جامعة بيرزيت

BIRZEIT UNIVERSITY

Mathematics Department

Math 332

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(1)

Second Exam 1st. Semester 2018/2019

Student name: .... ID no.: .... sec........

Do only three problems:

1)(17 points) solve the following PDE problem

$u_t = u_{xx} + t\,sin3x, \qquad x \in (0,\pi), t \in (0,\infty)$

$u(0,t) = 0,\ u(\pi,t) = 0$

$u(x,0) = 5\,sin8x - 10\,sin5x$

The solution of the homogenous part is

(2) $\phi_n(x) = \sin nx$, $\lambda_n = n^2$

We seek a solution of the form $u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin nx$

and expand $f(x,t) = t \sin 3x = \sum_{n=1}^{\infty} F_n(t) \sin nx$

(2) $\rightarrow F_3(t) = t$ but $F_n(t) = 0$ for $n \neq 3$

Substituting, we get

(2) $B_n' + n^2 B_n = F_n(t)$

(2) since $u(x,0) = \sum_{n=1}^{\infty} B_n(0) \sin nx = 5 \sin 8x - 10 \sin 5x$

$\rightarrow B_8(0) = 5$, $B_5(0) = -10$, $B_n(0) = 0$ for $n \neq 8,5$

So we have 4 cases,

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$n=3$, we have

$$B_3' + 9 B_3 = t$$
$$B_3(0) = 0$$

$$\Rightarrow \mu(t) = e^{9t}$$

2 $$B_3(t) = e^{-9t}\left[\int_0^t s e^{9s} ds + 0\right]$$

$$= e^{-9t}\left[\frac{s}{9}e^{9s} - \frac{1}{81}e^{9s}\Big|_0^t\right]$$

$s \;+\; e^{9s}$
$1 \;-\; \frac{1}{9}e^{9s}$
$0 \;\; \frac{1}{81}e^{9s}$

$$= e^{-9t}\left[\frac{t}{9}e^{9t} - \frac{1}{81}e^{9t} + \frac{1}{81}\right]$$

$$= \frac{t}{9} - \frac{1}{81} + \frac{1}{81}e^{-9t}$$

$n=5$, we have

2 $$B_5' + 25 B_5 = 0 \qquad \frac{dB_5}{dt} = -25 B_5$$

$$B_5(0) = -10 \qquad \Rightarrow \frac{dB_5}{B_5} = -25 dt$$

$\cancel{dB_5}$

$$\Rightarrow \ln|B_5| = -25t + C$$

$$\rightarrow B_5(t) = C e^{-25t}$$

$$B_5(0) = -10 \Rightarrow B_5(t) = -10 e^{-25t}$$

$n=8$, $$B_8' + 64 B_5 = 0 \quad B_8(0) = 5$$

2 $$B_8(t) = 5 e^{-64t}$$

$n \neq 3,5,8$ we have $\left.\begin{matrix} B_n' + n^2 B_n = 0 \\ B_n(0) = 0 \end{matrix}\right\} \Rightarrow B_n(t) = 0$

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So

1 $$u(x,t) = \left(\frac{t}{9} - \frac{1}{81} + \frac{1}{81}e^{-9t}\right)\sin 3x - 10 e^{-25t}\sin 5x + 5e^{-64t}\sin 8x$$

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2)(17 points)

a) Find u_{yy} in polar form.

b) solve the following PDE problem

$u_{rr} + \frac{1}{r} u_{rr} + \frac{1}{r^2} u_{\theta\theta} = 0$, $0 < r < 2$, $0 < \theta < \frac{\pi}{2}$,

$u(r,0) = 0$,

$u\left(r,\frac{\pi}{2}\right) = 0$,

$u(1,\theta) = 6 \sin 4\theta$,

$u(2,\theta) = 4 \sin 6\theta$

(a) $w = u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$ (2)

$u_{yy} = w_r \sin\theta + w_\theta \frac{\cos\theta}{r}$ (1)

$= \left(u_{rr} \sin\theta + u_{\theta r} \frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right) \sin\theta$ (2)

$+ \left(u_{r\theta} \sin\theta + u_r \cos\theta + u_{\theta\theta} \frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right) \frac{\cos\theta}{r}$ (2)

(b) $u = R(r)\,\Theta(\theta)$

(1) $\Theta'' + \lambda\Theta = 0$ & $r^2 R'' + r R' - \lambda R = 0$

$\Rightarrow \Theta(0) = 0 \quad \Theta(\frac{\pi}{2}) = 0$

(1) $\lambda = \left(\frac{n\pi}{\frac{\pi}{2}}\right)^2 = 4n^2$ $\Rightarrow r^2 R'' + r R' - 4n^2 R = 0$

$\phi_n(\theta) = \sin 2n\theta$ $\quad R(r) = A_n r^{2n} + B_n r^{-2n}$ (1)

(1) $u(r,\theta) = \sum_{n=1}^{\infty} (A_n r^{2n} + B_n r^{-2n}) \sin 2n\theta$

$u(1,\theta) = \sum_{n=1}^{\infty} (A_n + B_n) \sin 2n\theta = 6 \sin 4\theta$

(1) $\Rightarrow$ for $n = 2$, $A_2 + B_2 = 6$

for $n \neq 2$, $A_n + B_n = 0$

$$u(2,\theta) = \sum_{n=1}^{\infty}\left(A_n 4^n + B_n 4^{-n}\right)\sin 2n\theta = 4\sin 6\theta$$

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$\Rightarrow$ for $n=3$, $A_3 4^3 + B_3 4^{-3} = 4$ 1

$\rightarrow$ for $n \neq 3$ $A_n 4^n + B_n 4^{-n} = 0$

<u>3 cases</u>:

$n=2$, $\left.\begin{array}{l} A_2 + B_2 = 6 \\ 4^2 A_2 + 4^{-n} B_2 = 0 \end{array}\right\} \Rightarrow A_2, B_2$ 1

$n=3$, $\left.\begin{array}{l} A_3 + B_3 = 0 \\ 4^3 A_3 + \frac{1}{4^3} B_3 = 4 \end{array}\right\} A_3, B_3$ 1

$n \neq 2, 3$, $\left.\begin{array}{l} A_n + B_n = 0 \\ 4^n A_n + \frac{1}{4^n} B_n = 0 \end{array}\right\} A_n = B_n = 0$ 1

$$u(r,\theta) = \left(A_2 r^4 + B_2 r^{-4}\right)\sin 4\theta + \left(A_3 r^6 + B_3 r^{-6}\right)\sin 6\theta$$ 1

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3)(17points)a) Solve

$u_{yy} + u_{xx} = 0 \qquad x \in R(0,2), y \in (0,1)$ $\quad u(x,y) = X(x)Y(y)$

$u(x,0) = 0, \qquad x \in R(0,2),$ $\Rightarrow X(x)Y(0) = 0 \Rightarrow Y(0) = 0$

$u(x,1) = 0, \qquad x \in R(0,2),$ $\quad X(x)Y(1) = 0 \Rightarrow Y(1) = 0$

$u(0,y) = 3sin2\pi\, y, \qquad y \in (0,1)$

$u(2,y) = 4sin3\pi y\,, \qquad y \in (0,1)$

b) Only sketch out the solution if the first equation becomes $u_{yy} + u_{xx} = sinxcosy \qquad x \in R(0,2), y \in (0,1)$

a) $u(x,y) = X(x)Y(y)$

$\Rightarrow X'' \cancel{+} \lambda X = 0 \quad , Y'' + \lambda Y = 0$

$Y(0) = 0, \; Y(1) = 0$

2 $\quad \Rightarrow Y_n = \sin(n\pi y) \quad , \lambda_n = (n\pi)^2$

2 $\quad \Rightarrow X_n(x) = A_n \cosh n\pi x + B_n \sinh n\pi x$

1 $\quad u(x,y) = \sum_{n=1}^{\infty} (A_n \cosh n\pi x + B_n \sinh n\pi x) \sin n\pi y$

1 $\quad u(0,y) = \sum_{n=1}^{\infty} A_n \sin n\pi y = 3\sin 2\pi y$

$n=2, A_2 = 3, \quad A_n = 0, n \neq 3$

1 $\quad u(2,y) = \sum_{n=1}^{\infty} (A_n \cosh 2n\pi + B_n \sinh 2n\pi) \sin n\pi y = 4\sin 3\pi y$

$A_3 \cosh 6\pi + B_3 \sinh 6\pi = 4$

$A_n \cosh 2n\pi + B_n \sinh 2n\pi = 0 \quad , n \neq 3$

2 $\quad \boxed{A_3 = 0, \quad B_3 = \frac{4}{\sinh 6\pi}}$ $\quad n=2 \; B_2 = \frac{-3\cosh 4\pi}{\sinh 4\pi}$

$n \neq 2,3, \; A_n = 0, \; B_n = 0$

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(b) If $f(x,y) = \sin\pi x \sin\pi y$

1 We seek $u(x,y) = \sum_{n=1}^{\infty} B_n(x) \sin n\pi y$

1 and expand $f(x,y) = \sum_{n=1}^{\infty} F_n(x) \sin n\pi y$

~~where~~ $\cancel{F_n(x) = 2\int_0^1 \sin n\pi x \sin n\pi y \, dy}$

$$F_n(x) = 2\int_0^1 (\sin\pi x \sin\pi y) \sin n\pi y \, dy$$

$$F_1 = \sin\pi x, \quad F_n = 0, \quad n \neq 1$$

1 substituting $B_n''(x) - \pi^2 B_n(x) = F_n(x)$

with B.C. $u(0,y) = 3\sin 2\pi y = \sum_{n=1}^{\infty} B_n(0) \sin n\pi y$

1 $\Rightarrow B_2(0) = 3, \quad B_n(0) = 0, \; n \neq 2$

$u(2,y) = 4\sin 3\pi y = \sum_{n=1}^{\infty} B_n(2) \sin n\pi y$

1 $\Rightarrow B_3(2) = 4$

$B_n(2) = 0, \; n \neq 3$

$\boxed{n=1}$ $B_1'' - \pi^2 B_1 = \sin\pi x$

$B_1(0) = 0, \; B_1(2) = 0$

$\boxed{n=2}$ $B_2'' - 4\pi^2 B_2 = 0$

$B_2(0) = 3, \; B_2(2) = 0$

2

$\boxed{n=3}$ $B_3'' - 9\pi^2 B_2 = 0$

$B_3(0) = 0, \; B_3(2) = 4$

$n \neq 1,2,3$ $\left.\begin{array}{l} B_n'' - n^2\pi^2 B_n = 0 \\ B_n(0) = 0, \; B_n(2) = 0 \end{array}\right\} B_n = 0$

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4) (17 points) solve the following PDE problem

$$u_{xx} = u_t + 2u_x, \quad 0 < x < \pi, t > 0$$

$$u(0,t) = 0$$

$$u(\pi,t) = 0$$

$$u(x,0) = (x^{\cancel{2}} - \pi^{\cancel{2}})\,x$$

$$u(x,t) = X(x)T(t)$$

$$\frac{X''T}{XT} = \frac{XT'}{XT} + \frac{2X'T}{XT}$$

$$\frac{X''}{X} = \frac{T'}{T} + \frac{2X'}{X}$$

$$\Rightarrow \frac{X''}{X} - \frac{2X'}{X} = \frac{T'}{T} = -\lambda$$

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$$X'' - 2X' + \lambda X = 0 \qquad T' + \lambda T = 0$$

$$X(0) = 0, \quad X(\pi) = 0$$

$\underline{\lambda = 1}$ $\quad X'' - 2X' + 1 = 0$

$$r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$$

$$X(x) = c_1 e^x + c_2 x e^x$$

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$$X(0) = 0 = c_1$$

$$X(x) = c_2 x e^x$$

$X(\pi) = c_2 \pi e^{\pi} = 0 \Rightarrow c_2 = 0$, $\lambda = 1$ is not an eigenvalue.

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$\lambda < 1 \Rightarrow \lambda = 1 - k^2$

$X'' - 2X' + (1-k^2)X = 0$

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 4(1)(1-k^2)}}{2} = 1 \pm k$$

$X(x) = c_1 e^{(1+k)x} + c_2 e^{(1-k)x}$

$X(x) = e^x (c_1 e^{kx} + c_2 e^{-kx})$ 2

$X(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$X(\pi) = e^{\pi} (c_1 e^{k\pi} + c_2 e^{-k\pi}) = 0$

$c_1 (e^{k\pi} - e^{-k\pi}) = 0$

$\neq 0 \Rightarrow c_1 = 0$ since $k \neq 0$

$c_2 = 0$ $2 \sinh k\pi \neq 0$

$\lambda > 1 \Rightarrow \lambda = 1 + k^2$

$X'' - 2X' + (1+k^2)X = 0$

$r^2 - 2r + (1+k^2) = 0$

$$r_{1,2} = \frac{2 \pm \sqrt{4 - 4(1+k^2)}}{2} = 1 \pm ki$$

$X(x) = e^x (c_1 \cos kx + c_2 \sin kx)$ 4

$X(0) = 0 = c_1$

$X(x) = c_2 e^x \sin kx$

$X(\pi) = c_2 e^{\pi} \sin k\pi = 0$

$\sin k\pi = 0 \Rightarrow k\pi = n\pi \Rightarrow \boxed{k = n}$

$\Rightarrow$ Eigenvalue $\lambda_n = 1 + n^2$, Eigenfunction $\phi_n = e^x \sin nx$

solving T: $T' + (1+n^2)T = 0 \Rightarrow T_n(t) = c_n e^{-(1+n^2)t}$ 2

3 $u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(1+n^2)t} e^x \sin nx$ $\quad A_n = \frac{\langle x^2 - x\pi, e^x \sin nx \rangle}{\langle e^x \sin nx, e^x \sin nx \rangle}$

$u(x,0) = \sum A_n e^x \sin nx = x(x-\pi)$