infty$, $e^{-\sqrt{n}} \sin nx \to 0$, but for $x \neq 0$ the solution $e^{-\sqrt{n}}/n \sin nx \sinh ny \to \infty$ for any $y \neq 0$. The solution (1.27) fails to depend continuously on the initial data, and therefore is unstable.

*From Zach…

1.8. SECOND ORDER LINEAR PDEs WITH CONSTANT COEFFICIENTS

One of the simplest equations in this category is a second order partial derivative equal to a function of the independent variables. Illustrations of this type follow.

Example 1.6. Find a solution for the PDE

$$u_{xy}=xy^2$$

First integrate relative to y with x fixed. Then

$$u_x=\frac{xy^3}{3}+f'(x)$$

where $f'(x)$ is an arbitrary function of x only. A second integration relative to x with y fixed produces the solution

$$u=\frac{x^2y^3}{6}+f(x)+g(y)$$

where $g(y)$ is an arbitrary function of y alone. Anticipating an integration relative to x, we select an arbitrary function $f'(x)$ in derivative form in the first step.

Example 1.7. Solve the PDE

$$u_{yy}=e^y$$

with the supplementary conditions

$$u_y(x,0)=x^3 \quad \text{and} \quad u(x,0)=e^x$$

Integrating the PDE relative to y, one obtains

$$u_y=e^y+f(x)$$

Due to the nature of the first supplementary condition we determine $f(x)$ before finding u.

$$u_y(x,0)=x^3=1+f(x)$$

This implies that

$$f(x)=x^3-1$$

Therefore,

$$u_y=e^y+x^3-1$$

Integrating a second time relative to y, one finds

$$u=e^y+x^3y-y+g(x)$$

To determine $g(x)$ we use the second condition,

$$u(x,0)=e^x=1+g(x)$$

It follows that

$$g(x)=e^x-1$$

The solution for the problem is

$$u=e^y+x^3y-y+e^x-1$$

For a second type, we consider the equation with second partial derivatives only

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0 \quad (1.28)$$

where A, B, and C are real constants. Let

$$u=f(y+mx) \quad (1.29)$$

be a proposed solution. We attempt to find m so that (1.29) satisfies (1.28). If f is a solution of (1.28) it must be twice differentiable. Substituting (1.29) into (1.28), we obtain

$$Am^2f''(y+mx)+Bmf''(y+mx)+Cf''(y+mx)=0$$

If $f''(y+mx)\neq 0$,

$$Am^2+Bm+C=0 \quad (1.30)$$

The polynomial equation (1.30) is a characteristic equation. If it has distinct roots $m=m_1$ and $m=m_2$ then $u=f(y+m_1x)$ and $u=g(y+m_2x)$ are solutions of (1.28). The linear combination

$$u=f(y+m_1x)+g(y+m_2x) \quad (1.31)$$

If m_1 and m_2 are distinct and new variables

$$r=y+m_1x \quad \text{and} \quad s=y+m_2x \tag{1.32}$$

are introduced in (1.28), the new equation is

$$A\left[m_1^2u_{rr}+2m_1m_2u_{rs}+m_2^2u_{ss}\right]+B\left[m_1u_{rr}+(m_1+m_2)u_{rs}+m_2u_{ss}\right]$$
$$+C\left[u_{rr}+2u_{rs}+u_{ss}\right]=0 \tag{1.33}$$

assuming $u_{rs}=u_{sr}$. Equation (1.33) can be simplified so that the coefficients of u_{rr} and u_{ss} are both zero, and

$$u_{rs}=0 \tag{1.34}$$

Equation (1.34) is a special type solvable by integration. It has the solution

$$u=f(r)+g(s)$$

Replacing r and s as given in (1.32) one obtains the solution (1.31).

The d'Alembert solution of the wave equation

$$u_{tt}=c^2u_{xx}, c>0 \tag{1.35}$$

is a good illustration of the transformation described in (1.32). Equation (1.35) is hyperbolic. The auxiliary equation is

$$m^2-c^2=0 \tag{1.36}$$

The transformation (1.32) becomes

$$r=x+ct \quad \text{and} \quad s=x-ct \tag{1.37}$$

Using (1.37) as described above, we obtain

$$u=f(x+ct)+g(x-ct)$$

for the solution of the wave equation.

The solutions of the characteristic equation (1.30) may be (a) real and distinct, (b) double, or (c) conjugate (imaginary part nonzero) complex numbers. The discriminant for the quadratic equation (1.30) is the same as the discriminant for (1.28). Therefore, a hyperbolic PDE (1.28) is matched by real and distinct roots in (1.30); an elliptic equation (1.28) is paired with conjugate complex roots in (1.30); and a parabolic equation (1.28) is associated with a double root in (1.30).

If $m_1=m_2$ in (1.30), then $B^2-4AC=0$. The two roots are $m_1=-B/2A$. A second solution for (1.28) is

$$u=xg(y+m_1x)$$

This result can be verified if $m_1=m_2=-B/2A$ is employed. In this case

$$u=f(y+m_1x)+xg(y+m_1x) \tag{1.38}$$

is a general solution for (1.28). One can show that

$$u=f(y+m_1x)+yg(y+m_1x) \tag{1.39}$$

is a general solution of (1.28) also.

Example 1.8. Find a general solution for $u_{xx}+4u_{xy}+4u_{yy}=0$.

This equation is parabolic. The characteristic equation has a double root -2. A general solution using (1.38) is

$$u=f(y-2x)+xg(y-2x)$$

If (1.39) is used

$$u=f(y-2x)+yg(y-2x)$$

is a general solution.

Example 1.9. Determine a solution for $u_{xx}+4u_{yy}=0$.

The discriminant $B^2-4AC<0$. Therefore, the equation is elliptic. The characteristic equation has roots $\pm 2i$. The general solution is written in the same form as (1.31). For this PDE

$$u=f(y-2ix)+g(y+2ix)$$

is a general solution.

By comparison with an ODE one may suspect the existence of an exponential solution for the homogeneous PDE

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0 \tag{1.40}$$

where the coefficients $A,\ldots,F$ are real constants. Let

$$u=e^{\alpha x+\beta y} \tag{1.41}$$

where α and β are real, be a proposed solution. Substituting (1.41) in (1.40), one obtains the condition

$$A\alpha^2+B\alpha\beta+C\beta^2+D\alpha+E\beta+F=0 \tag{1.42}$$

In the quadratic equation (1.42), one may solve for β as a function of α or α as a function of β. Assume that we solve for β and obtain $\beta_1(\alpha)$ and $\beta_2(\alpha)$. A particular solution

$$u=K_1e^{\alpha x+\beta_1(\alpha)y}+K_2e^{\alpha x+\beta_2(\alpha)y}$$

is the result.

Example 1.10. Determine a solution for the PDE

$$u_{xx}-u_{yy}-2u_x+u=0 \tag{1.43}$$

Substitute the exponential function

$$u=e^{\alpha x+\beta y}$$

in (1.43). The characteristic equation

$$\alpha^2-\beta^2-2\alpha+1=0$$

has solutions

$$\beta=\alpha-1 \quad \text{and} \quad \beta=-\alpha+1$$

Using superposition of the two solutions one finds the particular solution

$$u=K_1e^{\alpha x+(\alpha-1)y}+K_2e^{\alpha x+(-\alpha+1)y}$$

This solution may be written

$$u=K_1e^{-y}e^{\alpha(x+y)}+K_2e^{y}e^{\alpha(x-y)}$$

We may conjecture that a general solution has the form

$$u=e^{-y}f(x+y)+e^{y}g(x-y) \tag{1.44}$$

where f and g are twice differentiable arbitrary functions. By substituting (1.44) into (1.43), we confirm that (1.44) is a solution.

When the left member of (1.42) has distinct linear factors, the type of simplification discussed is possible. The case of a repeated linear factor may be considered by using a result comparable to (1.38) or (1.39).

Example 1.11. Examine

$$u_{xx}-2u_{xy}+u_{yy}-2u_y+2u_x+u=0$$

for a general solution.

Let $u=e^{\alpha x+\beta y}$ and obtain a characteristic equation

$$\alpha^2-2\alpha\beta+\beta^2-2\beta+2\alpha+1=0$$

The double root is

$$\beta=\alpha+1$$

An exponential form of a solution is

$$u=e^{y}\left[K_1e^{\alpha(x+y)}+K_2xe^{\alpha(x+y)}\right]$$

A general solution

$$u=e^{y}\left[f(x+y)+xg(x+y)\right]$$

can be verified.

Certain cases may arise in (1.42) where linear factors with imaginary elements appear.

Example 1.12. Investigate a solution for the equation

$$u_{xx}+u_{yy}-2u_y+u=0 \tag{1.45}$$

Let

$$u=e^{\alpha x+\beta y}$$

be a proposed solution. The characteristic equation

$$\alpha^2+\beta^2-2\beta+1=0$$

has two linear factors with imaginary elements for which

$$\beta=1\pm i\alpha$$

An exponential solution is

$$u=e^{y}\left[e^{\alpha(x+iy)}+e^{\alpha(x-iy)}\right] \tag{1.46}$$

A general solution for (1.45) is suggested by (1.46)

$$u=e^{y}\left[f(x+iy)+g(x-iy)\right] \quad (1.47)$$

It is easy to verify that (1.47) is a solution of (1.45).

In some situations the exponential procedure may produce a set of useful particular solutions, but fail to suggest a general solution.

Example 1.13. Determine a solution for the equation

$$u_{xx}+u_{yy}+4u=0$$

One obtains a characteristic equation

$$\alpha^2+\beta^2+4=0$$

with

$$\beta=\pm i\sqrt{\alpha^2+4}$$

If the exponential substitution is followed then

$$u=e^{\alpha x}\left[K_1 e^{i\sqrt{\alpha^2+4}\,y}+K_2 e^{-i\sqrt{\alpha^2+4}\,y}\right]$$

This solution can be expressed

$$u=e^{\alpha x}\left[M_1\cos\sqrt{\alpha^2+4}\,y+M_2\sin\sqrt{\alpha^2+4}\,y\right]$$

if K_1 and K_2 are properly related to M_1 and M_2 using Euler's identity.

Equation (1.40) can be solved almost like an ODE if only partial derivatives with respect to one variable appear. Arbitrary constants of the ODE solution become arbitrary functions of the remaining variable.

Example 1.14. Solve the PDE

$$u_{yy}-4u_y+3u=0$$

The dependent variable u is a function of x and y, but the only derivatives involved are relative to y alone. The corresponding ODE, with u as a function of y,

$$\frac{d^2u}{dy^2}-4\frac{du}{dy}+3u=0$$

has a solution

$$u=c_1e^{3y}+c_2e^{y}$$

Arbitrary constants c_1 and c_2 are replaced by arbitrary functions of x alone. The general solution becomes

$$u=e^{3y}f(x)+e^{y}g(x)$$

Other PDEs may be solved by using comparable solutions of ODEs.

Example 1.15. Find a solution for the PDE

$$xu_{xy}+2u_y=y^2$$

We observe that the equation may be written

$$\frac{\partial}{\partial y}\left[xu_x+2u\right]=y^2$$

By integrating, we obtain

$$xu_x+2u=\frac{y^3}{3}+f(x)$$

Dividing by x, with y fixed, one recognizes a linear differential equation of first order

$$u_x+\frac{2}{x}u=\frac{y^3}{3x}+\frac{f(x)}{x}$$

The integrating factor is x^2. This equation may be displayed

$$\frac{\partial}{\partial x}(x^2u)=\frac{xy^3}{3}+xf(x)$$

Integrating the most recent equation, we obtain

$$x^2u=\frac{x^2y^3}{6}+f^*(x)+G(y)$$

An explicit form of the solution is

$$u=\frac{y^3}{6}+F(x)+\frac{1}{x^2}G(y)$$

For more information regarding Section 1.8, the reader may consult Hildebrand [18, Chapter 8].

1, 2, 3, 4, 8, 9, 10

Exercises 1.3

1. Solve the boundary value problem

$$u_{xy}=0, \qquad u_x(x,0)=\cos x, \qquad u\left(\frac{\pi}{2}, y\right)=\sin y$$

2. Find the solution for

$$u_{yx}=x^2y, \qquad u_y(0, y)=y^2, \qquad u(x,1)=\cos x$$

3. Determine a solution for $u_{xx}=\cos x$ if

$$u(0, y)=y^2 \quad \text{and} \quad u(\pi, y)=\pi \sin y.$$

4. Classify the following PDEs as hyperbolic, parabolic or elliptic:
   (a) $yu_{xx}+xu_{yy}=0$.
   (b) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}+u_x+u_y=0$.
   (c) $u_{xx}+2u_{xy}-3u_{yy}=0$.
   (d) $u_{xx}-2u_{xy}+u_{yy}=0$.
   (e) $u_{xx}+a^2u_{yy}=0$, $a>0$.
   (f) $u_{xx}-2u_{xy}+2u_{yy}=0$.
   Solve the equations (c)–(f).

5. The d'Alembert solution of the wave equation (1.35) is

$$u=f(x+ct)+g(x-ct)$$

Solve the wave equation if $u(x,0)=0$ and $u_t(x,0)=\phi(x)$.

6. (a) Determine a general solution for equation 4(c) by using the transformation $s=y-3x$, $r=y+x$.
   (b) If $u(0, y)=0$ and $u_x(0, y)=\phi(y)$ in (a), show that

$$u=\frac{1}{4}\int_{y-3x}^{y+x}\phi(\alpha)\,d\alpha$$

7. Determine a solution for $u_{xx}+2u_{xy}+u_{yy}+u_x+u_y=0$ by letting $u=e^{\alpha x+\beta y}$. After finding β as a function of α, propose a general solution. Verify the general solution.

8. Using the substitution $u=e^{\alpha x+\beta y}$ (a) find an exponential solution for $4u_{xx}-u_{yy}-2u_x+4u_y=0$; (b) propose and verify a general solution for the equation.

9. Solve the PDE $xu_{xy}+3u_y=y^3$.

10. If $Au_{xx}+Bu_{xy}+Cu_{yy}=F(x, y)$, A, B, and C are constants, then the equation has a general solution

$$u=u_c(x, y)+u_p(x, y)$$

where $u_c(x, y)$ is a general solution of $Au_{xx}+Bu_{xy}+Cu_{yy}=0$ and $u_p(x, y)$ is a particular solution of the original equation. Find a general solution for the following equations:
   (a) $u_{xx}-2u_{xy}+3u_{yy}=e^x$
   (b) $u_{xx}-u_{xy}-2u_{yy}=\sin y$.

1.9. SEPARATION OF VARIABLES

It is assumed in this method that the solution of a PDE can be expressed in the form of a product of functions of single independent variables. Using this procedure we produce an equation with one member a function of a single variable and the other member a function of the remaining variables. Each member can be a constant but not a function of all the original independent variables. This process is illustrated in the following examples.

Example 1.16. Find a solution for the PDE

$$u_t=4u_{xx} \tag{1.48}$$

using the separation of variables.

We assume that the solution of (1.48) has the form

$$u(x,t)=X(x)T(t) \tag{1.49}$$

where X is a function of x alone and T is a function of t alone. Inserting (1.49) into (1.48) we obtain

$$XT'=4X''T$$

After dividing by $4XT$, one has the variables separated in the form

$$\frac{T'}{4T}=\frac{X''}{X} \tag{1.50}$$

If (1.50) is differentiated partially relative to t, one attains the result

$$\frac{\partial}{\partial t}\left(\frac{T'}{4T}\right)=0 \tag{1.51}$$

Assuming ϕ is an arbitrary function of x alone, the solution of (1.51) is

$$\frac{T'}{4T}=\phi(x)$$

This violates the condition that T is a function of t alone unless $\phi(x)$ is a constant. A similar partial differentiation of (1.50) relative to x leads to a PDE which has a solution

$$\frac{X''}{X}=\psi(t)$$

valid only if $\psi(t)$ is constant. Therefore both members of (1.50) must be equal to the same constant, say α^2 or $-\alpha^2$.

If α^2 is used (1.50) becomes

$$\frac{T'}{4T}=\frac{X''}{X}=\alpha^2 \tag{1.52}$$

Result (1.52) is equivalent to two ODEs

$$T'-4\alpha^2T=0$$
$$X''-\alpha^2X=0 \tag{1.53}$$

The solutions of the two ODEs of (1.53) are respectively,

$$T=Ae^{4\alpha^2t}$$
$$X=B_1e^{\alpha x}+B_2e^{-\alpha x} \tag{1.54}$$

Inserting the solutions of (1.54) in (1.49) we find a solution

$$u(x,t)=e^{4\alpha^2t}[C_1e^{\alpha x}+C_2e^{-\alpha x}]$$

Where $C_1=AB_1$ and $C_2=AB_2$.

If $-\alpha^2$ is used instead of α^2 in (1.52) the two ODEs are

$$T'+4\alpha^2T=0$$
$$X''+\alpha^2X=0 \tag{1.55}$$

The solutions of (1.55) are

$$T=A^*e^{-4\alpha^2t}$$
$$X=B_1^*\cos\alpha x+B_2^*\sin\alpha x \tag{1.56}$$

Using the solutions of (1.56) in (1.49) we have

$$u=e^{-4\alpha^2t}[C_1^*\cos\alpha x+C_2^*\sin\alpha x]$$

In most of our BVPs a bounded solution will be necessary. The constants α^2 or $-\alpha^2$ must be selected to satisfy this requirement.

Example 1.17. Determine a solution for

$$u_t=a^2(u_{xx}+u_{yy}) \tag{1.57}$$

Since three independent variables appear in (1.57), we let

$$u(x,y,t)=T(t)X(x)Y(y) \tag{1.58}$$

Equation (1.57) has the form

$$T'XY=a^2(TX''Y+TXY'') \tag{1.59}$$

after substituting (1.58) in the PDE. Equation (1.59) has another form

$$\frac{T'}{a^2T}=\frac{X''}{X}+\frac{Y''}{Y} \tag{1.60}$$

Partially differentiating (1.60) relative to x, then y, and finally t, we have respectively

$$\frac{\partial}{\partial x}\left(\frac{X''}{X}\right)=0$$
$$\frac{\partial}{\partial y}\left(\frac{Y''}{Y}\right)=0 \tag{1.61}$$
$$\frac{\partial}{\partial t}\left(\frac{T'}{a^2T}\right)=0$$

Solutions of the three PDEs of (1.61) are

$$\frac{X''}{X}=-\alpha^2$$
$$\frac{Y''}{Y}=-\beta^2 \tag{1.62}$$
$$\frac{T'}{a^2T}=-(\alpha^2+\beta^2)$$

... that (1.60) be satisfied we select $-(\alpha^2+\beta^2)$ as the constant in the ... of the T equation.
... three associated ODEs

$$X''+\alpha^2 X=0$$

$$Y''+\beta^2 Y=0$$

$$T'+(\alpha^2+\beta^2)a^2T=0$$

... solutions

$$X=B_1\cos\alpha x+B_2\sin\alpha x$$

$$Y=C_1\cos\beta y+C_2\sin\beta y$$

$$T=A\exp\left[-(\alpha^2+\beta^2)a^2t\right]$$

Therefore,

$$u=\exp\left[-(\alpha^2+\beta^2)a^2t\right]\left[B_1^*\cos\alpha x+B_2^*\sin\alpha x\right]\left[C_1\cos\beta y+C_2\sin\beta y\right]$$

... a solution of (1.57). Other forms for the solution are available. The one displayed is a bounded solution.

The method of separation of variables is valuable for solving a number of important problems of mathematical physics, yet it fails for many PDEs and BVPs. Myint-U [25, pp. 128–129] shows that the second order PDE* with variable coefficients in x and y

$$A(x,y)u_{xx}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=0 \tag{1.63}$$

is separable when a functional multiplier $1/[\phi(x,y)]$ converts the new equation

$$A(x,y)X''Y+C(x,y)XY''+D(x,y)X'Y+E(x,y)XY'+F(x,y)XY=0$$

into the form

$$A_1(x)X''Y+B_1(y)XY''+A_2(x)X'Y+B_2(y)XY'+\left[A_3(x)+B_3(y)\right]XY=0$$

Explicit rules for the workability of this method are a bit elusive. Types of differential equations, kinds of coordinate systems, and forms of boundary conditions are all important items for the success of the procedure.

*The example that follows is from Myint-U [25], by permission of Elsevier North Holland, Inc.

Exercises 1.4 1(b, c, n, i, k), 2(b,c)

1. Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.
   (a) $u_{xy}-u=0$.
   (b) $u_{tt}-u_{xx}=0$.
   (c) $u_{xx}-u_{yy}-2u_y=0$.
   (d) $u_{xx}-u_{yy}+2u_x-2u_y+u=0$.
   (e) $t^2u_{tt}-x^2u_{xx}=0$.
   (f) $(t^2+x^2)u_{tt}+u_{xx}=0$.
   (g) $u_{xx}-y^2u_{yy}-yu_y=0$.
   (h) $u_{xy}=0$.
   (i) $u_{xx}-u_{xy}+u_{yy}=2x$.
   (j) $u_{xx}=u_{yy}-u_y=0$.
   (k) $u_t=u_{xx}$.

2. Find a solution for the boundary (or initial) value problems:
   (a) $u_{tt}-u_{xx}=0$, $u(x,0)=u(0,t)=0$.
   (b) $u_{xx}-u_{yy}-2u_y=0$, $u_x(0,y)=u(x,0)=0$.
   (c) $u_t=u_{xx}$, $u_x(0,t)=0$.

3. (a) Show that the equation with constant coefficients

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0$$

is separable if the coefficients meet proper conditions. Determine appropriate conditions. *Note*: Let $u(x,y)=X(x)Y(y)$ and show that a result

$$\left(\frac{X''}{X}\right)'+\frac{B}{A}\left(\frac{X'}{X}\right)'\left(\frac{Y'}{Y}\right)=0$$

is obtained from

$$\frac{X''}{X}+\frac{B}{A}\frac{X'}{X}\frac{Y'}{Y}+\frac{C}{A}\frac{Y''}{Y}=0$$

Finally, show that

$$Y'+\lambda Y=0 \quad \text{and} \quad X''-\lambda\frac{B}{A}X'+\lambda^2\frac{C}{A}X=0$$

are related ODEs.

(b) Find a solution for $u_{xx}-u_{xy}+u_{yy}=0$ by separating variables.