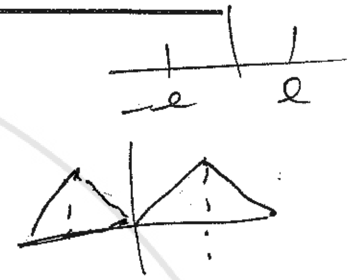
Key (1)

جامعة بيرزيت
BIRZEIT UNIVERSITY

Name:____________ **MATH 332** First Semester 2014/2015

Number:____________ Midterm



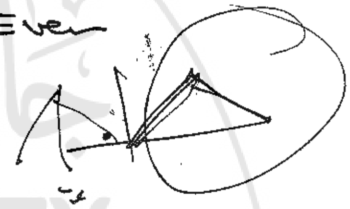
(1) (20 points) (a) Find the Fourier series of

$$f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1<x<2 \end{cases} \qquad f(x+2)=f(x)$$

$\ell = 1$

(b) Find Fourier series of $F(x)=\int_0^x f(t)\,dt$

Even



(c) Can we differentiable term by term to get $f'(x)$, if so find it.

(2) $a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx = \frac{2}{2(1)}\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos n\pi x\,dx = \frac{2}{1}\int_0^1 x\cos n\pi x\,dx$

$x \;+\; \cos n\pi x$; $1 \;-\; \frac{1}{n\pi}\sin n\pi x$; $0 \;\; -\frac{1}{n^2\pi^2}\cos n\pi x$

$= 2\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x \Big|_0^1\right]$

(6) $= 2\left[0 + \frac{1}{n^2\pi^2}(-1)^n - \left[0+\frac{1}{n^2\pi^2}\right]\right] = 2\left[\frac{(-1)^n - 1}{n^2\pi^2}\right]$

$= \begin{cases} \frac{-4}{n^2\pi^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

(2) $b_n = 0$ since f is even

$\Rightarrow f(x) = \frac{1}{2} - \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$

$= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2}\cos(2n-1)\pi x$

$$f(x)-\frac{1}{2}=\sum_{\substack{n=1\\ \cancel{odd}}}-\frac{4}{(2n+1)^2\pi^2}\cos(2n+1)\pi x$$

b)

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$$\int_0^x\left[f(s)-\frac{1}{2}\right]ds=\sum_{n=1}^{\infty}-\frac{4}{(2n+1)^3\pi^3}\sin(2n+1)\pi x$$

$$\int_0^x f(t)\,dt=\frac{1}{2}x-\sum_{n=1}^{\infty}\frac{4}{(2n+1)^3\pi^3}\sin(2n+1)\pi x+c$$

$c=0$ ($F(x)$ is odd)

$$\cancel{\left[\int_0^x x\,dx-\frac{1}{2}\int_0^1 x\,dx\right]}$$

c) Yes, The condition are satisfied

$f(x)$ is continu on $\mathbb{R}$

$f'(x)$ is piecewise contin

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so

$$f'(x)=\sum_{\substack{n=1\\ odd}}^{\infty}\frac{4}{n\pi}\sin n\pi x$$

(2) (10 points) Find the eigenvalues and Fourier of the BVP

$$x'' + \lambda x = 0 \qquad , 0 < x < \pi$$

$$x'(0) = 0 \qquad , x'(\pi) = 0$$

$\Rightarrow r^2 + \lambda = 0$

$\lambda = 0$, $y(x) = c_1 x + c_2$ $\quad y'(x) = c_1 \Rightarrow y'(0) = 0 = c_1$

2

$y'(\pi) = 0 \Rightarrow c_1 = 0 \Rightarrow y(x) = c_2 \Rightarrow \boxed{\phi_0(x) = 1}$

$\lambda < 0 \Rightarrow \lambda = -k^2$ $\quad y(x) = c_1 e^{kx} + c_2 e^{-kx}$ $\quad y'(x) = kc_1 e^{kx} - kc_2 e^{-kx} = 0$

$y'(0) = c_1 - c_2 = 0 \Rightarrow c_2 = c_1$

2

$y'(\pi) = c_1 e^{\pi k} - c_2 e^{-\pi k} = 0 \Rightarrow c_1 (2\sinh \pi k) = 0 \Rightarrow c_1 = 0, c_2 = 0$

No eigenvalues or vectors

$\lambda > 0 \Rightarrow \lambda = k^2$, $y(x) = c_1 \cos kx + c_2 \sin kx$

$y'(x) = -c_1 k \sin kx + c_2 k \cos kx$

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$y'(0) = 0 = c_2 \Rightarrow y'(\pi) = -c_1 k \sin k\pi = 0$ ($c_1 \neq 0$) $\Rightarrow k\pi = n\pi$ $\quad k = n$

$\Rightarrow$ eigenvalues $\lambda = k^2 = n^2$

$\Rightarrow$ eigenfunction $\phi_n(x) = \cos nx$

(3) (10 points) Find Fourier transform of $f(x) = e^{-x^2}$

$$f(x) = \frac{\langle \phi_0, f(x) \rangle}{\langle 1, 1 \rangle} + \sum \frac{\langle \phi_n, f \rangle}{\langle \phi_n, \phi_n \rangle} \phi_n$$

2

$$= \frac{\int_0^\pi f(x)\,dx}{\int_0^\pi 1\,dx} + \sum \frac{\int_0^\pi f(x) \cos nx\,dx}{\int_0^\pi \cos^2 nx\,dx} \cos nx$$

$\pi/2$

$$= \frac{1}{\pi} \int_0^\pi f(x)\,dx + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^\pi f(x) \cos nx\,dx \right) \cos nx$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{1}{\pi} \int_0^\pi f(x)\,dx$, $\quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx\,dx$

(3)

(4)

(~~2~~) $f(x) = e^{-x^2}$

$f'(x) = -2x\, e^{-x^2} = -2x f(x)$

(~~4~~ 3) $\phi(f'(x)) = -2\phi(x f(x))$

$\cancel{i}w\, F(w) = -2\cancel{i} F'(w)$, $F(w) = \phi(f(x))$

$F(w) = \int_{-\infty}^{\infty} e^{iwx} f(x)\, dx$

$\frac{F'(w)}{F(w)} = -\frac{1}{2} w$

(~~4~~ 3) $\int \frac{dF}{F} = \int -\frac{1}{2} w\, dw$

$\ln|F| = -\frac{1}{2}\frac{w^2}{2} + c = -\frac{1}{4} w^2 + c$

$F(w) = C e^{-\frac{w^2}{4}}$

(2) $\Rightarrow C = F(0) = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\Rightarrow F(w) = \sqrt{\pi}\, e^{-\frac{w^2}{4}}$

5

(4)(a) (15 points) Solve the PDE

$u_t = u_{xx}$

$u_x(x,0) = 0$

$u_x(x,\pi) = 0$

$u(x,0) = cos2x - \frac{1}{3}\cos 7x + 3$

$u(x,t) = X(x)T(t)$

$XT' = X''T \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$ 4

$\Rightarrow X'' + \lambda X = 0 \quad , T' + \lambda T = 0$

BCs ~~...~~ $X'(0) = 0 \quad X'(\pi) = 0$

solution eigenvalues $\lambda = 0, \lambda = n^2, n = 1 \ldots$

$\Rightarrow \phi_0(x) = 1, \quad \phi_n(x) = \cos nx, n = 1, 2 \ldots$

For $\lambda = 0, T' = 0, T(t) = A_1 \quad u(x,t) = X(x)T(t) = A \cdot A_1 = C$

For $\lambda = n^2, \quad T' + n^2 T = 0 \Rightarrow T_n(t) = c_n e^{-n^2 t}$

so $u(x,t) = \sum_{n=1}^{\infty} H_n \cos nx \, e^{-n^2 t} + H_0$

$f(x) = u(x,0) = \sum_{n=1}^{\infty} H_n \cos nx + H_0 = \cos 2x - \frac{1}{3}\cos 7x + 3$

$H_0 = 3, \quad H_2 = 1, \quad H_7 = -\frac{1}{3}, \quad H_n = 0, \quad n \neq 0, 2, 7$

$\Rightarrow u(x,t) = 3 + e^{-4t}\cos 2x - \frac{1}{3} e^{-49t} \cos 7x$

(b) (10 points) Use the above to solve

$u_t = u_{xx} + x$

$u_x(x,0) = 0$

$u_x(x,\pi) = 0$

$u(x,0) = cos2x - \frac{1}{3}\cos 7x + 3$

4(b). Let $u(x,t) = \sum_{n=0}^{\infty} B_n(t)\cos nx$ ---(1)   6

and $F(x,t) = \sum_{n=0}^{\infty} F_n(t)\cos nx$

3 where $F_n(t) = \frac{2}{\pi}\int_0^{\pi} F(x,t)\cos nx$

$\Rightarrow F_n(t) = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \begin{cases} \frac{-4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

substituting (1) in $u_t = u_{xx} + F(x,t)$

2 $\Rightarrow \sum_{n=0}^{\infty} B_n'(t)\cos nx = \sum -n^2 B_n(t)\cos nx + \begin{cases} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{-4}{n^2\pi}\cos nx, & n \text{ odd} \\ 0, & \text{o.w} \end{cases}$

n odd $\Rightarrow$

$B_n' + n^2 B_n = \frac{-4}{n^2\pi}$

$\mu(t) = e^{n^2 t}$

$B_n(t) = e^{-n^2 t}\left[\int \frac{-4}{n^2\pi} e^{n^2 t}\,dt + C\right]$ 3

3 $B_n(t) = \frac{-4}{n^4\pi} + c_n e^{-n^2 t}$

$\Rightarrow u(x,t) = \sum_{\substack{n=1\\ n \text{ odd}}} \left[\frac{-4}{n^4\pi} + c_n e^{-n^2 t}\right]\cos nx$

n even

$B_n' + n^2 B_n = 0 \qquad \frac{dB_n}{B_n} = -n^2 dt, \quad B_n = c_n e^{-n^2 t}.$

$u(x,t) = \sum_{n \text{ even}} c_n e^{-n^2 t}\cos nx$

2

$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

2 for $f(x) = 3 + \cos 2x - \frac{1}{3}\cos 7x \qquad c_0 = 3,\ c_2 = 2,\ c_7 = -\frac{1}{3}$

$\ldots e^{-4t}\cos 2x + \frac{1}{\ldots} e^{-49t} \ldots + \sum \frac{-4}{\ldots}\cos nx$

5

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5 $-w^2 \hat{U} - 4\hat{U} = \hat{f}(w)$

$$\hat{U} = \frac{-1}{w^2+4}\hat{f}(w)$$

$$= \frac{-1}{\cancel{4}}\,\frac{\cancel{4}}{w^2+4}\hat{f}(w)$$

$$\frac{2\alpha}{\alpha^2+w^2} \to e^{-\alpha|x|}$$

$e^{-\alpha|x|}$

5

$$= -\frac{1}{4}\,\phi(e^{-2|x|})\,\phi(f(x))$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} f(\xi)\,e^{-2|x-\xi|}\,d\xi$$

5

$$= -\frac{1}{4}\int_{0}^{1} e^{-2|x-\xi|}\,d\xi.$$

6 $u_t + 2u_x = u_{xx}$ 8

$u(x,t) = X(x)T(t)$

$\Rightarrow XT' + 2X'T = X''T \quad / XT$

$\Rightarrow \frac{T'}{T} + \frac{2X'}{X} = \frac{X''}{X}$

$\Rightarrow \frac{X''}{X} - \frac{2X'}{X} = -\lambda \quad , \quad \frac{T'}{T} = -\lambda$

$\Rightarrow X'' - 2X' + \lambda X = 0 \qquad T' + \lambda T = 0$ 2

$\Rightarrow r^2 - 2r + \lambda = 0 \qquad u_x(0,t) = 0 \Rightarrow X'(0) = 0$

$u_x(\pi,t) = 0 \Rightarrow X'(\pi) = 0$

$r_{1,2} = \frac{2 \pm \sqrt{4 - 4(1)(\lambda)}}{2} = 1 \pm \sqrt{1-\lambda}$

$\lambda = 1 \Rightarrow X(x) = c_1 e^{2x} + c_2 x e^{2x} = e^{2x}(c_1 + c_2 x)$

$r = 2, 2$

$X'(x) = c_2 e^{2x} + 2(c_1 + c_2 x)e^{2x}$

$X'(0) = c_2 + 2c_1 = 0$ 2

$X'(\pi) = c_2 e^{2\pi} + (2c_1 + 2c_2\pi)e^{2\pi} = 0$

$c_2 = -2c_1 \qquad \cancel{-2c_1 e^{2\pi}} + \cancel{2c_1 e^{2\pi}} - 4c_1 \pi e^{2\pi} = 0$

$c_1 = 0 \Rightarrow c_2 = 0$

No eigen values. or vectors

$\lambda < 1 \Rightarrow \lambda - 1 < 0 \Rightarrow 1 - \lambda > 0 \Rightarrow$ let $1 - \lambda = k^2 > 0$ i.e. $\lambda = 1 - k^2$

$r_1 = 1+k, \quad r_2 = 1-k$

$X(x) = c_1 e^{(1+k)x} + c_2 e^{(1-k)x}$

$X'(x) = c_1(1+k)e^{(1+k)x} + c_2(1-k)e^{(1-k)x}$ 2

$X'(0) = c_1(1+k) + c_2(1-k) = 0$

$X'(\pi) = c_1(1+k)e^{(1+k)\pi} + c_2(1-k)e^{(1-k)\pi}$

$\Rightarrow c_1 = 0, \ c_2 = 0$

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If $\lambda > 1 \Rightarrow 1-\lambda < 0 \Rightarrow 1-\lambda = -k^2$

$r_{1,2} = 1 \pm ki$

$X(x) = c_1 e^x e^{kix} + c_2 e^x e^{-kix} = e^x (c_1 e^{kix} + c_2 e^{-kix})$

$X'(x) = e^x (c_1 e^{kix} + c_2 e^{-kix}) + e^x (kic_1 e^{kix} - kic_2 e^{-kix})$

$X'(0) = c_1 + c_2 + kic_1 - kic_2 = 0 = c_1(1+ki) + c_2(1-ki) = 0$

$c_1 = A c_2 \quad A = \frac{1-ki}{1+ki}$

$X'(\pi) = \cancel{e^{\pi}} (A c_2 e^{ki\pi} + c_2 e^{-ki\pi}) + \cancel{e^{\pi}} (A c_2 ki e^{ki\pi} - ki c_2 e^{-ki\pi}) = 0$

$A e^{ki\pi} (1+ki) + e^{-ki\pi} (1-ki) = 0$

$\frac{1-ki}{\cancel{1+ki}} e^{ki\pi} \cancel{(1+ki)} + (1-ki) e^{-ki\pi} = 0$

3

$e^{ki\pi} + e^{-ki\pi} = 0$

$2 \sinh k\pi = 0$

$k\pi = n\pi$

$\lambda = 1 + k^2$ $\quad$ $k = n$

$\Rightarrow T_n(t) = c_n e^{-\lambda t}$

$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} e^{-(1+k^2)t} (A_n e^x \cos nx + B_n e^x \sin nx)$

1

$u(x,0) = f(x) = \sum A_n e^x \cos nx + B_n e^x \sin nx$

$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x) \cos nx \, dx}{e^x}$

$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{e^x} \sin nx \, dx$