truncated cone, has a linear taper $y = cx$ as shown in cross section in Figure 5.2.1(b), the moment of inertia of a cross section with respect to an axis perpendicular to the xy-plane is $I = \frac{1}{4}\pi r^4$, where $r = y$ and $y = cx$. Hence we can write $I(x) = I_0(x/b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi(cb)^4$. Substituting $I(x)$ into the differential equation in (24), we see that the deflection in this case is determined from the BVP

$$x^4\frac{d^2y}{dx^2} + \lambda y = 0, \quad y(a) = 0, \quad y(b) = 0,$$

where $\lambda = Pb^4/EI_0$. Use the results of Problem 33 to find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

(b) Use a CAS to plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when $b = 11$ and $a = 1$.

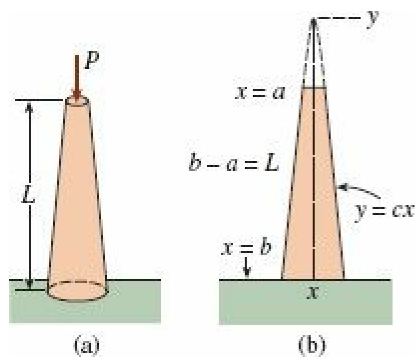

FIGURE 5.2.1 Tapered column in Problem 34

Discussion Problems

35. Discuss how you would define a regular singular point for the linear third-order differential equation

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

36. Each of the differential equations

$$x^3y'' + y = 0 \quad \text{and} \quad x^2y'' + (3x - 1)y' + y = 0$$

has an irregular singular point at $x = 0$. Determine whether the method of Frobenius yields a series solution of each differential equation about $x = 0$. Discuss and explain your findings.

37. We have seen that $x = 0$ is a regular singular point of any Cauchy–Euler equation $ax^2y'' + bxy' + cy = 0$. Are the indicial equation (14) for a Cauchy–Euler equation and its auxiliary equation related? Discuss.

5.3 Special Functions

Introduction The two differential equations

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \qquad (1)$$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad (2)$$

occur frequently in advanced studies in applied mathematics, physics, and engineering. They are called **Bessel's equation of order** v and **Legendre's equation of order** $\boldsymbol{n}$, respectively. Naturally, solutions of (1) are called **Bessel functions** and solutions of (2) are called **Legendre functions.** When we solve (1) we shall assume that $v \geq 0$, whereas in (2) we shall consider only the case when v is a nonnegative integer. Since we shall seek series solutions of each equation about $x = 0$, we observe that the origin is a regular singular point of Bessel's equation but is an ordinary point of Legendre's equation.

5.3.1 Bessel Functions

The Solution Because $x = 0$ is a regular singular point of Bessel's equation, we know that there exists at least one solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting the last expression into (1) then gives

$$\begin{aligned}
x^2y'' + xy' + (x^2 - \nu^2)y &= \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= c_0(r^2 - r + r - \nu^2)x^r \\
&\quad + x^r \sum_{n=1}^{\infty} c_n[(n+r)(n+r-1) + (n+r) - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} \\
&= c_0(r^2 - \nu^2)x^r + x^r \sum_{n=1}^{\infty} c_n[(n+r)^2 - \nu^2]x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2}. \qquad (3)
\end{aligned}$$

From (3) we see that the indicial equation is $r^2 - v^2 = 0$ so that the indicial roots are $r_1 = v$ and $r_2 = -v$. When $r_1 = v$, (3) becomes

$$\begin{aligned}
&x^\nu \sum_{n=1}^{\infty} c_n n(n+2\nu)x^n + x^\nu \sum_{n=0}^{\infty} c_n x^{n+2} \\
&= x^\nu \Bigg[(1+2\nu)c_1 x + \underbrace{\sum_{n=2}^{\infty} c_n n(n+2\nu)x^n}_{k = n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{k = n}\Bigg] \\
&= x^\nu \left[(1+2\nu)c_1 x + \sum_{k=0}^{\infty} [(k+2)(k+2+2\nu)c_{k+2} + c_k]x^{k+2}\right] = 0.
\end{aligned}$$

Therefore, by the usual argument we can write $(1 + 2v)c_1 = 0$ and

$$(k+2)(k+2+2\nu)c_{k+2} + c_k = 0$$

or

$$c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)}, \quad k = 0, 1, 2, \ldots. \qquad (4)$$

The choice $c_1 = 0$ in (4) implies $c_3 = c_5 = c_7 = \ldots = 0$, so for $k = 0, 2, 4, \ldots$ we find, after letting $k + 2 = 2n$, $n = 1, 2, 3, \ldots$, that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n + \nu)}. \tag{5}$$

Thus

$$c_2 = -\frac{c_0}{2^2 \cdot 1 \cdot (1 + \nu)}$$

$$c_4 = -\frac{c_2}{2^2 \cdot 2(2 + \nu)} = \frac{c_0}{2^4 \cdot 1 \cdot 2(1 + \nu)(2 + \nu)}$$

$$c_6 = -\frac{c_4}{2^2 \cdot 3(3 + \nu)} = -\frac{c_0}{2^6 \cdot 1 \cdot 2 \cdot 3(1 + \nu)(2 + \nu)(3 + \nu)}$$

$$\vdots$$

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1 + \nu)(2 + \nu) \cdots (n + \nu)}, \qquad n = 1, 2, 3, \ldots. \tag{6}$$

It is standard practice to choose c_0 to be a specific value—namely,

$$c_0 = \frac{1}{2^\nu \Gamma(1 + \nu)},$$

where $\Gamma(1 + \nu)$ is the gamma function. See Appendix II. Since this latter function possesses the convenient property $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$, we can reduce the indicated product in the denominator of (6) to one term. For example,

$$\Gamma(1 + \nu + 1) = (1 + \nu)\Gamma(1 + \nu)$$
$$\Gamma(1 + \nu + 2) = (2 + \nu)\Gamma(2 + \nu) = (2 + \nu)(1 + \nu)\Gamma(1 + \nu).$$

Hence we can write (6) as

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!(1 + \nu)(2 + \nu) \cdots (n + \nu)\Gamma(1 + \nu)} = \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1 + \nu + n)}$$

Bessel Functions of the First Kind The series solution $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$ is usually denoted by $J_\nu(x)$:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu}. \tag{7}$$

If $\nu \geq 0$, the series converges at least on the interval $[0, \infty)$. Also, for the second exponent $r_2 = -\nu$ we obtain, in exactly the same manner,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1 - \nu + n)} \left(\frac{x}{2}\right)^{2n-\nu}. \tag{8}$$

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order ν and $-\nu$, respectively. Depending on the value of ν, (8) may contain negative powers of x and hence converge on the interval $(0, \infty)$.*

Now some care must be taken in writing the general solution of (1). When $\nu = 0$, it is apparent that (7) and (8) are the same. If $\nu > 0$ and $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not a positive integer, it follows from Case I of Section 5.2 that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of (1) on $(0, \infty)$, and so the general solution on the interval is $y = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$. But we also know from Case II o

Section 5.2 that when $r_1 - r_2 = 2\nu$ is a positive integer, a second series solution of (1) *may* exist. In this second case we distinguish two possibilities. When $\nu = m$ = positive integer, $J_{-m}(x)$ defined by (8) and $J_m(x)$ are not linearly independent solutions. It can be shown that J_{-m} is a constant multiple of J_m (see Property (*i*) on page 277). In addition, $r_1 - r_2 = 2\nu$ can be a positive integer when ν is half an odd positive integer. It can be shown in this latter event that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. In other words, the general solution of (1) on $(0, \infty)$ is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(x), \quad \nu \neq \text{integer}. \tag{9}$$

The graphs of $y = J_0(x)$ (blue) and $y = J_1(x)$ (red) are given in **FIGURE 5.3.1**.

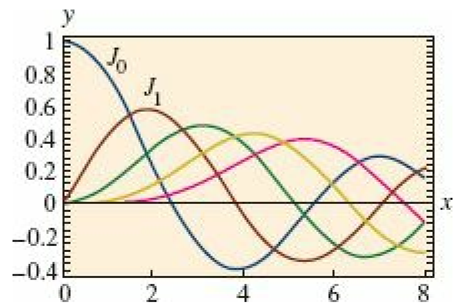

FIGURE 5.3.1 Bessel functions of the first kind for $n = 0, 1, 2, 3, 4$

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*When we replace x by $|x|$, the series given in (7) and (8) converge for $0 < |x| < \infty$.

EXAMPLE 1 General Solution: ν Not an Integer

By identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$ we can see from (9) that the general solution of the equation $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ on $(0, \infty)$ is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$. ≡

Bessel Functions of the Second Kind If $\nu \neq$ integer, the function defined by the linear combination

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{10}$$

and the function $J_n(x)$ are linearly independent solutions of (1). Thus another form of the general solution of (1) is $y = c_1 J_\nu(x) + c_2 Y_\nu(x)$, provided $\nu \neq$ integer. As $\nu \to m$, m an integer, (10) has the indeterminate form 0/0. However, it can be shown by L'Hôpital's rule that $\lim_{\nu \to m} Y_\nu(x)$ exists. Moreover, the function

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of $x^2y'' + xy' + (x^2 - m^2)y = 0$. Hence for *any* value of ν the general solution of (1) on the interval $(0, \infty)$ can be written as

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{11}$$

$Y_\nu(x)$ is called the **Bessel function of the second kind** of order ν. **FIGURE 5.3.2** shows the graphs of $Y_0(x)$ (blue) and $y_1(x)$ (red).

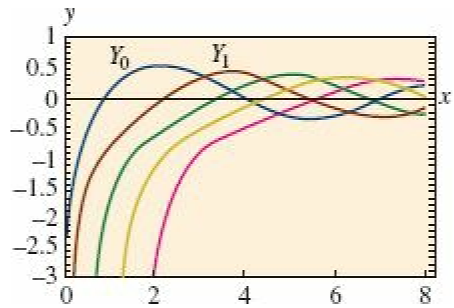

FIGURE 5.3.2 Bessel functions of the second kind for $n = 0, 1, 2, 3, 4$

EXAMPLE 2 General Solution: v an Integer

By identifying $v^2 = 9$ and $v = 3$ we see from (11) that the general solution of the equation $x^2y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$ is $y = c_1J_3(x) + c_2Y_3(x)$. ≡

DEs Solvable in Terms of Bessel Functions Sometimes it is possible to transform a differential equation into equation (1) by means of a change of variable. We can then express the solution of the original equation in terms of Bessel functions. For example, if we let $t = \alpha x$, $\alpha > 0$, in

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \tag{12}$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

Accordingly (12) becomes

$$\left(\frac{t}{\alpha}\right)^2\alpha^2\frac{d^2y}{dt^2} + \left(\frac{t}{\alpha}\right)\alpha\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \quad \text{or} \quad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

The last equation is Bessel's equation of order v with solution $y = c_1J_v(t) + c_2Y_v(t)$. By resubstituting $t = \alpha x$ in the last expression we find that the general solution of (12) on the interval $(0, \infty)$ is

$$y = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x). \tag{13}$$

Equation (12), called the **parametric Bessel equation of order** v, and its general solution (13) are very important in the study of certain boundary-value problems involving partial differential equations that are expressed in cylindrical coordinates.
Another equation that bears a resemblance to (1) is the **modified Bessel equation of order** v,

$$x^2y'' + xy' - (x^2 + \nu^2)\,y = 0. \tag{14}$$

This DE can be solved in the manner just illustrated for (12). This time if we let $t = ix$, where $i^2 = -1$, then (14) becomes

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0.$$

Since solutions of the last DE are $J_v(t)$ and $Y_v(t)$, *complex-valued* solutions of equation (14) are

$J_v(ix)$ and $Y_v(ix)$. A real-valued solution, called the **modified Bessel function of the first kind** of order v, is defined in terms of $J_v(ix)$:

$$I_\nu(x) = i^{-\nu} J_\nu(ix). \tag{15}$$

See Problem 21 in Exercises 5.3. Analogous to (10), the **modified Bessel function of the second kind** of order $v \neq$ integer is defined to be

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}, \tag{16}$$

and for integral $v = n$,

$$K_n(x) = \lim_{\nu \to n} K_\nu(x).$$

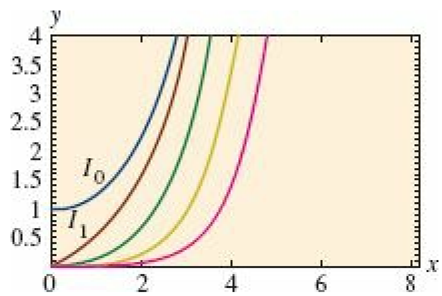

FIGURE 5.3.3 Modified Bessel function of the first kind for $n = 0, 1, 2, 3, 4$

Because I_v and K_v are linearly independent on the interval $(0, \infty)$ for any value of v, the general solution of (14) is

$$y = c_1 I_\nu(x) + c_2 K_\nu(x). \tag{17}$$

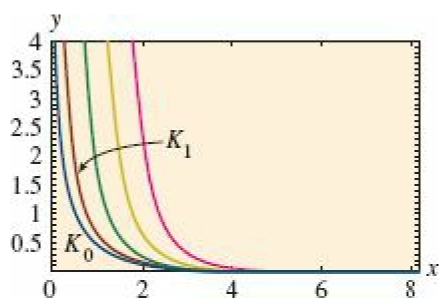

FIGURE 5.3.4 Modified Bessel function of the second kind for $n = 0, 1, 2, 3, 4$

The graphs of $I_0(x)$ (blue) and $I_1(x)$ (red) are given in **FIGURE 5.3.3** and the graphs $K_0(x)$ (blue) and $K_1(x)$ (red) are shown in **FIGURE 5.3.4** Unlike the Bessel functions of the first and second kinds, the graphs of the modified Bessel functions of the first kind and second kind are not oscillatory. Moreover, the graphs in Figures 5.3.3 and 5.3.4 illustrate the fact that the modified Bessel functions $I_n(x)$ and $K_n(x)$, $n = 0, 1, 2, \ldots$ have no real zeros in the interval $(0, \infty)$. Also, note that $K_n(x) \to \infty$ as $x \to 0^+$.

Proceeding as we did in (12) and (13), we see that the general solution of the **parametric form** of the modified Bessel equation of order v

$$x^2y'' + xy' - (\alpha^2x^2 + v^2)\, y = 0$$

on the interval $(0, \infty)$ is

$$y = c_1 I_v(\alpha x) + c_2 K_v(\alpha x).$$

Yet another equation, important because many differential equations fit into its form by appropriate choices of the parameters, is

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)y = 0, \quad p \geq 0. \tag{18}$$

Although we shall not supply the details, the general solution of (18),

$$y = x^a\left[c_1 J_p(bx^c) + c_2 Y_p(bx^c)\right], \tag{19}$$

can be found by means of a change in both the independent and the dependent variables: $z = bx^c,\ y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$. If p is not an integer, then Y_p in (19) can be replaced by J_{-p}.

EXAMPLE 3 Using (18)

Find the general solution of $xy'' + 3y' + 9y = 0$ on $(0, \infty)$.

Solution By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0$$

we can make the following identifications with (18):

$$1 - 2a = 3,\ b^2c^2 = 9,\ 2c - 2 = -1, \quad \text{and} \quad a^2 - p^2c^2 = 0.$$

The first and third equations imply $a = -1$ and $c = \frac{1}{2}$. With these values the second and fourth equations are satisfied by taking $b = 6$ and $p = 2$. From (19) we find that the general solution of the given DE on the interval $(0, \infty)$ is $y = x^{-1}[c_1 J_2(6x^{1/2}) + c_2 Y_2(6x^{1/2})]$. ≡

EXAMPLE 4 The Aging Spring Revisited

Recall that in Section 3.8 we saw that one mathematical model for the free undamped motion of a mass on an aging spring is given by $mx'' + ke^{-at}x = 0$, $\alpha > 0$. We are now in a position to find the general solution of the equation. It is left as a problem to show that the change of variables $s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}$ transforms the differential equation of the aging spring into

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0.$$

The last equation is recognized as (1) with $v = 0$ and where the symbols x and s play the roles of y and x, respectively. The general solution of the new equation is $x = c_1 J_0(s) + c_2 Y_0(s)$. If we resubstitute s, then the general solution of $mx'' + ke^{-at}x = 0$ is seen to be

$$x(t) = c_1 J_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right) + c_2 Y_0\left(\frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}\right).$$

See Problems 33 and 43 in Exercises 5.3.

The other model discussed in Section 5.1 of a spring whose characteristics change with time was $mx'' + ktx = 0$. By dividing through by m we see that the equation $x'' + (k/m)tx = 0$ is Airy's equation, $y'' + \alpha^2 xy = 0$. See Example 2 in Section 5.1. The general solution of Airy's differential equation can also be written in terms of Bessel functions. See Problems 34, 35, and 44 in Exercises 5.3.

Properties We list below a few of the more useful properties of Bessel functions of the first and second kinds of order m, $m = 0, 1, 2, \ldots$:

$$(i)\ J_{-m}(x) = (-1)^m J_m(x) \qquad (ii)\ J_m(-x) = (-1)^m J_m(x)$$

$$(iii)\ J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases} \qquad (iv)\ \lim_{x\to 0^+} Y_m(x) = -\infty.$$

Note that Property (ii) indicates that $J_m(x)$ is an even function if m is an even integer and an odd function if m is an odd integer. The graphs of $Y_0(x)$ and $Y_1(x)$ in Figure 5.3.2 illustrate Property (iv): $Y_m(x)$ is unbounded at the origin. This last fact is not obvious from (10). The solutions of the Bessel equation of order 0 can be obtained using the solutions $y_1(x)$ in (21) and $y_2(x)$ in (22) of Section 5.2. It can be shown that (21) of Section 5.2 is $y_1(x) = J_0(x)$, whereas (22) of that section is

$$y_2(x) = J_0(x)\ln x - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k}.$$

The Bessel function of the second kind of order 0, $Y_0(x)$, is then defined to be the linear combination $Y_0(x) = \frac{2}{\pi}(\gamma - \ln 2)\, y_1(x) + \frac{2}{\pi} y_2(x)$ for $x > 0$. That is,

$$Y_0(x) = \frac{2}{\pi} J_0(x)\left[\gamma + \ln\frac{x}{2}\right] - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)\left(\frac{x}{2}\right)^{2k},$$

where $\gamma = 0.57721566 \ldots$ is **Euler's constant.** Because of the presence of the logarithmic term, it is apparent that $Y_0(x)$ is discontinuous at $x = 0$.

Numerical Values The first five nonnegative zeros of $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ are given in Table 5.3.1. Some additional functional values of these four functions are given in Table 5.3.2.

TABLE 5.3.1 Zeros of J_0, J_1, Y_0, and Y_1

$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.0156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

TABLE 5.3.2 Numerical Values of J_0, J_1, Y_0, and y_1

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0	1.0000	0.0000	—	—
1	0.7652	0.4401	0.0883	−0.7812
2	0.2239	0.5767	0.5104	−0.1070
3	−0.2601	0.3391	0.3769	0.3247
4	−0.3971	−0.0660	−0.0169	0.3979
5	−0.1776	−0.3276	−0.3085	0.1479
6	0.1506	−0.2767	−0.2882	−0.1750
7	0.3001	−0.0047	−0.0259	−0.3027
8	0.1717	0.2346	0.2235	−0.1581
9	−0.0903	0.2453	0.2499	0.1043
10	−0.2459	0.0435	0.0557	0.2490
11	−0.1712	−0.1768	−0.1688	0.1637
12	0.0477	−0.2234	−0.2252	−0.0571
13	0.2069	−0.0703	−0.0782	−0.2101
14	0.1711	0.1334	0.1272	−0.1666
15	−0.0142	0.2051	0.2055	0.0211

Differential Recurrence Relation Recurrence formulas that relate Bessel functions of different orders are important in theory and in applications. In the next example we derive a **differential recurrence relation.**

EXAMPLES 5 Derivation Using the Series Definition

Derive the formula $xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$.

Solution It follows from (7) that

$$xJ'_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+\nu)}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \nu\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu} + 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \nu J_\nu(x) + x\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\Gamma(1+\nu+n)}\left(\frac{x}{2}\right)^{2n+\nu-1}}_{k=n-1}$$

$$= \nu J_\nu(x) - x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(2+\nu+k)}\left(\frac{x}{2}\right)^{2k+\nu+1} = \nu J_\nu(x) - xJ_{\nu+1}(x).$$ ≡

The result in Example 5 can be written in an alternative form. Dividing $xJ'_\nu(x) - \nu J_\nu(x) = -xJ_{\nu+1}(x)$ by x gives

$$J'_\nu(x) - \frac{\nu}{x}J_\nu(x) = -J_{\nu+1}(x).$$

This last expression is recognized as a linear first-order differential equation in $J_\nu(x)$. Multiplying both sides of the equality by the integrating factor $x^{-\nu}$ then yields

$$\frac{d}{dx}[x^{-\nu}J_\nu(x)] = -x^{-\nu}J_{\nu+1}(x). \tag{20}$$

It can be shown in a similar manner that

$$\frac{d}{dx}[x^{\nu}J_{\nu}(x)] = x^{\nu}J_{\nu-1}(x). \tag{21}$$

See Problem 27 in Exercises 5.3. The differential recurrence relations (20) and (21) are also valid for the Bessel function of the second kind $Y_\nu(x)$. Observe that when $\nu = 0$ it follows from (20) that

$$J_0'(x) = -J_1(x) \quad \text{and} \quad Y_0'(x) = -Y_1(x). \tag{22}$$

An application of these results is given in Problem 43 in Exercises 5.3.

Bessel Functions of Half-Integral Order When the order ν is half an odd integer, that is, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots,$ Bessel functions of the first and second kinds can be expressed in terms of the elementary functions $\sin x$, $\cos x$, and powers of x. To see this let's consider the case when $\nu = \frac{1}{2}$. From (7) we have

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n+1/2}.$$

In view of the properties the gamma function, $\Gamma(1 + \alpha) = \alpha\Gamma(\alpha)$ and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ the values of $\Gamma(1 + \frac{1}{2} + n)$ for $n = 0$, $n = 1$, $n = 2$, and $n = 3$ are, respectively,

▶ See Appendix II.

$$\Gamma(\tfrac{3}{2}) = \Gamma(1+\tfrac{1}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma(\tfrac{5}{2}) = \Gamma(1+\tfrac{3}{2}) = \tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma(\tfrac{7}{2}) = \Gamma(1+\tfrac{5}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \frac{5\cdot 3}{2^3}\sqrt{\pi} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2^3 4\cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma(\tfrac{9}{2}) = \Gamma(1+\tfrac{7}{2}) = \tfrac{7}{2}\Gamma(\tfrac{7}{2}) = \frac{7\cdot 5!}{2^6 2!}\sqrt{\pi} = \frac{7\cdot 6\cdot 5!}{2^6\cdot 6\cdot 2!}\sqrt{\pi} = \frac{7!}{2^7 3!}\sqrt{\pi}.$$

In general,

$$\Gamma(1+\tfrac{1}{2}+n) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}.$$

Hence,

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\dfrac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

The infinite series in the last line is the Maclaurin series for $\sin x$, and so we have shown that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x. \tag{23}$$

We leave it as an exercise to show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x. \tag{24}$$

See Problems 31 and 32 in Exercises 5.3.

If n is an integer, then the order $\nu = n + \frac{1}{2}$ is half an odd integer. Because $\cos(n + \frac{1}{2})\pi = 0$ and $\sin(n + \frac{1}{2})\pi = \cos n\pi = (-1)^n$, we see from (10) that

$$Y_{n+1/2}(x) = (-1)^{n+1}J_{-(n+1/2)}(x). \tag{25}$$

For $n = 0$ and $n = -1$ in the last formula, we get, in turn, $Y_{1/2}(x) = -J_{-1/2}(x)$ and $Y_{-1/2}(x) = J_{1/2}(x)$. In view of (23) and (24) these results are the same as

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x \tag{26}$$

and

$$Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \tag{27}$$

□ **Spherical Bessel Functions** Bessel functions of half-integral order are used to define two more important functions:

$$j_n(x) = \sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) \quad \text{and} \quad y_n(x) = \sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x). \tag{28}$$

The function $j_n(x)$ is called the **spherical Bessel function of the first kind** and $y_n(x)$ is the **spherical Bessel function of the second kind.** For example, by using (23) and (26) we see that for $n = 0$ the expressions in (28) become

$$j_0(x) = \sqrt{\frac{\pi}{2x}}J_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}}\sin x = \frac{\sin x}{x}$$

and

$$y_0(x) = \sqrt{\frac{\pi}{2x}}Y_{1/2}(x) = -\sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}}\cos x = -\frac{\cos x}{x}.$$

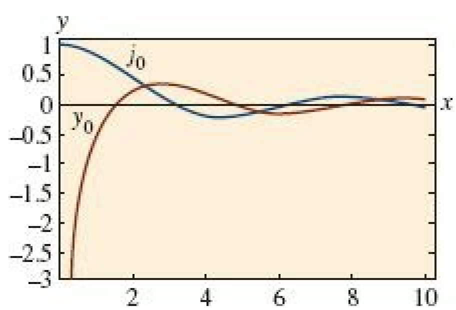

FIGURE 5.3.5 Spherical Bessel functions $j_0(x)$ and $y_0(x)$

The graphs of $j_n(x)$ and $y_n(x)$ for $n \geq 0$ are very similar to those given in Figures 5.3.1 and 5.3.2, that is, both functions are oscillatory, and $y_n(x)$ becomes unbounded as $x \to 0^+$. The graphs of $j_0(x)$ (blue) and $y_0(x)$ (red) are given in **FIGURE 5.3.5**. See Problems 39 and 40 in Exercises 5.3.

Spherical Bessel functions arise in the solution of a special partial differential equation expressed in spherical coordinates. See Problems 41 and 42 in Exercises 5.3 and Problem 14 in Exercises 14.3.

5.3.2 Legendre Functions

The Solution Since $x = 0$ is an ordinary point of Legendre's equation (2), we substitute the series $y = \sum_{k=0}^{\infty} c_k x^k$, shift summation indices, and combine series to get

$$(1 - x^2)y'' - 2xy' + n(n+1)y = [n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x$$

$$+ \sum_{j=2}^{\infty}[(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j]x^j = 0,$$

which implies that

$$n(n+1)c_0 + 2c_2 = 0$$

$$(n-1)(n+2)c_1 + 6c_3 = 0$$

$$(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0$$

or

$$c_2 = -\frac{n(n+1)}{2!}c_0$$

$$c_3 = -\frac{(n-1)(n+2)}{3!}c_1$$

$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_j, \quad j = 2, 3, 4, \ldots. \qquad (29)$$

Letting j take on the values 2, 3, 4, …, recurrence relation (29) yields

$$c_4 = -\frac{(n-2)(n+3)}{4\cdot 3}c_2 = \frac{(n-2)n(n+1)(n+3)}{4!}c_0$$

$$c_5 = -\frac{(n-3)(n+4)}{5\cdot 4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$

$$c_6 = -\frac{(n-4)(n+5)}{6\cdot 5}c_4 = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_0$$

$$c_7 = -\frac{(n-5)(n+6)}{7\cdot 6}c_5 = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_1$$

and so on. Thus for at least $|x| < 1$ we obtain two linearly independent power series solutions:

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

$$y_2(x) = c_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]. \tag{30}$$

Notice that if n is an even integer, the first series terminates, whereas $y_2(x)$ is an infinite series. For example, if $n = 4$, then

$$y_1(x) = c_0\left[1 - \frac{4 \cdot 5}{2!}x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!}x^4\right] = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right].$$

Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n; that is, *when n is a nonnegative integer, we obtain an nth-degree polynomial solution* of Legendre's equation.

Since we know that a constant multiple of a solution of Legendre's equation is also a solution, it is traditional to choose specific values for c_0 or c_1, depending on whether n is an even or odd positive integer, respectively. For $n = 0$ we choose $c_0 = 1$, and for $n = 2, 4, 6, \ldots$,

$$c_0 = (-1)^{n/2}\frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n};$$

whereas for $n = 1$ we choose $c_1 = 1$, and for $n = 3, 5, 7, \ldots$,

$$c_1 = (-1)^{(n-1)/2}\frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}.$$

For example, when $n = 4$ we have

$$y_1(x) = (-1)^{4/2}\frac{1 \cdot 3}{2 \cdot 4}\left[1 - 10x^2 + \frac{35}{3}x^4\right] = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Legendre Polynomials These specific nth-degree polynomial solutions are called **Legendre polynomials** and are denoted by $P_n(x)$. From the series for $y_1(x)$ and $y_2(x)$ and from the above choices of c_0 and c_1 we find that the first several Legendre polynomials are

$$\begin{aligned} &P_0(x) = 1, && P_1(x) = x, \\ &P_2(x) = \frac{1}{2}(3x^2 - 1), && P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ &P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), && P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned} \tag{31}$$

Remember, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, …, are, in turn, particular solutions of the differential equations

$$n = 0: \quad (1 - x^2)y'' - 2xy' = 0$$

$$n = 1: \quad (1 - x^2)y'' - 2xy' + 2y = 0$$

$$n = 2: \quad (1 - x^2)y'' - 2xy' + 6y = 0 \tag{32}$$

$$n = 3: \quad (1 - x^2)y'' - 2xy' + 12y = 0$$

$$\vdots \qquad\qquad \vdots$$

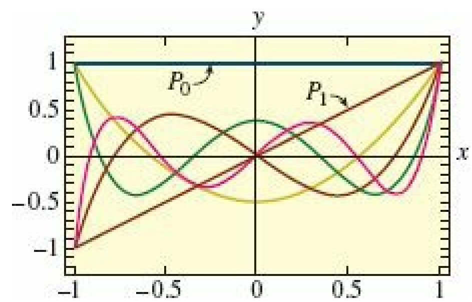

FIGURE 5.3.6 Legendre polynomials for $n = 0, 1, 2, 3, 4, 5$

The graphs, on the interval $[-1, 1]$, of the six Legendre polynomials in (31) are given in **FIGURE 5.3.6**.

Properties You are encouraged to verify the following properties for the Legendre polynomials in (31):

$$(i) \;\; P_n(-x) = (-1)^n P_n(x)$$

$$(ii) \;\; P_n(1) = 1 \qquad\qquad (iii) \;\; P_n(-1) = (-1)^n$$

$$(iv) \;\; P_n(0) = 0, \;\; n \text{ odd} \qquad (v) \;\; P_n'(0) = 0, \;\; n \text{ even.}$$

Property (i) indicates, as is apparent in Figure 5.3.6, that $P_n(x)$ is an even or odd function according to whether n is even or odd.

Recurrence Relation Recurrence relations that relate Legendre polynomials of different degrees are also important in some aspects of their applications. We state, without proof, the following three-term recurrence relation

$$(k + 1)P_{k+1}(x) - (2k + 1)xP_k(x) + kP_{k-1}(x) = 0, \tag{33}$$

which is valid for $k = 1, 2, 3, \ldots$. In (31) we listed the first six Legendre polynomials. If, say, we wish to find $P_6(x)$, we can use (33) with $k = 5$. This relation expresses $P_6(x)$ in terms of the known $P_4(x)$ and $P_5(x)$. See Problem 49 in Exercises 5.3.

Another formula, although not a recurrence relation, can generate the Legendre polynomials by differentiation. **Rodrigues' formula** for these polynomials is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n, \quad n = 0, 1, 2, \ldots. \tag{34}$$

See Problem 53 in Exercises 5.3.

Remarks

Although we have assumed that the parameter n in Legendre's differential equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

represented a nonnegative integer, in a more general setting n can represent any real number. If n is *not* a nonnegative integer, then both Legendre functions $y_1(x)$ and $y_2(x)$ given in (30) are infinite series convergent on the open interval $(-1, 1)$ and divergent (unbounded) at $x = \pm 1$. If n is a nonnegative integer, then as we have just seen one of the Legendre functions in (30) is a polynomial and the other is an infinite series convergent for $-1 < x < 1$. You should be aware of the fact that Legendre's equation possesses solutions that are bounded on the *closed* interval $[-1, 1]$ only in the case when $n = 0, 1, 2, \ldots$ More to the point, the only Legendre functions that are bounded on the closed interval $[-1, 1]$ are the Legendre polynomials $P_n(x)$ or constant multiples of these polynomials. See Problem 51 in Exercises 5.3 and Problem 24 in Chapter 5 in Review.

5.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-12.

5.3.1 Bessel Functions

In Problems 1–6, use (1) to find the general solution of the given differential equation on $(0, \infty)$.

1. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$

2. $x^2y'' + xy' + (x^2 - 1)y = 0$

3. $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$

4. $16x^2y'' + 16xy' + (16_x^2 - 1)y = 0$

5. $xy'' + y' + xy = 0$

6. $\frac{d}{dx}[xy'] + \left(x - \frac{4}{x}\right)y = 0$

In Problems 7–10, use (12) to find the general solution of the given differential equation on the interval $(0, \infty)$.

7. $x^2y'' + xy' + (9x^2 - 4)y = 0$

8. $x^2y'' + xy' + (36x^2 - \frac{1}{4})y = 0$

9. $x^2y'' + xy' + (25x^2 - \frac{4}{9})\,y = 0$

10. $x^2y'' + xy' + (2x^2 - 64)y = 0$

In Problems 11 and 12, use the indicated change of variable to find the general solution of the given differential equation on the interval $(0, \infty)$.

11. $x^2y'' + 2xy' + \alpha^2x^2y = 0;\ y = x^{-1/2}u(x)$

12. $x^2y'' + (\alpha^2x^2 - \nu^2 + \frac{1}{4})y = 0;\quad y = \sqrt{x}\,u(x)$

In Problems 13–20, use (18) to find the general solution of the given differential equation on the interval $(0, \infty)$.

13. $xy'' + 2y' + 4y = 0$

14. $xy'' + 3y' + xy = 0$

15. $xy'' - y' + xy = 0$

16. $xy'' - 5y' + xy = 0$

17. $x^2y'' + (x^2 - 2)y = 0$

18. $4x^2y'' + (16x^2 + 1)y = 0$

19. $xy'' + 3y' + x^3y = 0$

20. $9x^2y + 9xy' + (x^6 - 36)y = 0$

21. Use the series in (7) to verify that $I_v(x) = i^{-v}J_v(ix)$ is a real function.

22. Assume that b in equation (18) can be pure imaginary; that is, $b = \beta i$, $\beta > 0$, $i^2 = -1$. Use this assumption to express the general solution of the given differential equation in terms of the modified Bessel functions I_n and K_n.

(a) $y'' - x^2y = 0$

(b) $xy'' + y' - 7x^3y = 0$

In Problems 23–26, first use (18) to express the general solution of the given differential equation in terms of Bessel functions. Then use (23) and (24) to express the general solution in terms of elementary functions.

23. $y'' + y = 0$

24. $x^2y'' + 4xy' + (x^2 + 2)y = 0$

25. $16x^2y'' + 32xy' + (x^4 - 12)y = 0$

26. $4x^2y'' - 4xy' + (16x^2 + 3)y = 0$

27. **(a)** Proceed as in Example 5 to show that

$$xJ'_{v}(x) = -vJ_v(x) + x\,J_{v-1}(x).$$

[*Hint*: Write $2n + v = 2(n + v) - v$.]

(b) Use the result in part (a) to derive (21).

28. Use the formula obtained in Example 5 along with part (a) of Problem 27 to derive the recurrence relation

$$2v\,J_n(x) = x\,J_{v+1}(x) + xJ_{v-1}(x).$$

In Problems 29 and 30, use (20) or (21) to obtain the given result.

29. $\int_0^x rJ_0(r)\,dr = xJ_1(x)$

30. $J'_0(x) = J_{-1}(x) = -J_1(x)$

31. **(a)** Proceed as on pages 279–280 to derive the elementary form of $J_{-1/2}(x)$ given in (24).

(b) Use $v = -\frac{1}{2}$ along with (23) and (24) in the recurrence relation in Problem 28 to express $J_{-3/2}(x)$ in terms of sin x, cos x, and powers of x.

(c) Use a graphing utility to plot the graph of $J_{-3/2}(x)$.

32. **(a)** Use the recurrence relation in Problem 28 to express $J_{3/2}(x)$, $J_{5/2}(x)$, and $J_{7/2}(x)$ in terms of sin x, cos x, and powers of x.

(b) Use a graphing utility to plot the graphs of $J_{3/2}(x)$, $J_{5/2}(x)$, and $J_{7/2}(x)$ in the same coordinate plane.

33. Use the change of variables $s = \frac{2}{\alpha}\sqrt{\frac{k}{m}}e^{-\alpha t/2}$ to show that the differential equation of the aging spring $mx'' + ke^{-at}x = 0$, $\alpha > 0$, becomes

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0.$$

34. Show that $y = x^{1/2}w(\frac{2}{3}\alpha x^{3/2})$ is a solution of Airy's differential equation $y'' + \alpha^2xy = 0, x > 0$, whenever w is a solution of Bessel's equation of order $\frac{1}{3}$; that is, $t^2w'' + tw' + (t^2 - \frac{1}{9})\,w = 0, t > 0$. [Hint: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$.]

35. **(a)** Use the result of Problem 34 to express the general solution of Airy's differential equation for $x > 0$ in terms of Bessel functions.

(b) Verify the results in part (a) using (18).

36. Use Table 5.3.1 to find the first three positive eigenvalues and corresponding eigenfunctions of the boundary-value problem

$$xy'' + y' + \gamma xy = 0,$$

$$y(x), y'(x) \text{ bounded as } x \to 0^+, y(2) = 0.$$

[*Hint*: By identifying $\gamma = \alpha^2$, the DE is the parametric Bessel equation of order zero.]

37. **(a)** Use (18) to show that the general solution of the differential equation $xy'' + \lambda y = 0$ on the interval $(0, \infty)$ is

$$y = c_1\sqrt{x}J_1(2\sqrt{\lambda x}) + c_2\sqrt{x}Y_1(2\sqrt{\lambda x}).$$

(b) Verify by direct substitution that $y = \sqrt{x}J_1(2\sqrt{x})$ is a particular solution of the DE in the case $\lambda = 1$.

38. **(a)** Use (15) and (7) to show that

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sinh x.$$

(b) Use (15) and (8) to show that

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cosh x.$$

(c) Use (16) to express $K_{1/2}(x)$ in terms of elementary functions.

39. **(a)** Use the first formula in (28) to find the spherical Bessel functions $j_1(x)$, $j_2(x)$, and $j_3(x)$.

(b) Use a graphing utility to plot the graphs of $j_1(x)$, $j_2(x)$ and $j_3(x)$ in the same coordinate plane.

40. **(a)** Use the second formula in (28) to find the spherical Bessel functions $y_1(x)$, $y_2(x)$, and $y_3(x)$.

(b) Use a graphing utility to plot the graphs of $y_1(x)$, $y_2(x)$ and $y_3(x)$ in the same coordinate plane.

41. If n is an integer, use the substitution $R(x) = (\alpha x)^{-1/2} Z(x)$ to show that the differential equation

$$x^2 \frac{d^2R}{dx^2} + 2x\frac{dR}{dx} + [\alpha^2x^2 - n(n+1)]R = 0 \qquad (35)$$

becomes

$$x^2 \frac{d^2Z}{dx^2} + x\frac{dZ}{dx} + [\alpha^2x^2 - (n+\tfrac{1}{2})^2]Z = 0. \qquad (36)$$

42. **(a)** In Problem 41, find the general solution of the DE in (36) on the interval $(0, \infty)$.

(b) Use part (a) to find the general solution of the DE in (35) on the interval $(0, \infty)$.

(c) Use part (b) to express the general solution of (35) in terms of the spherical Bessel functions of the first and second kind defined in (28).

≡ Computer Lab Assignments

43. **(a)** Use the general solution given in Example 4 to solve the IVP

$$4x'' + e^{-0.1t}x = 0, \; x(0) = 1, \; x'(0) = -\tfrac{1}{2}.$$

Also use $J'_0(x) = -J_1(x)$ and $Y'_0(x) = -Y_1(x)$ along with Table 5.3.1 or a CAS to evaluate coefficients.

(b) Use a CAS to graph the solution obtained in part (a) for $0 \le t < \infty$.

44. **(a)** Use the general solution obtained in Problem 35 to solve the IVP

$$4x'' + tx = 0, \; x(0.1) = 1, \; x'(0.1) = -\tfrac{1}{2}.$$

Use a CAS to evaluate coefficients.

(b) Use a CAS to graph the solution obtained in part (a) for $0 \le t \le 200$.

45. **Column Bending Under Its Own Weight** A uniform thin column of length L, positioned vertically with one end embedded in the ground, will deflect, or bend away, from the vertical under the influence of its own weight when its length or height exceeds a certain critical value. It can be shown that the angular deflection $\theta(x)$ of the column from the vertical at a point $P(x)$ is a solution of the boundary-value problem

$$EI\frac{d^2\theta}{dx^2} + \delta g(L - x)\theta = 0, \quad \theta(0) = 0, \quad \theta'(L) = 0,$$

where E is Young's modulus, I is the cross-sectional moment of inertia, δ is the constant linear density, and x is the distance along the column measured from its base. See **FIGURE 5.3.7**.

The column will bend only for those values of L for which the boundary-value problem has a nontrivial solution.

(a) Restate the boundary-value problem by making the change of variables $t = L - x$. Then use the results of a problem earlier in this exercise set to express the general solution of the differential equation in terms of Bessel functions.

(b) Use the general solution found in part (a) to find a solution of the BVP and an equation that defines the critical length L; that is, the smallest value of L for which the column will start to bend.

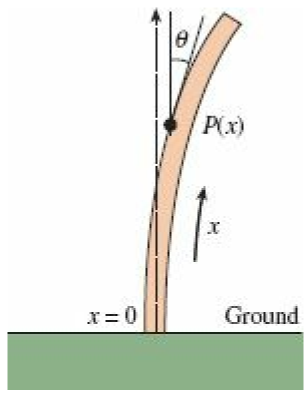

FIGURE 5.3.7 Column in Problem 45

(c) With the aid of a CAS, find the critical length L of a solid steel rod of radius $r = 0.05$ in., $\delta g = 0.28\,A$ lb/in., $E = 2.6 \times 10^7$ lb/in.^{2}, $A = \pi r^2$, and $I = \frac{1}{4}\pi r^4$.

46. Buckling of a Thin Vertical Column In Example 4 of Section 3.9 we saw that when a constant vertical compressive force, or load, P was applied to a thin column of uniform cross section and hinged at both ends, the deflection $y(x)$ is a solution of the BVP:

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

(a) If the bending stiffness factor EI is proportional to x, then $EI(x) = kx$, where k is a constant of proportionality. If $EI(L) = kL = M$ is the maximum stiffness factor, then $k = M/L$ and so $EI(x) = Mx/L$. Use the information in Problem 37 to find a solution of

$$M\frac{x}{L}\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

if it is known that $\sqrt{x}Y_1(2\sqrt{\lambda x})$ is *not* zero at $x = 0$.

(b) Use Table 5.3.1 to find the Euler load P_1 for the column.

(c) Use a CAS to graph the first buckling mode $y_1(x)$ corresponding to the Euler load P_1. For simplicity assume that $c_1 = 1$ and $L = 1$.

47. Pendulum of Varying Length For the simple pendulum described on page 187 of Section 3.11, suppose that the rod holding the mass m at one end is replaced by a flexible wire or string and that the wire is strung over a pulley at the point of support O in Figure 3.11.3. In this manner, while it is in motion in a vertical plane, the mass m can be raised or lowered. In other words, the length $l(t)$ of the pendulum varies with time. Under the same assumptions leading to equation (6) in Section 3.11, it can be shown* that the differential equation for the displacement angle θ is now

$$l\theta'' + 2l'\theta' + g\sin\theta = 0.$$

(a) If l increases at a constant rate v and if $l(0) = l_0$, show that a linearization of the foregoing DE is

$$(l_0 + vt)\theta'' + 2v\theta' + g\theta = 0. \qquad (37)$$

(b) Make the change of variables $x = (l_0 + vt)/v$ and show that (37) becomes

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{g}{vx}\theta = 0.$$

(c) Use part (b) and (18) to express the general solution of equation (37) in terms of Bessel functions.

(d) Use the general solution obtained in part (c) to solve the initial-value problem consisting of equation (37) and the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$. [*Hints*: To simplify calculations use a further change of variable $u = \frac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\frac{g}{v}}x^{1/2}$. Also, recall (20) holds for both $J_1(u)$ and $y_1(u)$. Finally, the identity

$$J_1(u)Y_2(u) - J_2(u)Y_1(u) = -\frac{2}{\pi u}$$

will be helpful.]

(e) Use a CAS to graph the solution $\theta(t)$ of the IVP in part (d) when $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s. Experiment with the graph using different time intervals such as [0, 10], [0, 30], and so on.

(f) What do the graphs indicate about the displacement angle $\theta(t)$ as the length l of the wire increases with time?

5.3.2 Legendre Functions

48. **(a)** Use the explicit solutions $y_1(x)$ and $y_2(x)$ of Legendre's equation given in (30) and the appropriate choice of c_0 and c_1 to find the Legendre polynomials $P_6(x)$ and $P_7(x)$.

(b) Write the differential equations for which $P_6(x)$ and $P_7(x)$ are particular solutions.

49. Use the recurrence relation (33) and $P_0(x) = 1$, $P_1(x) = x$, to generate the next six Legendre polynomials.

*See *Mathematical Methods in Physical Sciences*, Mary Boas, John Wiley & Sons, 1966; Also see the article by Borelli, Coleman, and Hobson in *Mathematics Magazine*, vol. 58, no. 2, March 1985.

50. Show that the differential equation

$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$

can be transformed into Legendre's equation by means of the substitution $x = \cos\theta$.

51. Find the first three positive values of λ for which the problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

$$y(0) = 0,\ y(x),\ y'(x) \text{ bounded on } [-1, 1]$$

has nontrivial solutions.

52. The differential equation

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]y = 0,$$

is known as the **associated Legendre equation.** When $m = 0$ this equation reduces to Legendre's equation (2). A solution of the associated equation is

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x),$$

where $P_n(x)$, $n = 0, 1, 2, \ldots$ are the Legendre polynomials given in (31). The solutions $P_n^m(x)$ for $m = 0, 1, 2, \ldots$, are called **associated Legendre functions**.

(a) Find the associated Legendre functions $P_0^0(x)$, $P_1^0(x)$,, $P_1^1(x)$, $P_2^1(x)$, $P_2^2(x)$, $P_3^1(x)$, $P_3^2(x)$, and $P_3^3(x)$..

(b) What can you say about $P_n^m(x)$ when m is an even nonnegative integer?

(c) What can you say about $P_n^m(x)$ when m is an nonnegative integer and $m > n$?

(d) Verify that $y = P_1^1(x)$ satisfies the associated Legendre equation when $n = 1$ and $m = 1$.

≡ Computer Lab Assignments

53. For purposes of this problem, ignore the list of Legendre polynomials given on page 282 and the graphs given in Figure 5.3.6. Use Rodrigues' formula (34) to generate the Legendre polynomials $P_1(x)$, $P_2(x)$, …, $P_7(x)$. Use a CAS to carry out the differentiations and simplifications.

54. Use a CAS to graph $P_1(x)$, $P_2(x)$, …, $P_7(x)$ on the closed interval $[-1, 1]$.

55. Use a root-finding application to find the zeros of $P_1(x)$, $P_2(x)$, …, $P_7(x)$. If the Legendre polynomials are built-in functions of your CAS, find the zeros of Legendre polynomials of higher degree. Form a conjecture about the location of the zeros of any Legendre polynomial $P_n(x)$, and then investigate to see whether it is true.

5 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-12.

In Problems 1 and 2, answer true or false without referring back to the text.

1. The general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$ is $y = c_1J_1(x) + c_2J_{-1}(x)$.______