Bessel and Legendre series

Fourier - Bessel Series

Df: the Fourier - Bessel series of f defined on $(0,b)$ is given by

(i) $\boxed{f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)}$, where

$$\boxed{c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\,dx}$$

where α_i are defined by $\boxed{J_n(\alpha b) = 0}$.

(ii) $\boxed{f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x)}$

$$\boxed{c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\,dx}$$

where α_i are defined by

$$\boxed{h J_n(\alpha b) + \alpha b J_n'(\alpha b) = 0}$$

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(iii) $$\boxed{f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x)}$$

$$\boxed{c_1 = \frac{2}{b^2}\int_0^b x f(x)\,dx}$$

$$\boxed{c_i = \frac{2}{b^2 J_0^2(\alpha_i b)}\int_0^b x J_0(\alpha_i x) f(x)\,dx}$$

where α_i are defined by $\boxed{J_0'(\alpha b) = 0}$

Convergence of a Fourier-Bessel Series

Thm. Let f & f' be piecewise cont- on $[0,b]$. Then for all $x \in (0,b)$, the Fourier-Bessel series of f converges to $f(x)$ at a point where f is continuous and to $\dfrac{f(x^+) + f(x^-)}{2}$ at a point where f is discontinuous.

Ex. Expand $f(x) = x$, $0 < x < 3$ in a Fourier-Bessel series, using Bessel function of order one that satisfy the boundary condition $J_1(3\alpha) = 0$.

Sol. we use (i) in the last def'n.

with $\boxed{b = 3}$, $\boxed{n = 1}$

$$f(x) = \sum_{i=1}^{\infty} c_i \, J_1(\alpha_i x) \quad \text{, where}$$

$$c_i = \frac{2}{9 J_2^2(3\alpha_i)} \int_0^3 x \, J_1(\alpha_i x) \cdot x \, dx .$$

$$= \frac{2}{9 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x) \, dx .$$

let $\boxed{t = \alpha_i x} \Rightarrow dt = \alpha_i \, dx$

or $\boxed{dx = \frac{dt}{\alpha_i}}$, $\boxed{\begin{array}{l} x = 0 \Rightarrow t = 0 \\ x = 3 \Rightarrow t = 3\alpha_i \end{array}}$

and use $\frac{d}{dt}[t^2 J_2(t)] = t^2 J_1(t)$

$$= \frac{2}{9 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} \left(\frac{t}{\alpha_i}\right)^2 J_1(t) \, \frac{dt}{\alpha_i}$$

$$= \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \int_0^{3\alpha_i} t^2 J_1(t)\, dt$$

$$= \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \, t^2 J_2(t) \Big|_0^{3\alpha_i}$$

$$= \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \left[9\alpha_i^2 J_2(3\alpha_i) - 0 \right]$$

$$\boxed{c_i = \frac{2}{\alpha_i J_2(3\alpha_i)}}$$

$$\therefore f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$$

$$= 2 \sum_{i=1}^{\infty} \frac{J_1(\alpha_i x)}{\alpha_i J_2(3\alpha_i)}$$

<u>Ex.</u> If α_i in the last example are defined by $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$, then we multiply both sides by 3,

$$\boxed{3 J_1(3\alpha) + 3\alpha J_1'(3\alpha) = 0}$$

Now, we use (ii) in the def'n with $\boxed{h=3}$, $\boxed{b=3}$, $\boxed{n=1}$. Thus

$$c_i = \frac{2\alpha_i^2}{(9\alpha_i^2 - 1^2 + 3^2)\, J_1^2(3\alpha_i)} \int_0^3 x\, J_1(\alpha_i x) \cdot x \, dx$$

$$= \frac{2\alpha_i^2}{(9\alpha_i^2 + 8)\, J_1^2(3\alpha_i)} \int_0^3 \underbrace{x^2 J_1(\alpha_i x)\, dx}_{\substack{\text{let } t = \alpha_i x \\ dt = \alpha_i dx}}$$

$$= \frac{2\cancel{\alpha_i^2}}{(9\alpha_i^2 + 8)\, J_1^2(3\alpha_i)} \int_0^{3\alpha_i} \frac{t^2}{\cancel{\alpha_i^2}} J_1(t) \frac{dt}{\alpha_i}$$

$$= \frac{2}{\alpha_i (9\alpha_i^2 + 8)\, J_1^2(3\alpha_i)} \int_0^{3\alpha_i} t^2 J_1(t)\, dt$$

$$= \frac{2}{\alpha_i (9\alpha_i^2 + 8)\, J_1^2(3\alpha_i)}\, t^2 J_2(t) \Big|_0^{3\alpha_i}$$

$$= \frac{2 (9\alpha_i^2 J_2(3\alpha_i))}{\alpha_i (9\alpha_i^2 + 8)\, J_1^2(3\alpha_i)}$$

$$= \frac{18\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2+8) J_1^2(3\alpha_i)}$$

$$\therefore f(x) = \sum_{i=1}^{\infty} c_i J_1(\alpha_i x)$$

$$= 18 \sum_{i=1}^{\infty} \frac{\alpha_i J_2(3\alpha_i)}{(9\alpha_i^2+8) J_1^2(3\alpha_i)} J_1(\alpha_i x).$$

Fourier-Legendre Series:

Df: the Fourier-Legendre series of a function f defined on $(-1,1)$ is given by

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n P_n(x)}, \text{ where}$$

$$\boxed{c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx},$$

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left((x^2-1)^n\right), \quad n=0,1,2,\ldots}$$

- Convergence of a Fourier-Legendre series.

Thm Let f & f' be piece-wise cont. on $[-1,1]$. Then, $\forall x \in (-1,1)$, the Fourier-Legendre series of f converges to $f(x)$ at a point where f is continuous and to the average $\frac{f(x^+) + f(x^-)}{2}$ at a point where f is discontinuous.

ex. Find the first three non-zero terms in the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}.$$

Sol. we have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

So,

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

$$= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \cdots$$

where

$$C_0 = \frac{1}{2}\int_{-1}^{1} f(x)\, P_0(x)\, dx = \frac{1}{2}\int_{0}^{1} x\, dx = \boxed{\frac{1}{4}}.$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} f(x)\, P_1(x)\, dx = \frac{3}{2}\int_{0}^{1} x \cdot x\, dx = \boxed{\frac{1}{2}}.$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} f(x)\, P_2(x)\, dx$$

$$= \frac{5}{2}\int_{0}^{1} x \cdot \frac{1}{2}(3x^2-1)\, dx$$

$$= \frac{5}{4}\int_{0}^{1} (3x^3 - x)\, dx = \frac{5}{4}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right|_0^1$$

$$= \boxed{\frac{5}{16}}$$

that is $f(x) \sim C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x) + \cdots$

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$$= \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) + \cdots$$

$$= \frac{1}{4}\cdot 1 + \frac{1}{2}\cdot x + \frac{5}{16}\cdot \frac{1}{2}(3x^2-1) + \cdots$$

Alternative Form of Series

If we let $x = \cos\theta$, then $x = 1 \Rightarrow \theta = 0$.
$x = -1 \Rightarrow \theta = \pi$.

Since $dx = -\sin\theta \, d\theta$, then Df page 6 becomes:

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta) \quad , \text{where}$$

$$c_n = \frac{2n+1}{2} \int_0^{\pi} F(\theta) P_n(\cos\theta) \sin\theta \, d\theta ,$$

where $F(\theta) = f(\cos\theta)$.

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(x) = \frac{1}{2}(3\cos^2\theta - 1)$, ----

H.W. Write the first three nonzero terms of the Fourier Legendre Series of

$$f(x) = |x| , \quad -1 < x < 1.$$

(Simplify your final answer as a polynomial of degree 4)

Ans. $f(x) = -\frac{105}{128}x^4 + \frac{105}{64}x^2 + \frac{15}{128} + \cdots$