• Bessel's and Legendre Functions

the d.e $\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0}$ (1)

is called Bessel's eq. of order ν. $(\nu \geq 0)$.

the solution of eq (1) is given by

$$y = \begin{cases} c_1 J_\nu(x) + c_2 J_{-\nu}(x) & , \nu \neq \text{integer} \\ a_1 J_\nu(x) + a_2 Y_\nu(x) & , \nu \text{ integer}, \end{cases}$$

where $\boxed{J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}}$

(Bessel functions of the first kind)

$$\boxed{Y_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}}$$

(Bessel's function of the second kind)

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ex. Solve $x^2y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$.

sol. $\nu^2 = 9 \Rightarrow \boxed{\nu = 3}$

$\therefore y = c_1 J_3(x) + c_2 Y_3(x)$.

ex. Solve $x^2 y'' + x y' + (x^2 - \frac{1}{9}) y = 0$.

Sol. $\nu^2 = \frac{1}{9} \Rightarrow \boxed{\nu = \frac{1}{3}}$

$\therefore y = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$.

• The d.e $\boxed{x^2 y'' + x y' + (\alpha^2 x^2 - \nu^2) y = 0}$ (2)

is called the parametric Bessel eq. of order ν. (where ν is integer)

The general eq. for eq (2) is given by

$\boxed{y = c_1 J_\nu(\alpha x) + c_2 Y_{-\nu}(\alpha x)}$, ν = integer

[by letting $t = \alpha x$, $\alpha > 0$ and using chain Rule see the pdf file .]

ex. Solve $x^2 y'' + x y' + (9x^2 - 4) y = 0$

Sol. $\alpha^2 = 9 \Rightarrow \boxed{\alpha = 3}$, $\nu^2 = 4 \Rightarrow \boxed{\nu = 2}$

$\therefore y = c_1 J_2(3x) + c_2 Y_{-2}(3x)$.

ex. Solve $x^2 y'' + x y' + (2x^2 - 64) y = 0$.

Sol. $\alpha^2 = 2 \Rightarrow \boxed{\alpha = \sqrt{2}}$, $\nu^2 = 64 \Rightarrow \boxed{\nu = 8}$

$\therefore y = c_1 J_8(\sqrt{2} x) + c_2 Y_{-8}(\sqrt{2} x)$.

Properties of Bessel functions

(1) $J_{-m}(x) = (-1)^m J_m(x)$, $m = 0, 1, 2, \ldots$

(2) $J_m(-x) = (-1)^m J_m(x)$, $m = 0, 1, 2, \ldots$

(3) $J_m(0) = \begin{cases} 0, & m > 0 \\ 1, & m = 0 \end{cases}$, $m = 0, 1, 2, \ldots$

(4) $\lim\limits_{x \to 0^+} Y_m(x) = -\infty$, $m = 0, 1, 2, \ldots$

(5) $x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x)$.

(6) $\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x)$,

or $\int x^\nu J_{\nu-1}(x)\, dx = x^\nu J_\nu(x) + c$.

ex. $\int x^3 J_2(x)\, dx = x^3 J_3(x) + c$.

ex. $\frac{d}{dx}\left[x J_1(x)\right] = x J_0(x)$.

(7) $\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$

or $\int x^{-\nu} J_{\nu+1}(x)\, dx = -x^{-\nu} J_\nu(x) + c$.

(8) Bessel function of half-integral order $J_{1/2}(x)$.

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\frac{3}{2}+n)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sqrt{\frac{2}{\pi x}} \sin x.$$

(9) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

Legendre Functions

The d.e $\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$ (3)

is called Legendre's eq. of order n, $-1 \le x \le 1$

The solution for eq (3) is given by

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right], \quad n = 0, 1, 2, \ldots} \quad (4)$$

(4) is called Rodrigues' formula or Legendre polynomials.

for ex., $P_0(x) = \frac{1}{2^0\, 0!} \frac{d^0}{dx^0}\left[(x^2-1)^0\right] = 1$

$\Rightarrow \boxed{P_0(x) = 1}$

$$P_1(x) = \frac{1}{2 \cdot 1!} \frac{d}{dx}\left[(x^2-1)^1\right] = \frac{1}{2} \cdot 2x = x$$

$$\Rightarrow \boxed{P_1(x) = x}$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2}\left[(x^2-1)^2\right]$$

$$= \frac{1}{8} \frac{d}{dx}\left[2(x^2-1)(2x)\right]$$

$$= \frac{1}{2} \frac{d}{dx}\left[x^3-x\right] = \frac{1}{2}(3x^2-1).$$

$$\Rightarrow \boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$$

$$\boxed{P_3(x) = \frac{1}{2}(5x^3-3x)}$$ اشتقها

$$\boxed{P_4(x) = \frac{1}{8}(35x^4 - 30x^2+3)}$$ اشتقها

ملاحظة: مطلوب حفظ P_0 ، P_1 ، P_2 ، P_3 فقط
لانها ستُستخدم لاحقاً.

Particular solutions of the d.e's (3) :

$\boxed{n=0}$: $(1-x^2)y'' - 2xy' = 0 \Rightarrow \boxed{P_0(x) = 1}$.

$\boxed{n=1}$: $(1-x^2)y'' - 2xy' + 2y = 0 \Rightarrow \boxed{P_1(x) = x}$

$\boxed{n=2}$: $(1-x^2)y'' - 2xy' + 6y = 0$

$$\boxed{P_2(x) = \frac{1}{2}(3x^2-1)}.$$

$\boxed{n=3}$: $(1-x^2)y'' - 2xy + 12y = 0$

$$\boxed{P_3(x) = \frac{1}{2}(5x^3-3x)}.$$

$\vdots$

Properties:

(1) $P_n(-x) = (-1)^n P_n(x)$.

(2) $P_n(1) = 1$ (3) $P_n(-1) = (-1)^n$.

(4) $P_n(0) = 0$, n odd

(5) $P_n'(0) = 0$, n even.

(6) $\displaystyle\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$. (orthogonality)