10. Hermite's differential equation

$$y'' - 2xy' + 2ny = 0, \; n = 0, 1, 2, \ldots,$$

has polynomial solutions $H_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

11. Consider the regular Sturm–Liouville problem:

$$\frac{d}{dx}[(1+x^2)y'] + \frac{\lambda}{1+x^2}y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

(a) Find the eigenvalues and eigenfunctions of the boundary-value problem. [*Hint*: Let $x = \tan \pi$ and then use the Chain Rule.]

(b) Give an orthogonality relation.

12. (a) Find the eigenfunctions and the equation that defines the eigenvalues for the boundary-value problem

$$x^2y'' + xy' + (\lambda x^2 - 1)y = 0,$$
$$y \text{ is bounded at } x = 0, \; y(3) = 0.$$

(b) Use Table 5.3.1 of Section 5.3 to find the approximate values of the first four eigenvalues α_1, α_2, α_3, and α_4.

≡ Discussion Problem

13. Consider the special case of the regular Sturm–Liouville problem on the interval $[a, b]$:

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0, \quad y'(a) = 0, \quad y'(b) = 0.$$

Is $\alpha = 0$ an eigenvalue of the problem? Defend your answer.

≡ Computer Lab Assignments

14. (a) Give an orthogonality relation for the Sturm–Liouville problem in Problem 1.

(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues α_1 and α_2, respectively.

15. (a) Give an orthogonality relation for the Sturm–Liouville problem in Problem 2.

(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues α_1 and α_2, respectively.

12.6 Bessel and Legendre Series

≡ **Introduction** Fourier series, Fourier cosine series, and Fourier sine series are three ways of expanding a function in terms of an orthogonal set of functions. But such expansions are by no means limited to orthogonal sets of trigonometric functions. We saw in Section 12.1 that a function f defined on an interval (a, b) could be expanded, at least in a formal manner, in terms of any set of functions $\{f_n(x)\}$ that is orthogonal with respect to a weight function on $[a, b]$. Many of these orthogonal series expansions or generalized Fourier series derive from Sturm–Liouville problems that, in turn, arise from attempts to solve linear partial differential equations serving as models for physical systems. Fourier series and orthogonal series expansions (the latter includes the two series considered in this section) will appear in the subsequent consideration of these applications in Chapters 13 and 14.

12.6.1 Fourier–Bessel Series

We saw in Example 3 of Section 12.5 that for a fixed value of n the set of Bessel functions $\{J_n(\alpha_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$ when the α_i are defined by means of a boundary condition of the form

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0. \tag{1}$$

The eigenvalues of the corresponding Sturm–Liouville problem are $\lambda_i = \alpha^2_i$. From (7) and (8) of Section 12.1 the orthogonal series expansion or generalized Fourier series of a function f defined on the interval $(0, b)$ in terms of this orthogonal set is

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x), \tag{2}$$

where

$$c_i = \frac{\int_0^b x J_n(\alpha_i x) f(x)\, dx}{\|J_n(\alpha_i x)\|^2}. \tag{3}$$

The square norm of the function $J_n(\alpha_i x)$ is defined by (11) of Section 12.1:

$$\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)\, dx. \tag{4}$$

The series (2) with coefficients (3) is called a **Fourier–Bessel series.**

Differential Recurrence Relations The differential recurrence relations that were given in (20) and (21) of Section 5.3 are often useful in the evaluation of the coefficients (3). For convenience we reproduce those relations here:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \tag{5}$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x). \tag{6}$$

Square Norm The value of the square norm (4) depends on how the eigenvalues $\lambda_i = \alpha^2_i$ are defined. If $y = J_n(\alpha x)$, then we know from Example 3 of Section 12.5 that

$$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0.$$

After we multiply by $2xy'$, this equation can be written as

$$\frac{d}{dx}[xy']^2 + (\alpha^2 x^2 - n^2)\frac{d}{dx}[y]^2 = 0.$$

Integrating the last result by parts on $[0, b]$ then gives

$$2\alpha^2\int_0^b xy^2\,dx = \left([xy']^2 + (\alpha^2x^2 - n^2)y^2\right)\Big]_0^b.$$

Since $y = J_n(\alpha x)$, the lower limit is zero for $n > 0$ because $J_n(0) = 0$. For $n = 0$, the quantity $[xy']^2 + \alpha^2x^2y^2$ is zero at $x = 0$. Thus

$$2\alpha^2\int_0^b xJ_n^2(\alpha x)\,dx = \alpha^2b^2[J_n'(\alpha b)]^2 + (\alpha^2b^2 - n^2)[J_n(\alpha b)]^2, \qquad (7)$$

where we have used the Chain Rule to write $y' = \alpha J_n'(\alpha x)$.

We now consider three cases of the boundary condition (1).

Case I: If we choose $A_2 = 1$ and $B_2 = 0$, then (1) is

$$J_n(\alpha b) = 0. \qquad (8)$$

There are an infinite number of positive roots $x_t = \alpha_i b$ of (8) (see Figure 5.3.1) that define the α_i as $\alpha_i = x_i/b$. The eigenvalues are positive and are then $\lambda_i = \alpha^2 i = x^2 i/b^2$. No new eigenvalues result from the negative roots of (8) since $J_n(-x) = (-1)^n J_n(x)$. (See page 277.) The number 0 is not an eigenvalue for any n since $J_n(0) = 0$ for $n = 1, 2, 3, \ldots$ and $J_0(0) = 1$. In other words, if $\lambda = 0$, we get the trivial function (which is never an eigenfunction) for $n = 1, 2, 3, \ldots$, and for $n = 0$, $\lambda = 0$ (or equivalently, $\alpha = 0$) does not satisfy the equation in (8). When (6) is written in the form $x\,J'_n(x) = n\,J_n(x) - x\,J_{n+1}(x)$, it follows from (7) and (8) that the square norm of $J_n(\alpha_i x)$ is

$$\|J_n(\alpha_i x)\|^2 = \frac{b^2}{2}J_{n+1}^2(\alpha_i b). \qquad (9)$$

Case II: If we choose $A_2 = h \le 0$, $B_2 = b$, then (1) is

$$hJ_n(\alpha b) + \alpha bJ_n'(\alpha b) = 0. \qquad (10)$$

Equation (10) has an infinite number of positive roots $x_i = \alpha_i b$ for each positive integer $n = 1, 2, 3, \ldots$. As before, the eigenvalues are obtained from $\lambda_i = \alpha^2{}_i = x^2{}_i/b^2$. $\lambda = 0$ is not an eigenvalue for $n = 1, 2, 3, \ldots$. Substituting $\lambda_i bJ'_n(\alpha_i b) = -hJ_n(\alpha_i b)$ into (7), we find that the square norm of $J_n(\alpha_i x)$ is now

$$\|J_n(\alpha_i x)\|^2 = \frac{\alpha_i^2b^2 - n^2 + h^2}{2\alpha_i^2}J_n^2(\alpha_i b). \qquad (11)$$

Case III: If $h = 0$ and $n = 0$ in (10), the α_i are defined from the roots of

$$J_0'(\alpha b) = 0. \qquad (12)$$

Even though (12) is just a special case of (10), it is the only situation for which $\lambda = 0$ is an eigenvalue. To see this, observe that for $n = 0$, the result in (6) implies that $J'_0(\alpha b) = 0$ is equivalent to $J_1(\alpha b) = 0$. Since $x_1 = \alpha i b = 0$ is a root of the last equation, $\alpha_1 = 0$, and because $J_0(0) = 1$ is nontrivial, we conclude from $\lambda_1 = \alpha^2{}_1 = x^2{}_1/b^2$ that $\lambda_1 = 0$ is an eigenvalue. But obviously we cannot use (11) when $\alpha_1 = 0$, $h = 0$, and $n = 0$. However, from the square norm (4) we have

$$\|1\|^2 = \int_0^b x\,dx = \frac{b^2}{2}. \tag{13}$$

For $\alpha_i > 0$ we can use (11) with $h = 0$ and $n = 0$:

$$\|J_0(\alpha_i x)\|^2 = \frac{b^2}{2} J_0^2(\alpha_i b). \tag{14}$$

The following definition summarizes three forms of the series (2) corresponding to the square norms in the three cases.

Definition 12.6.1 Fourier–Bessel Series

The **Fourier–Bessel series** of a function f defined on the interval $(0, b)$ is given by

(i)

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{15}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\,dx, \tag{16}$$

where the α_i are defined by $J_n(\alpha b) = 0$.

(ii)

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{17}$$

$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\,dx, \tag{18}$$

where the α_i are defined by $hJ_n(\alpha b) + \alpha b J'_n(\alpha b) = 0$.

(iii)

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \tag{19}$$

$$c_1 = \frac{2}{b^2} \int_0^b x f(x)\,dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x)\,dx, \tag{20}$$

where the α_i are defined by $J'_0(\alpha b) = 0$.

□ **Convergence of a Fourier–Bessel Series** Sufficient conditions for the convergence of a Fourier–Bessel series are not particularly restrictive.

Theorem 12.6.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[0, b]$. Then for all x in the interval $(0, b)$, the Fourier–Bessel series of f converges to $f(x)$ at a point where f is continuous and to the average

$$\frac{f(x+) + f(x-)}{2}$$

at a point where f is discontinous.

EXAMPLE 1 Expansion in a Fourier–Bessel Series

Expand $f(x) = x$, $0 < x < 3$ in a Fourier–Bessel series, using Bessel functions of order on that satisfy the boundary condition $J_1(3\alpha) = 0$.

SOLUTION We use (15) where the coefficients c_i are given by (16) with $b = 3$:

$$c_i = \frac{2}{3^2 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x)\, dx.$$

To evaluate this integral we let $t = \alpha_i x$, $dx = dt/\alpha_i$, $x^2 = t^2/\alpha_i^2$, and use (5) in the form $\frac{d}{dt}[t^2 J_2(t)] = t^2 J_1(t)$:

$$c_i = \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} \frac{d}{dt}[t^2 J_2(t)]\, dt = \frac{2}{\alpha_i J_2(3\alpha_i)}.$$

Therefore the desired expansion is

$$f(x) = 2\sum_{i=1}^{\infty} \frac{1}{\alpha_i J_2(3\alpha_i)} J_1(\alpha_i x). \qquad \equiv$$

You are asked to find the first four values of the α_i for the foregoing Bessel series in Problem 1 in Exercises 12.6.

EXAMPLE 2 Expansion in a Fourier–Bessel Series

If the α_i in Example 1 are defined by $J_1(3\alpha) + \alpha J'_1(3\alpha) = 0$, then the only thing that changes in the expansion is the value of the square norm. Multiplying the boundary condition by 3 gives $3J_1(3\alpha) + 3\alpha J'_1(3\alpha) = 0$, which now matches (10) when $h = 3$, $b = 3$, and $n = 1$. Thus (18) and (17) yield, in turn,

$$c_i = \frac{18\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8)J_1^2(3\alpha_i)}$$

and

$$f(x) = 18\sum_{i=1}^{\infty} \frac{\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2 + 8)J_1^2(3\alpha_i)} J_1(\alpha_i x). \qquad \equiv$$

Use of Computers Since Bessel functions are "built-in functions" in a CAS, it is a straightforward task to find the approximate values of the α_i and the coefficients c_i in a Fourier–Bessel series. For example, in (9) we can think of $x_i = \alpha_i b$ as a positive root of the equation $hJ_n(x) + x J'_n(x) = 0$. Thus in Example 2 we have used a CAS to find the first five positive roots x_i of $3J_1(x) + xJ'_1(x) = 0$ and from these roots we obtain the first five values of α_i: $\alpha_1 = x_1/3 = 0.98320$, $\alpha_2 = x_2/3 = 1.94704$, $\alpha_3 = x_3/3 = 2.95758$, $\alpha_4 = x_4/3 = 3.98538$, and $\alpha_5 = x_5/3 = 5.02078$. Knowing the roots $x_i = 3\alpha_i$ and the α_i, we again use a CAS to calculate the numerical values of $J_2(3\alpha_i)$, $J^2_1(3\alpha_i)$, and finally the coefficients c_i. In this manner we find that the fifth partial sum $S_5(x)$ for the Fourier–Bessel series representation of $f(x) = x$, $0 < x < 3$ in Example 2 is

$$S_5(x) = 4.01844\, J_1(0.98320x) - 1.86937\, J_1(1.94704x)$$

$$+ 1.07106\, J_1(2.95758x) - 0.70306\, J_1(3.98538x) + 0.50343\, J_1(5.02078x).$$

The graph of $S_5(x)$ on the interval (0, 3) is shown in **FIGURE 12.6.1(a).** In Figure 12.6.1(b) we have graphed $S_{10}(x)$ on the interval (0, 50). Notice that outside the interval of definition (0, 3) the series does not converge to a periodic extension of f because Bessel functions are not periodic functions. See Problems 11 and 12 in Exercises 12.6.

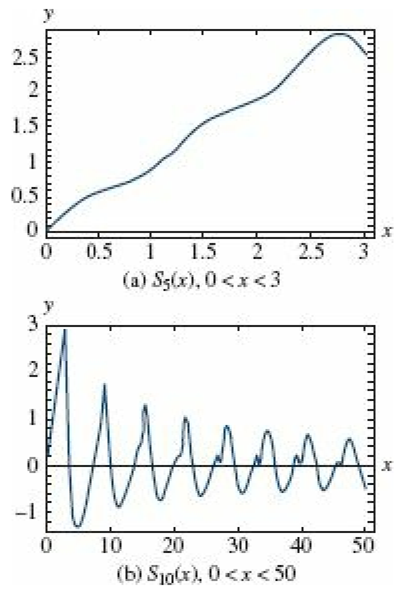

FIGURE 12.6.1 Partial sums of a Fourier–Bessel series

12.6.2 Fourier–Legendre Series

From Example 4 of Section 12.5 we know that the set of Legendre polynomials $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on the interval [–1, 1]. Furthermore, it can be proved that the square norm of a polynomial $P_n(x)$ depends on n in the following manner:

$$\|P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

The orthogonal series expansion of a function in terms of the Legendre polynomials is summarized in the next definition.

Definition 12.6.2 Fourier–Legendre Series

The **Fourier–Legendre series** of a function f defined on the interval (–1, 1) is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{21}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{22}$$

Convergence of a Fourier–Legendre Series Sufficient conditions for convergence of a Fourier–Legendre series are given in the next theorem.

Theorem 12.6.2 Conditions for Convergence

Let f and f' be piecewise continuous on the interval [– 1, 1]. Then for all x in the interval (–1, 1), the Fourier–Legendre series of f converges to $f(x)$ at a point where f is continuous and to the average

$$\frac{f(x+) + f(x-)}{2}$$

at a point where f is discontinuous.

EXAMPLE 3 Expansion in a Fourier–Legendre Series

Write out the first four nonzero terms in the Fourier–Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$

SOLUTION The first several Legendre polynomials are listed on page 282. From these and (22) we find

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)\,dx = \frac{1}{2}\int_0^1 1\cdot 1\,dx = \frac{1}{2}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = \frac{3}{2}\int_0^1 1\cdot x\,dx = \frac{3}{4}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{2}\int_0^1 1\cdot\frac{1}{2}(3x^2-1)\,dx = 0$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)\,dx = \frac{7}{2}\int_0^1 1\cdot\frac{1}{2}(5x^3-3x)\,dx = -\frac{7}{16}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} f(x)P_4(x)\,dx = \frac{9}{2}\int_0^1 1\cdot\frac{1}{8}(35x^4-30x^2+3)\,dx = 0$$

$$c_5 = \frac{11}{2}\int_{-1}^{1} f(x)P_5(x)\,dx = \frac{11}{2}\int_0^1 1\cdot\frac{1}{8}(63x^5-70x^3+15x)\,dx = \frac{11}{32}.$$

Hence

$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots.$$ ≡

Like the Bessel functions, Legendre polynomials are built-in functions in computer algebra systems such as *Maple* and *Mathematica,* and so each of the coefficients just listed can be found using the integration application of such a program. Indeed, using a CAS, we further find that $c_6 = 0$ and $c_7 = -\frac{65}{256}$. The fifth partial sum of the Fourier–Legendre series representation of the function f defined in Example 3 is then

$$S_5(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) - \frac{65}{256}P_7(x).$$

The graph of $S_5(x)$ on the interval (–1, 1) is given in FIGURE 12.6.2.

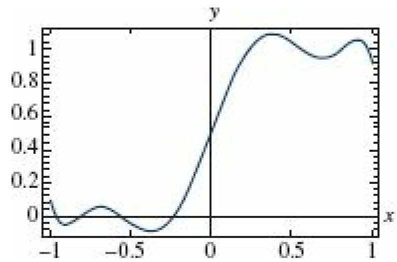

FIGURE 12.6.2 Partial sum $S_5(x)$ of Fourier–Legendre series in Example 3

Alternative Form of Series In applications, the Fourier–Legendre series appears in an alternative form. If we let $x = \cos\theta$, then $x = 1$ implies $\theta = 0$, whereas $x = -1$ implies $\theta = \pi$. Since $dx = -\sin\theta\, d\theta$, (21) and (22) become, respectively,

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta) \tag{23}$$

$$c_n = \frac{2n+1}{2}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\, d\theta, \tag{24}$$

where $f(\cos\theta)$ has been replaced by $F(\theta)$

12.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

12.6.1 Fourier–Bessel Series

In Problems 1 and 2, use Table 5.3.1 in Section 5.3.

1. Find the first four $\alpha_i > 0$ defined by $J_1(3\alpha) = 0$.
2. Find the first four $\alpha_i \leq 0$ defined by $J'_0(2\alpha) = 0$

In Problems 3–6, expand $f(x) = 1$, $0 < x < 2$, in a Fourier–Bessel series using Bessel functions of order zero that satisfy the given boundary condition.

3. $J_0(2\alpha) = 0$
4. $J'_0(2\alpha) = 0$
5. $J_0(2\alpha) + 2\alpha J'_0(2\alpha) = 0$
6. $J_0(2\alpha) + \alpha J'_0(2\alpha) = 0$

In Problems 7–10, expand the given function in a Fourier–Bessel series using Bessel functions of the same order as in the indicated boundary condition.

7. $f(x) = 5x,\ 0 < x < 4$
   $3J_1(4\alpha) + 4\alpha J'_1(4\alpha) = 0$
8. $f(x) = x^2,\ 0 < x < 1$
   $J_2(\alpha) = 0$
9. $f(x) = x^2,\ 0 < x < 3$
   $J'_0(3\alpha) = 0$
   [*Hint*: $t^3 = t^2 \cdot t$.]
10. $f(x) = 1 - x^2,\ 0 < x < 1$
    $J_0(\alpha) = 0$

Computer Lab Assignments

11. **(a)** Use a CAS to graph $y = 3J_1(x) + xJ'_1(x)$ on an interval so that the first five positive x-intercepts of the graph are shown.

(b) Use the root-finding capability of your CAS to approximate the first five roots x_i of the equation

$$3J_1(x) + xJ_1'(x) = 0.$$

(c) Use the data obtained in part (b) to find the first five positive values of ai that satisfy

$$3J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0.$$

See Problem 7.

(d) If instructed, find the first 10 positive values of α_i.

12. **(a)** Use the values of α_i in part (c) of Problem 11 and a CAS to approximate the values of the first five coefficients c_i of the Fourier–Bessel series obtained in Problem 7.

(b) Use a CAS to graph the partial sums $S_N(x)$, $N = 1, 2, 3, 4, 5$, of the Fourier–Bessel series in Problem 7.

(c) If instructed, graph the partial sum $S_{10}(x)$ for $0 < x < 4$ and for $0 < x < 50$.

Discussion Problems

13. If the partial sums in Problem 12 are plotted on a symmetric interval such as (–30, 30), would the graphs possess any symmetry? Explain.

14. **(a)** Sketch, by hand, a graph of what you think the Fourier–Bessel series in Problem 3 converges to on the interval (–2, 2).

(b) Sketch, by hand, a graph of what you think the Fourier–Bessel series would converge to on the interval (–4, 4) if the values a i in Problem 7 were defined by $3J_2(4\alpha) + 4\alpha J'_2(4\alpha) = 0$.

12.6.2 Fourier–Legendre Series

In Problems 15 and 16, write out the first five nonzero terms in the Fourier–Legendre expansion of the given function. If instructed, use a CAS as an aid in evaluating the coefficients. Use a CAS to graph the partial sum $S_5(x)$.

15. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

16. $f(x) = e^x$, $-1 < x < 1$

17. The first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. If $x = \cos\theta$, then $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$. Show that $P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$.

18. Use the results of Problem 17 to find a Fourier–Legendre expansion (23) of $F(\theta) = 1 - \cos 2\theta$.

19. A Legendre polynomial $P_n(x)$ is an even or odd function, depending on whether n is even or odd. Show that if f is an even function on the interval (–1, 1), then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n} P_{2n}(x) \qquad (25)$$

$$c_{2n} = (4n + 1) \int_0^1 f(x) P_{2n}(x)\, dx. \qquad (26)$$

20. Show that if f is an odd function on the interval (–1, 1), then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x) \qquad (27)$$

$$c_{2n+1} = (4n + 3) \int_0^1 f(x) P_{2n+1}(x)\, dx. \qquad (28)$$

The series (25) and (27) can also be used when f is defined on only the interval (0, 1). Both series represent f on (0, 1); but on the interval (–1, 0), (25) represents an even extension, whereas (27) represents an odd extension. In Problems 21 and 22, write out the first four nonzero terms in the indicated expansion of the given function. What function does the series represent on the interval (–1, 1)? Use a CAS to graph the partial sum $S_4(x)$.

21. $f(x) = x$, $0 < x < 1$; (25)

22. $f(x) = 1$, $0 < x < 1$; (27)

Discussion Problems

23. Why is a Fourier–Legendre expansion of a polynomial function that is defined on the interval (–1, 1) necessarily a finite series?

24. Use your conclusion from Problem 23 to find the finite Fourier–Legendre series of $f(x) = x^2$. The series of $f(x) = x^3$. Do *not* use (21) and (22).

12 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1–10, fill in the blank or answer true/false without referring back to the text.

1. The functions $f(x) = x^2 - 1$ and $g(x) = x^5$ are orthogonal on the interval $[-\pi, \pi]$. ________
2. The product of an odd function f with an odd function g is an _______ function.
3. To expand $f(x) = |x| + 1$, $-\pi < x < \pi$, in an appropriate trigonometric series we would use a _____ series.
4. $y = 0$ is never an eigenfunction of a Sturm–Liouville problem._
5. $\lambda = 0$ is never an eigenvalue of a Sturm–Liouville problem. ______________
6. If the function

$$f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ -x, & 0 < x < 1 \end{cases}$$

is expanded in a Fourier series, the series will converge to _______ at $x = -1$, to _______ at $x = 0$, and to _______ at $x = 1$.

7. Suppose the function $f(x) = x^2 + 1$, $0 < x < 3$, is expanded in a Fourier series, a cosine series, and a sine series. At $x = 0$, the Fourier series will converge to ___________, the cosine series