CHAPTER 14

CHAPTER CONTENTS

In the previous chapter we utilized Fourier series to solve boundary-value problems that were described in the Cartesian or rectangular coordinate system. In this chapter we will finally put to practical use the theory of Fourier–Bessel series (Section 14.2) and Fourier–Legendre series (Section 14.3) in the solution of boundary-value problems described, respectively in **cylindrical coordinates** and in **spherical coordinates**.

14.1 Problems in Polar Coordinates

Introduction All the boundary-value problems that have been considered so far have been expressed in terms of rectangular coordinates. If, however, we wish to find temperatures in a circular disk, a circular cylinder, or in a sphere, we would naturally try to describe the problems in polar coordinates, cylindrical coordinates, or spherical coordinates, respectively.

Because we consider only steady-state temperature problems in polar coordinates in this section, the first thing that must be done is to convert the familiar Laplace's equation in rectangular coordinates to polar coordinates.

Laplacian in Polar Coordinates The relationships between polar coordinates in the plane and rectangular coordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta \qquad \text{and} \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

See **FIGURE 14.1.1** The first pair of equations transform polar coordinates (r, θ) into rectangular coordinates (x, y); the second pair of equations enable us to transform rectangular coordinates into polar coordinates. These equations also make it possible to convert the two-dimensional Laplacian of the function u, $\nabla^2 u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, into polar coordinates. You are encouraged to work through the details of the Chain Rule and show that

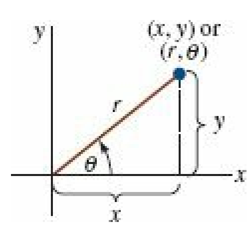

FIGURE 14.1.1 Polar coordinates of a point (x, y) are (r, θ)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \qquad (1)$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}. \qquad (2)$$

Adding (1) and (2) and simplifying yields the Laplacian of u in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

In this section we shall concentrate only on boundary-value problems involving Laplace's equation in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \qquad (3)$$

Our first example is the Dirichlet problem for a circular disk. We wish to solve Laplace's equation (3) for the steady-state temperature $u(r, \theta)$ in a circular disk or plate of radius c when the temperature of the circumference is $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$. See **FIGURE 14.1.2**. It is assumed that the two faces of the plate are insulated. This seemingly simple problem is unlike any we have encountered in the previous chapter.

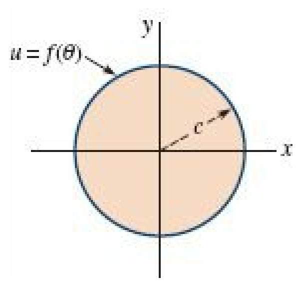

FIGURE 14.1.2 Dirichlet problem for a circular plate

EXAMPLE 1 Steady Temperatures in a Circular Plate

Solve Laplace's equation (3) subject to $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$.

SOLUTION Before attempting separation of variables we note that the single boundary condition is nonhomogeneous. In other words, there are no explicit conditions in the statement of the problem that enable us to determine either the coefficients in the solutions of the separated ODEs or the required eigenvalues. However, there are some implicit conditions.

First, our physical intuition leads us to expect that the temperature $u(r, \theta)$ should be continuous and therefore bounded inside the circle $r = c$. In addition, the temperature $u(r, \theta)$ should be single-valued; this means that the value of u should be the same at a specified point in the plate regardless of the polar description of that point. Since $(r, \theta + 2\pi)$ is an equivalent description of the point (r, θ), we must have $u(r, \theta) = u(r, \theta + 2\pi)$. That is, $u(r, \theta)$ must be periodic in θ with period 2π. If we seek a product solution $u = R(r)\Theta(\theta)$, then $\Theta(\theta)$ needs to be 2π-periodic.

With all this in mind, we choose to write the separation constant in the separation of variables as λ:

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

The separated equations are then

$$r^2R'' + rR' - \lambda R = 0 \tag{4}$$

$$\Theta'' + \lambda\Theta = 0 \tag{5}$$

We are seeking a solution of the problem

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi). \tag{6}$$

Although (6) is not a regular Sturm–Liouville problem, nonetheless the problem generates eigenvalues and eigenfunctions. The latter form an orthogonal set on the interval $[0, 2\pi]$. Of the three possible general solutions of (5),

$$\Theta(\theta) = c_1 + c_2\theta, \qquad \lambda = 0 \tag{7}$$

$$\Theta(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta, \qquad \lambda = -\alpha^2 < 0 \tag{8}$$

$$\Theta(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta, \qquad \lambda = \alpha^2 > 0 \tag{9}$$

we can dismiss (8) as inherently non-periodic unless $c_1 = c_2 = 0$. Similarly, solution (7) is non-periodic unless we define $c_2 = 0$. The remaining constant solution $\Theta(\theta) = c_1$, $c_1 \neq 0$, can be assigned any period and so $\lambda = 0$ is an eigenvalue. Finally, solution (9) will be 2π-periodic if we take $\alpha = n$, where $n = 1, 2, \ldots$. * The eigenvalues of (6) are then $\lambda_0 = 0$ and $\lambda_n = n^2$, $n = 1, 2, \ldots$. If we correspond $\lambda_0 = 0$ with $n = 0$, the eigenfunctions of (6) are

$$\Theta(\theta) = c_1,\ n = 0, \qquad \text{and} \qquad \Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta,\ n = 1, 2, \ldots.$$

When $\lambda_n = n^2$, $n = 0, 1, 2, \ldots$ the solutions of the Cauchy–Euler DE (4) are

$$R(r) = c_3 + c_4 \ln r,\ n = 0, \tag{10}$$

$$R(r) = c_3 r^n + c_4 r^{-n},\ n = 1, 2, \ldots. \tag{11}$$

Now observe in (11) that $r^{-n} = 1/r^n$. In either of the solutions (10) or (11), we must define $c_4 = 0$ in order to guarantee that the solution u is bounded at the center of the plate (which is $r = 0$). Thus product solutions $u_n = R(r)\Theta(\theta)$ for Laplace's equation in polar coordinates are

$$u_0 = A_0,\ n = 0, \qquad \text{and} \qquad u_n = r^n(A_n \cos n\theta + B_n \sin n\theta),\ n = 1, 2, \ldots,$$

where we have replaced c_3c_1 by A_0 for $n = 0$ and by A_n for $n = 1, 2, \ldots$; the product c_3c_2 has been replaced by B_n. The superposition principle then gives

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta). \tag{12}$$

By applying the boundary condition at $r = c$ to the result in (12) we recognize

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n(A_n \cos n\theta + B_n \sin n\theta)$$

For example, note that $\cos n(\theta + 2\pi) = \cos(n\theta + 2n\pi) = \cos n\theta$.

as an expansion of f in a full Fourier series. Consequently we can make the identifications

$$A_0 = \frac{a_0}{2}, \quad c^n A_n = a_n, \quad \text{and} \quad c^n B_n = b_n.$$

That is,

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta \tag{13}$$

$$A_n = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \tag{14}$$

$$B_n = \frac{1}{c^n\pi}\int_0^{2\pi} f(\theta) \sin n\theta\, d\theta. \tag{15}$$

The solution of the problem consists of the series given in (12), where the coefficients A_0, A_n, and B_n are defined in (13), (14), and (15), respectively. ≡

Observe in Example 1 that corresponding to each *positive* eigenvalue, $\lambda_n = n^2$, $n = 1, 2, \ldots$, there are two different eigenfunctions—namely, $\cos n\theta$ and $\sin n\theta$. In this situation the eigenvalues are sometimes called **double eigenvalues**.

EXAMPLE 2 Steady Temperatures in a Semicircular Plate

Find the steady-state temperature $u(r, \theta)$ in the semicircular plate shown in **FIGURE 14.1.3**.

SOLUTION The boundary-value problem is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \; 0 < \theta < \pi, 0 < r < c$$

$$u(c, \theta) = u_0, \; 0 < \theta < \pi$$

$$u(r, 0) = 0, \quad u(r, \pi) = 0, \; 0 < r < c.$$

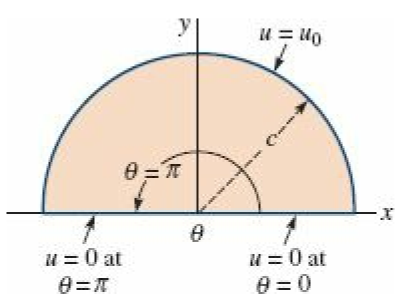

FIGURE 14.1.3 Semicircular plate in Example 2

Defining $u = R(r)\Theta(\theta)$ and separating variables gives

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

and

$$r^2R'' + rR' - \lambda R = 0 \tag{16}$$

$$\Theta'' + \lambda\Theta = 0. \tag{17}$$

The homogeneous conditions stipulated at the boundaries $\theta = 0$ and $\theta = \pi$ translate into $\Theta(0) = 0$ and $\Theta(\pi) = 0$. These conditions together with equation (17) constitute a regular Sturm–Liouville problem:

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\pi) = 0. \tag{18}$$

This familiar problem* possesses eigenvalues $\lambda_n = n^2$ and eigenfunctions $\Theta(\theta) = c_2 \sin n\theta$, $n = 1, 2, \ldots$ Also, by replacing λ by n^2 the solution of (16) is $R(r) = c_3 r^n + c_4 r^{-n}$. The reasoning used in Example 1; namely, that we expect a solution u of the problem to be bounded at $r = 0$, prompts us to define $c_4 = 0$. Therefore $u_n = R(r)\Theta(\theta) = A_n r^n \sin n\theta$ and

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

The remaining boundary condition at $r = c$ gives the Fourier sine series

$$u_0 = \sum_{n=1}^{\infty} A_n c^n \sin n\theta.$$

Consequently
$$A_n c^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta \, d\theta,$$

and so
$$A_n = \frac{2u_0}{\pi c^n} \frac{1 - (-1)^n}{n}.$$

Hence the solution of the problem is given by

$$u(r, \theta) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n}\left(\frac{r}{c}\right)^n \sin n\theta.$$

14.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-33.

In Problems 1–4, find the steady-state temperature $u(r, \theta)$ in a circular plate of radius 1 if the temperature on the circumference is as given.

1. $u(1, \theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$

2. $u(1, \theta) = \begin{cases} \theta, & 0 < \theta < \pi \\ \pi - \theta, & \pi < \theta < 2\pi \end{cases}$

3. $u(1, \theta) = 2\pi\theta - \theta^2, 0 < \theta < 2\pi$

4. $u(1, \theta) = \theta, 0 < \theta < 2\pi$

5. If the boundaries $\theta = 0$ and $\theta = \pi$ of a semicircular plate of radius 2 are insulated, we then have

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=0} = 0, \quad \left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi} = 0, \quad 0 < r < 2.$$

Find the steady-state temperature $u(r, \theta)$ if

$$u(2, \theta) = \begin{cases} u_0, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi, \end{cases}$$

where u_0 is a constant.

6. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius 1 if the boundary-conditions are

$$u(1, \theta) = u_0, \quad 0 < \theta < \pi$$
$$u(r, 0) = 0, \quad u(r, \pi) = u_0, \quad 0 < r < 1,$$

where u_0 is a constant.

7. Find the steady-state temperature $u(r, \theta)$ in the plate in the shape of an annulus bounded between two concentric circles of radius a and b, $a < b$, shown in **FIGURE 14.1.4**. [*Hint:* Proceed as in Example 1.]

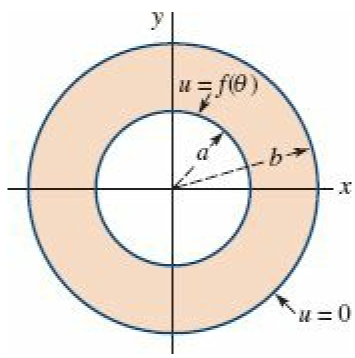

FIGURE 14.1.4 Annular plate in Problem 7

8. If the boundary-conditions for the annular plate in Figure 14.1.4 are

$$u(a, \theta) = u_0, \quad u(b, \theta) = u_1, \quad 0 < \theta < 2\pi,$$

where u_0 and u_1 are constants, show that the steady-state temperature is given by

$$u(r, \theta) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}.$$

[*Hint*: Try a solution of the form $u(r, \theta) = v(r, \theta) + \psi(r)$.]

9. Find the steady-state temperature $u(r, \theta)$ in the annular plate shown in Figure 14.1.4 if the boundary conditions are

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = 0, \quad u(b, \theta) = f(\theta), \quad 0 < \theta < 2\pi.$$

10. Find the steady-state temperature $u(r, \theta)$ in the annular plate shown in Figure 14.1.4 if $a = 1$, $b = 2$, and

$$u(1, \theta) = 75 \sin \theta, \quad u(2, \theta) = 60 \cos \theta, \quad 0 < \theta < 2\pi.$$

11. Find the steady-state temperature $u(r, \theta)$ in the semiannular plate shown in **FIGURE 14.1.5** if the boundary conditions are

$$u(a, \theta) = \theta(\pi - \theta), \quad u(b, \theta) = 0, \quad 0 < \theta < \pi$$
$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad a < r < b.$$

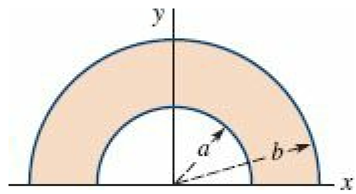

FIGURE 14.1.5 Semiannular plate in Problem 11

12. Find the steady-state temperature $u(r, \theta)$ in the semiannular plate shown in Figure 14.1.5 if $a = 1$, $b = 2$, and

$$u(1, \theta) = 0, \quad u(2, \theta) = u_0, \quad 0 < \theta < \pi$$
$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad 1 < r < 2,$$

where u_0 is a constant.

13. Find the steady-state temperature $u(r, \theta)$ in the quarter-circular plate shown in **FIGURE 14.1.6**.

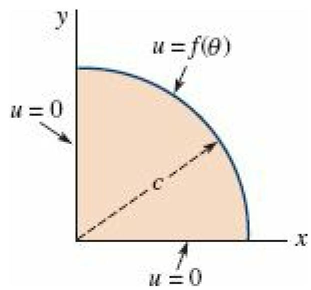

FIGURE 14.1.6 Quarter plate in Problem 13

14. Find the steady-state temperature $u(r, \theta)$ in the quarter-circular plate shown in Figure 14.1.6 if the boundaries $\theta = 0$ and $\theta = \pi/2$ are insulated, and

$$u(c, \theta) = \begin{cases} 1, & 0 < \theta < \pi/4 \\ 0, & \pi/4 < \theta < \pi. \end{cases}$$

15. Find the steady-state temperature $u(r, \theta)$ in the infinite wedge-shaped plate shown in **FIGURE 14.1.7**. [*Hint:* Assume that the temperature is bounded as $r \to 0$ and as $r \to \infty$.]

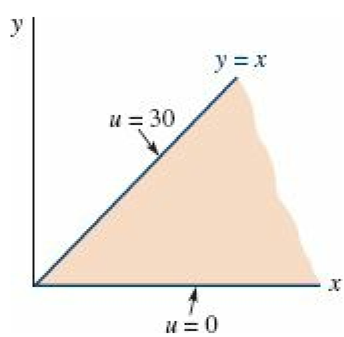

FIGURE 14.1.7 Wedge-shaped plate in Problem 15

16. The plate in the first quadrant shown in **FIGURE 14.1.8** is one-eighth of the annular plate in

Figure 14.1.4. Find the steady-steady temperature $u(r, \theta)$.

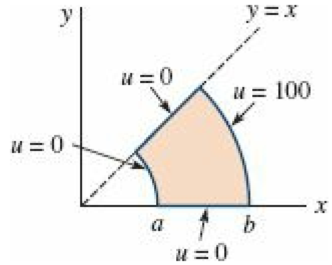

FIGURE 14.1.8 Plate in Problem 16

17. Solve the *exterior* Dirichlet problem for a circular disk of radius c shown in **FIGURE 14.1.9**. In other words, find the steady-state temperature $u(r, \theta)$ in a plate that coincides with the entire xy-plane in which a circular hole of radius c has been cut out around the origin and the temperature on the circumference of the hole is $f(\theta)$. [Hint: Assume that the temperature u is bounded as $r \to \infty$.]

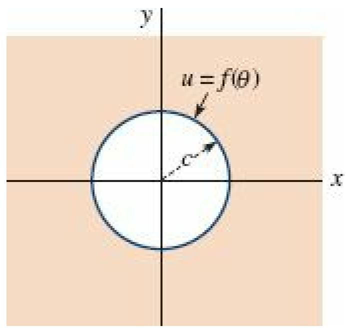

FIGURE 14.1.9 Infinite plate in Problem 17

18. Consider the steady-state temperature $u(r, \theta)$ in the semiannular plate shown in Figure 14.1.5 with $a = 1$, $b = 2$, and boundary conditions

$$u(1, \theta) = 0, \quad u(2, \theta) = 0, \quad 0 < \theta < \pi$$
$$u(r, 0) = 0, \quad u(r, \pi) = r, \quad 1 < r < 2.$$

Show that in this case the choice of $\lambda = -\alpha^2$ in (4) and (5) leads to eigenvalues and eigenfunctions. Find the steady-steady temperature $u(r, \theta)$.

≡ Computer Lab Assignment

19. **(a)** Find the series solution for $u(r, \theta)$ in Example 1 when

$$u(1, \theta) = \begin{cases} 100, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi. \end{cases}$$

See Problem 1.

(b) Use a CAS or a graphing utility to plot the partial sum $S_5(r, \theta)$ consisting of the first five nonzero terms of the solution in part (a) for $r = 0.9$, $r = 0.7$, $r = 0.5$, $r = 0.3$, $r = 0.1$. Superimpose the graphs on the same coordinate axes.

(c) Approximate the temperatures $u(0.9, 1.3)$, $u(0.7, 2)$, $u(0.5, 3.5)$, $u(0.3, 4)$, $u(0.1, 5.5)$. Then approximate $u(0.9, 2\pi - 1.3)$, $u(0.7, 2\pi - 2)$, $u(0.5, 2\pi - 3.5)$, $u(0.3, 2\pi - 4)$, $u(0.1, 2\pi - 5.5)$.

(d) What is the temperature at the center of the circular plate? Why is it appropriate to call this value the *average temperature* in the plate? [*Hint*: Look at the graphs in part (b) and look at the numbers in part (c).]

≡ Discussion Problems

20. Solve the Neumann problem for a circular plate:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < 2\pi, \quad 0 < r < c$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=c} = f(\theta), \quad 0 < \theta < 2\pi.$$

Give the compatibility condition. [Hint: See Problem 21 of Exercises 13.5.]

21. Consider the annular plate shown in Figure 14.1.4. Discuss how the steady-state temperature $u(r, \theta)$ can be found when the boundary conditions are

$$u(a, \theta) = f(\theta), \quad u(b, \theta) = g(\theta), \quad 0 \le \theta \le 2\pi.$$

14.2 Problems in Cylindrical Coordinates

≡ **Introduction** In this section we are going to consider boundary-value problems involving forms of the heat and wave equation in polar coordinates and a form of Laplace's equation in cylindrical coordinates. There is a commonality throughout the examples and most of the exercises—the boundary-value problem possesses radial symmetry.

Radial Symmetry The two-dimensional heat and wave equations

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \quad \text{and} \quad a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

expressed in polar coordinates are, in turn,

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t} \quad \text{and} \quad a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where $u = u(r, \theta, t)$. To solve a boundary-value problem involving either of these equations by separation of variables we must define $u = R(r)\Theta(\theta)T(t)$. As in Section 13.8, this assumption leads to multiple infinite series. See Problem 17 in Exercises 14.2. In the discussion that follows we shall consider the simpler, but still important, problems that possess **radial symmetry**—that is, problems in which the unknown function u is independent of the angular coordinate θ. In this case the heat and wave equations in (1) take, in turn, the forms

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t} \quad \text{and} \quad a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \tag{2}$$

where $u = u(r, t)$. Vibrations described by the second equation in (2) are said to be **radial vibrations**.

The first example deals with the free undamped radial vibrations of a thin circular membrane. We assume that the displacements are small and that the motion is such that each point on the membrane moves in a direction perpendicular to the *xy*-plane (transverse vibrations)—that is, the u-axis is perpendicular to the *xy*-plane. A physical model to keep in mind while studying this example is a vibrating drumhead.

EXAMPLE 1 Radial Vibrations of a Circular Membrane

Find the displacement $u(r, t)$ of a circular membrane of radius c clamped along its circumference if its initial displacement is $f(r)$ and its initial velocity is $g(r)$. See **FIGURE 14.2.1**.

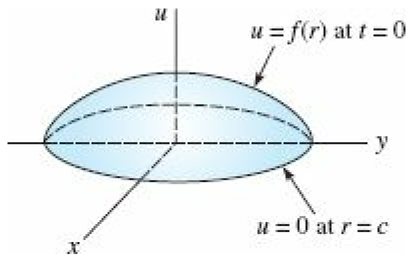

FIGURE 14.2.1 Initial displacement of circular membrane in Example 1

SOLUTION The boundary-value problem to be solved is

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, t > 0$$

$$u(c, t) = 0, \quad t > 0$$

$$u(r, 0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < c.$$

Substituting $u = R(r)T(t)$ into the partial differential equation and separating variables gives

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{a^2T} = -\lambda. \tag{3}$$

Note in (3) we have returned to our usual separation constant $-\lambda$. The two equations obtained from (3) are

$$rR'' + R' + \lambda rR = 0 \tag{4}$$

and

$$T'' + a^2\lambda T = 0. \tag{5}$$

Because of the vibrational nature of the problem, equation (5) suggests that we use only $\lambda = \alpha^2 > 0$, $\alpha > 0$. Now (4) is not a Cauchy–Euler equation but is the parametric Bessel differential equation of order $v = 0$; that is, $rR'' + R' + \alpha^2 rR = 0$. From (13) of Section 5.3 the general solution of the last equation is

$$R(r) = c_1J_0(\alpha r) + c_2Y_0(\alpha r). \tag{6}$$

The general solution of the familiar equation (5) is

$$T(t) = c_3 \cos a\alpha t + c_4 \sin a\alpha t$$

Now recall, the Bessel function of the second kind of order zero has the property that $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$, and so the implicit assumption that the displacement $u(r, t)$ should be bounded at $r = 0$ forces us to define $c_2 = 0$ in (6). Thus $R(r) = c_1 J_0(\alpha r)$.

Since the boundary condition $u(c, t) = 0$ is equivalent to $R(c) = 0$, we must have $c_1 J_0(\alpha c) = 0$. We rule out $c_1 = 0$ (this would lead to a trivial solution of the PDE), so consequently

$$J_0(\alpha c) = 0. \qquad (7)$$

If $x_n = \alpha_n c$ are the positive roots of (7), then $\alpha_n = x_n/c$ and so the eigenvalues of the problem are $\lambda_n = \alpha^2_n, = x^2_n/c^2$ and the eigenfunctions are $c_1 J_0(\alpha_n r)$. Product solutions that satisfy the partial differential equation and the boundary condition are

$$u_n = R(r)T(t) = (A_n \cos a\alpha_n t + B_n \sin a\alpha_n t)J_0(\alpha_n r), \qquad (8)$$

where we have done the usual relabeling of the constants. The superposition principle then gives

$$u(r, t) = \sum_{n=1}^{\infty}(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t)J_0(\alpha_n r). \qquad (9)$$

The given initial conditions determine the coefficients A_n and B_n.

Setting $t = 0$ in (9) and using $u(r, 0) = f(r)$ gives

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r). \qquad (10)$$

This last result is recognized as the Fourier–Bessel expansion of the function f on the interval (0, c). Hence by a direct comparison of (7) and (10) with (8) and (15) of Section 12.6 we can identify the coefficients A_n with those given in (16) of Section 12.6:

$$A_n = \frac{2}{c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) f(r)\, dr. \qquad (11)$$

Next, we differentiate (9) with respect to t, set $t = 0$, and use $u_t(r, 0) = g(r)$:

$$g(r) = \sum_{n=1}^{\infty} a\alpha_n B_n J_0(\alpha_n r).$$

This is now a Fourier–Bessel expansion of the function g. By identifying the total coefficient $a\alpha_n B_n$ with (16) of Section 12.6 we can write

$$B_n = \frac{2}{a\alpha_n c^2 J_1^2(\alpha_n c)} \int_0^c r J_0(\alpha_n r) g(r)\, dr. \qquad (12)$$

Finally, the solution of the given boundary-value problem is the series (9) with coefficients A_n and B_n defined in (11) and (12), respectively.

≡

 Standing Waves Analogous to (11) of Section 13.4, the product solutions (8) are called **standing waves**. For $n = 1, 2, 3, \ldots$, the standing waves are basically the graph of $J_0(\alpha_n r)$ with the time varying amplitude

$$A_n \cos a\alpha_n t + B_n \sin a\alpha_n t.$$

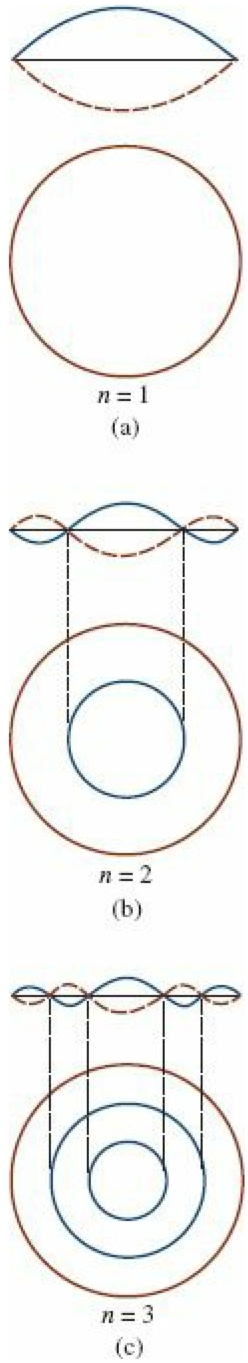

FIGURE 14.2.2 Standing waves

The standing waves at different values of time are represented by the dashed graphs in **FIGURE 14.2.2**. The zeros of each standing wave in the interval $(0, c)$ are the roots of $J_0(\alpha_n r) = 0$ and correspond to the set of points on a standing wave where there is no motion. This set of points is called a **nodal line**. If (as in Example 1) the positive roots of $J_0(\alpha_n c) = 0$ are denoted by x_n, then $x_n = \alpha_n c$ implies $\alpha_n = x_n/c$ and consequently the zeros of the standing waves are determined from

$$J_0(\alpha_n r) = J_0\left(\frac{x_n}{c}r\right) = 0.$$

Now from Table 5.3.1, the first three positive zeros of J_0 are (approximately) $x_1 = 2.4$, $x_2 = 5.5$, and $x_3 = 8.7$. Thus for $n = 1$, the first positive root of

$$J_0\left(\frac{x_1}{c}r\right) = 0 \quad \text{is} \quad \frac{2.4}{c}r = 2.4 \quad \text{or} \quad r = c.$$

Since we are seeking zeros of the standing waves in the open interval $(0, c)$, the last result means that the first standing wave has no nodal line. For $n = 2$, the first two positive roots of

$$J_0\left(\frac{x_2}{c}r\right) = 0 \qquad \text{are determined from} \qquad \frac{5.5}{c}r = 2.4 \quad \text{and} \quad \frac{5.5}{c}r = 5.5.$$

Thus the second standing wave has one nodal line defined by $r = x_1c/x_2 = 2.4c/5.5$. Note that $r \approx 0.44c < c$. For $n = 3$, a similar analysis shows that there are two nodal lines defined by $r = x_1c/x_3 = 2.4c/8.7$ and $r = x_2c/x_3 = 5.5c/8.7$. In general, the nth standing wave has $n - 1$ nodal lines $r = x_1c/x_n$, $r = x_2c/x_n, \ldots, r = x_{n-1}c/x_n$. Since $r = constant$ is an equation of a circle in polar coordinates, we see in Figure 14.2.2 that the nodal lines of a standing wave are concentric circles.

□ **Use of Computers** It is possible to see the effect of a single drumbeat for the model solved in Example 1 by means of the animation capabilities of a computer algebra system. In Problem 20 in Exercises 14.2 you are asked to find the solution given in (9) when

$$c = 1,\ f(r) = 0, \quad \text{and} \quad g(r) = \begin{cases} -v_0, & 0 \le r < b \\ 0, & b \le r < 1. \end{cases}$$

Some frames of a "movie" of the vibrating drumhead are given in **FIGURE 14.2.3**.

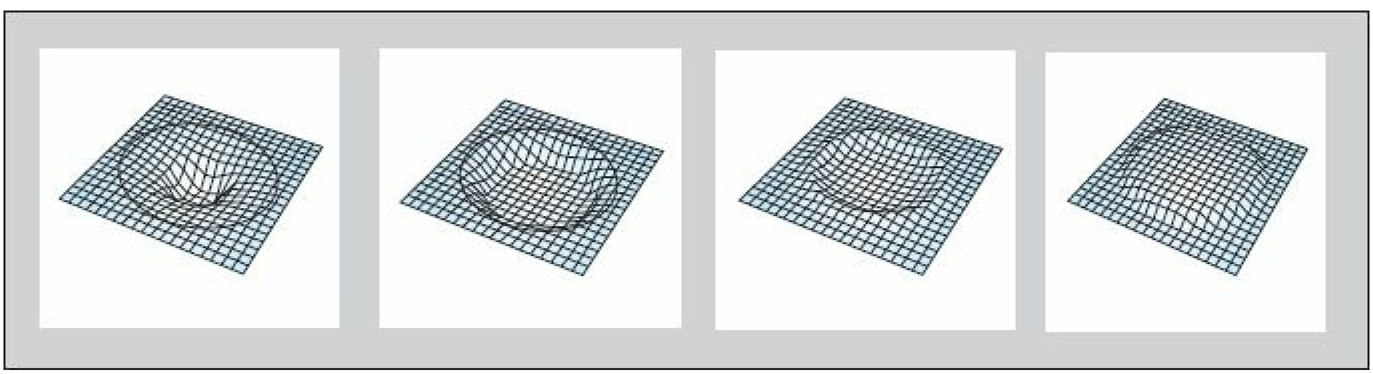

FIGURE 14.2.3 Frames of a CAS "movie"

□ **Laplacian in Cylindrical Coordinates** From **FIGURE 14.2.4** we can see that the relationship between the cylindrical coordinates of a point in space and its rectangular coordinates is given by

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z.$$

It follows immediately from the derivation of the Laplacian in polar coordinates (see Section 14.1) that the Laplacian of a function u in cylindrical coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

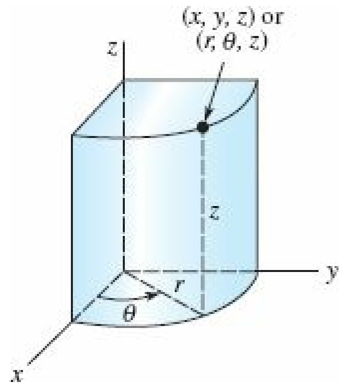

FIGURE 14.2.4 Cylindrical coordinates of a point (x, y, z) are (r, θ, z)

EXAMPLE 2 Steady Temperatures in a Circular Cylinder

Find the steady-state temperature in the circular cylinder shown in **FIGURE 14.2.5**.

SOLUTION The boundary conditions suggest that the temperature u has radial symmetry. Accordingly, $u(r, z)$ is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 2, 0 < z < 4$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2.$$

Using $u = R(r)Z(z)$ and separating variables gives

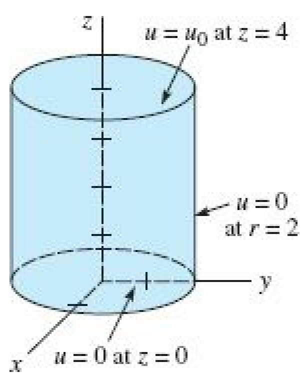

FIGURE 14.2.5 Finite cylinder in Example 2

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda. \tag{13}$$

and

$$rR'' + R' + \lambda rR = 0 \tag{14}$$

$$Z'' - \lambda Z = 0. \tag{15}$$

For the choice $\lambda = \alpha^2 > 0$, $\alpha > 0$, the general solution of (14) is

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

and since a solution of (15) is defined on the finite interval [0, 2], we write its general solution as

$$Z(z) = c_3 \cosh \alpha z + c_4 \sinh \alpha z.$$

As in Example 1, the assumption that the temperature u is bounded at $r = 0$ demands that $c_2 = 0$. The condition $u(2, z) = 0$ implies $R(2) = 0$. This equation,

$$J_0(2\alpha) = 0, \tag{16}$$

defines the positive eigenvalues $\lambda_n = \alpha_2^{\,n}$ of the problem. Last, $Z(0) = 0$ implies $c_3 = 0$. Hence we have $R = c_1 J_0(\alpha_n r)$, $Z) = c_4 \sinh \alpha_n z$,

$$u_n = R(r)Z(z) = A_n \sinh \alpha_n z J_0(\alpha_n r)$$

and

$$u(r, z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z J_0(\alpha_n r).$$

The remaining boundary condition at $z = 4$ then yields the Fourier–Bessel series

$$u_0 = \sum_{n=1}^{\infty} A_n \sinh 4\alpha_n J_0(\alpha_n r),$$

so that in view of (16) the coefficients are defined by (16) of Section 12.6,

$$A_n \sinh 4\alpha_n = \frac{2u_0}{2^2 J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r)\, dr.$$

To evaluate the last integral we first use the substitution $t = \alpha_n r$, followed by $\frac{d}{dt}[tJ_0(t)] = tJ_0(t)$. From

$$A_n \sinh 4\alpha_n = \frac{u_0}{2\alpha_n^2 J_1^2(2\alpha_n)} \int_0^{2\alpha_n} \frac{d}{dt}[tJ_1(t)]\, dt = \frac{u_0}{\alpha_n J_1(2\alpha_n)},$$

we obtain

$$A_n = \frac{u_0}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)}.$$

Finally, the temperature in the cylinder is

$$u(r, z) = u_0 \sum_{n=1}^{\infty} \frac{\sinh \alpha_n z}{\alpha_n \sinh 4\alpha_n J_1(2\alpha_n)} J_0(\alpha_n r).$$

≡

Do not conclude from two examples that every boundary-value problem in cylindrical coordinates gives rise to a Fourier–Bessel series.

EXAMPLE 3 Steady Temperatures in a Circular Cylinder

Find the steady-state temperatures $u(r, z)$ in the circular cylinder defined by $0 \le r \le 1$, $0 \le z \le 1$ if the boundary conditions are

$$u(1, z) = 1 - z, \; 0 < z < 1$$

$$u(r, 0) = 0, \; u(r, 1) = 0, \; 0 < r < 1.$$

SOLUTION Because of the nonhomogeneous condition specified at $r = 1$ we do not expect the eigenvalues of the problem to be defined in terms of zeros of a Bessel function of the first kind. As we did in Section 14.1 it is convenient in this problem to use λ as the separation constant. Thus from (13) of Example 2 we see that separation of variables now gives the two ordinary differential equations

$$rR'' + R' - \lambda rR = 0 \text{ and } Z'' + \lambda Z = 0.$$

You should verify that the two cases $\lambda = 0$ and $\lambda = -\alpha^2 < 0$ lead only to the trivial solution $u = 0$. In the case $\lambda = \alpha^2 > 0$ the DEs are

$$rR'' + R' - \alpha^2 rR = 0 \text{ and } Z'' + \alpha^2 Z = 0.$$

▶ Review pages 275-276 of Section 5.3. See also Figures 5.3.3 and 5.3.4.

The first equation is the parametric form of Bessel's modified DE of order $n = 0$. The solution of this equation is $R(r) = c_1 I_0(\alpha r) + c_2 K_0(\alpha r)$. We immediately define $c_2 = 0$ because the modified Bessel function of the second kind $K_0(\alpha r)$ is unbounded at $r = 0$. Therefore, $R(r) = c_1 I_0(\alpha r)$.

Now the eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$Z'' + \alpha^2 Z = 0, \; Z(0) = 0, \; Z(1) = 0$$

are $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$ and $Z(z) = c_3 \sin n\pi z$. Thus product solutions that satisfy the PDE and the homogeneous boundary conditions are

$$u_n = R(r)Z(z) = A_n I_0(n\pi r) \sin n\pi z.$$

Next we form

$$u(r, z) = \sum_{n=1}^{\infty} A_n I_0(n\pi r) \sin n\pi z.$$

The remaining condition at $r = 1$ yields the Fourier sine series

$$u(1, z) = 1 - z = \sum_{n=1}^{\infty} A_n I_0(n\pi) \sin n\pi z.$$

From (5) of Section 12.3 we can write

$$A_n I_0(n\pi) = 2\int_0^1 (1 - z) \sin n\pi z \, dz = \frac{2}{n\pi} \qquad \leftarrow \text{integration by parts}$$

and

$$A_n = \frac{2}{n\pi I_0(n\pi)}.$$

The steady-state temperature is then

$$u(r, z) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{I_0(n\pi r)}{nI_0(n\pi)}\sin n\pi z.$$

14.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-33.

1. Find the displacement $u(r, t)$ in Example 1 if $f(r) = 0$ and the circular membrane is given an initial unit velocity in the upward direction.

2. A circular membrane of radius 1 is clamped along its circumference. Find the displacement $u(r, t)$ if the membrane starts from rest from the initial displacement $f(r) = 1 - r^2$, $0 < r < 1$. [*Hint*: See Problem 10 in Exercises 12.6.]

3. Find the steady-state temperature $u(r, z)$ in the cylinder in Example 2 if the boundary conditions are $u(2, z) = 0$, $0 < z < 4$, $u(r, 0) = u_0$, $u(r, 4) = 0$, $0 < r < 2$.

4. If the lateral side of the cylinder in Example 2 is insulated, then

$$\left.\frac{\partial u}{\partial r}\right|_{r=2} = 0, \quad 0 < z < 4.$$

(a) Find the steady-state temperature $u(r, z)$ when $u(r, 4) = f(r)$, $0 < r < 2$.

(b) Show that the steady-state temperature in part (a) reduces to $u(r, z) = u_0 z/4$ when $f(r) = u_0$. [*Hint*: Use (12) of Section 12.6.]

In Problems 5–8, find the steady-state temperature $u(r, z)$ in a finite cylinder defined by $0 \le r \le 1$, $0 \le z \le 1$ if the boundary conditions are as given.

5. $u(1, z) = z, \quad 0 < z < 1$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \quad 0 < r < 1$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=1} = 0, \quad 0 < r < 1$$

6. $u(1, z) = z, \quad 0 < z < 1$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \quad 0 < r < 1$$

7. $u(1, z) = u_0, \quad 0 < z < 1$

$$u(r, 0) = 0, \quad 0 < r < 1$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=1} = 0, \quad 0 < r < 1$$

8. $u(1, z) = 0, \quad 0 < z < 1$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \quad 0 < r < 1$$

$u(r, 1) = u_0, \quad 0 < r < 1$

9. The temperature in a circular plate of radius c is determined from the boundary-value problem

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \quad 0 < r < c, t > 0$$

$$u(c, t) = 0, \quad t > 0$$

$$u(r, 0) = f(r), \quad 0 < r < c.$$

Solve for $u(r, t)$.

10. Solve Problem 9 if the edge $r = c$ of the plate is insulated.

11. When there is heat transfer from the lateral side of an infinite circular cylinder of radius 1 (see **FIGURE 14.2.6**) into a surrounding medium at temperature zero, the temperature inside the cylinder is determined from

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \quad 0 < r < 1, t > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = -hu(1, t), \quad h > 0, t > 0$$

$$u(r, 0) = f(r), \quad 0 < r < 1.$$

Solve for $u(r, t)$.

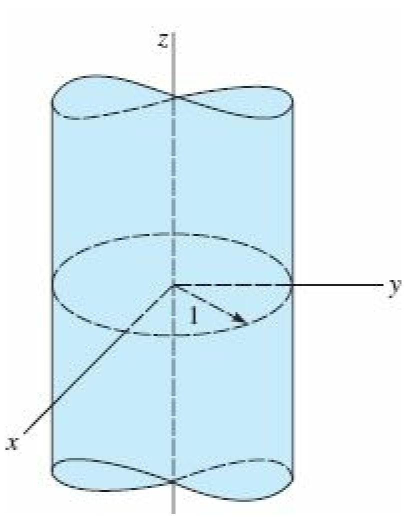

FIGURE 14.2.6 Infinite cylinder in Problem 11

12. Find the steady-state temperature $u(r, z)$ in a semi-infinite cylinder of radius 1 ($z \geq 0$) if there is heat transfer from its lateral side into a surrounding medium at temperature zero and if the temperature of the base $z = 0$ is held at a constant temperature u_0.

In Problems 13 and 14, use the substitution $u(r, t) = v(r, t) + \psi(r)$ to solve the given boundary-value problem. [*Hint*: Review Section 13.6.]

13. A circular plate is a composite of two different materials in the form of concentric circles. See **FIGURE 14.2.7** The temperature $u(r, t)$ in the plate is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 2, t > 0$$

$$u(2, t) = 100, \quad t > 0$$

$$u(r, 0) = \begin{cases} 200, & 0 < r < 1 \\ 100, & 1 < r < 2. \end{cases}$$

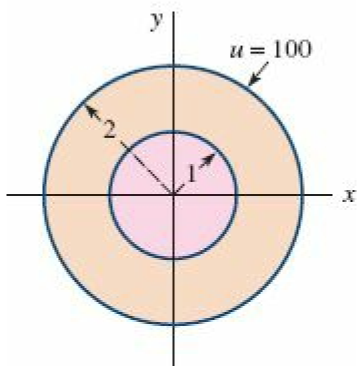

FIGURE 14.2.7 Circular plate in Problem 13

14. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \beta = \frac{\partial u}{\partial t}$, $0 < r < 1$, $t > 0$, β constant
$u(1, t) = 0$, $t > 0$
$u(r, 0) = 0$, $0 < r < 1$.

15. The horizontal displacement $u(x, t)$ of a heavy chain of length L oscillating in a vertical plane satisfies the partial differential equation

$$g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, t > 0.$$

See **FIGURE 14.2.8**.

(a) Using $-\lambda$ as a separation constant, show that the ordinary differential equation in the spatial variable x is $xX'' + X' + \lambda X = 0$. Solve this equation by means of the substitution $x = \tau^2/4$.

(b) Use the result of part (a) to solve the given partial differential equation subject to

$$u(L, t) = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L.$$

[*Hint*: Assume the oscillations at the free end $x = 0$ are finite.]

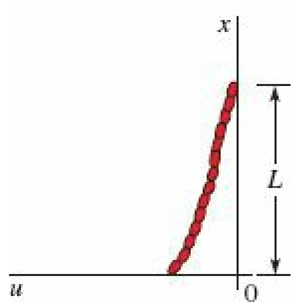

FIGURE 14.2.8 Oscillating chain in Problem 15

16. Consider the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1, t > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 1, \quad t > 0$$

$$u(r, 0) = 0, \quad 0 < r < 1.$$

(a) Use the substitution $u(r, t) = \mathrm{v}(r, t) + Bt$ in the preceding problem to show that $v(r, t)$ satisfies

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{\partial v}{\partial t} + B, \quad 0 < r < 1, t > 0$$

$$\left.\frac{\partial v}{\partial r}\right|_{r=1} = 1, \quad t > 0$$

$$v(r, 0) = 0, \quad 0 < r < 1.$$

Here B is a constant to be determined.

(b) Now use the substitution $v(r, t) = w(r, t) + \psi(r)$ to solve the boundary-value problem in part (a). [*Hint*: You may need to review Section 3.5.]

(c) What is the solution $u(r, t)$ of the first problem?

17. In this problem we consider the general case—that is, with u dependence—of the vibrating circular membrane of radius c:

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, t > 0$$

$$u(c, \theta, t) = 0, \quad 0 < \theta < 2\pi, t > 0$$

$$u(r, \theta, 0) = f(r, \theta), \quad 0 < r < c, 0 < \theta < 2\pi$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r, \theta), \quad 0 < r < c, 0 < \theta < 2\pi.$$

(a) Assume that $u = R(r)\Theta(u)T(t)$ and the separation constants are $-\lambda$ and $-v$. Show that the separated differential equations are

$$T'' + a_2\lambda T = 0, \ \Theta'' + v\Theta = 0$$

$$r_2R_2'' + rR' + (\lambda r_2 - v)R = 0.$$

(b) Let $\lambda = \alpha^2$ and $v = \beta_2$ and solve the separated equations in part (a).

(c) Show that the eigenvalues and eigenfunctions of the problem are as follows:

Eigenvalues: $v = n$, $n = 0, 1, 2, \ldots$
eigenfunctions: 1, $\cos n\,\theta$, $\sin n\,\theta$.
Eigenvalues: $\lambda_{ni} = x_{ni}/c$, $i = 1, 2, \ldots$, where, for each n, xni are the positive roots of $Jn(\lambda c) = 0$; eigenfunctions: $J_n(\lambda_{ni}r) = 0$.

(d) Use the superposition principle to determine a multiple series solution. Do not attempt to evaluate the coefficients.

≡ Computer Lab Assignments

18. **(a)** Consider Example 1 with $a = 1$, $c = 10$, $g(r) = 0$, and $f(r) = 1 - r/10$, $0 < r < 10$. Use a CAS as an aid in finding the numerical values of the first three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the boundary-value problem and the first three coefficients $\lambda_1, \lambda_2, \lambda_3$ of the solution $u(r, t)$ given in (9). Write the third partial sum $S_3(r, t)$ of the series solution.

(b) Use a CAS to plot the graph of $S_3(r, t)$ for $t = 0, 4, 10, 12, 20$.

19. Solve Problem 9 with boundary conditions $u(c, t) = 200$, $u(r, 0) = 0$. With these imposed conditions, one would expect intuitively that at any interior point of the plate, $u(r, t) \to 200$ as $t \to \infty$. Assume that $c = 10$ and that the plate is cast iron so that $k = 0.1$ (approximately). Use a CAS as an aid in finding the numerical values of the first five eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ of the boundary-value problem and the five coefficients A_1, A_2, A_3, A_4, A_5 in the solution $u(r, t)$. Let the corresponding approximate solution be denoted by $S_5(r, t)$. Plot $S_5(5, t)$ and $S_5(0, t)$ on a sufficiently large time interval $[0, T]$. Use the plots of $S_5(5, t)$ and $S_5(0, t)$ to estimate the times (in seconds) for which $u(5, t) \approx 100$ and $u(0, t) \approx 100$. Repeat for $u(5, t) \approx 200$ and $u(0, t) \approx 200$.

20. Consider an idealized drum consisting of a thin membrane stretched over a circular frame of radius 1. When such a drum is struck at its center, one hears a sound that is frequently described as a dull thud rather than a melodic tone. We can model a single drumbeat using the boundary-value problem solved in Example 1.

(a) Find the solution $u(r, t)$ given in (9) when $c = 1$, $f(r) = 0$, and

$$g(r) = \begin{cases} -v_0, & 0 \le r < b \\ 0, & b \le r < 1. \end{cases}$$

(b) Show that the frequency of the standing wave $u_n(r, t)$ is $f_n = a\lambda_n/2\pi$, where λ_n is the nth positive zero of $J_0(x)$. Unlike the solution of the one-dimensional wave equation in Section 13.4, the frequencies are not integer multiples of the fundamental frequency f_1. Show that $f_2 = 2.295f_1$ and $f_3 = 3.598f_1$. We say that the drumbeat produces anharmonic overtones. As a result the displacement function $u(r, t)$ is not periodic, and so our ideal drum cannot produce a sustained tone.

(c) Let $a = 1$, $b = ¼$, and $v_0 = 1$ in your solution in part (a). Use a CAS to graph the fifth partial sum $S_5(r, t)$ at the times $t = 0, 0.1, 0.2, 0.3, \ldots, 5.9, 6.0$ on the interval [–1, 1]. Use the animation capabilities of your CAS to produce a movie of these vibrations.

(d) For a greater challenge, use the 3D plotting capabilities of your CAS to make a movie of the motion of the circular drumhead that is shown in cross section in part (c). [*Hint:* There are several ways of proceeding. For a fixed time, either graph u as a function of x and y using $r = \sqrt{x^2 + y^2}$ or use the equivalent of *Mathematica's* **RevolutionPlot3D**.]

14.3 Problems in Spherical Coordinates

Introduction In this section we continue our examination of boundary-value problems in different coordinate systems. This time we are going to consider problems involving the heat, wave, and Laplace's equation in spherical coordinates.

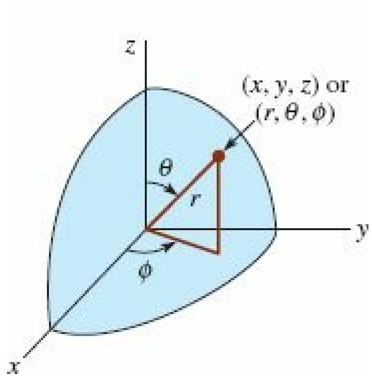

FIGURE 14.3.1 Spherical coordinates of a point (x, y, z) are (r, θ, ϕ)

Laplacian in Spherical Coordinates As shown in **FIGURE 14.3.1**, a point in 3-space is described in terms of rectangular coordinates and in spherical coordinates. The rectangular coordinates x, y, and z of the point are related to its spherical coordinates r, θ, and ϕ through the equations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta. \tag{1}$$

By using the equations in (1) it can be shown that the Laplacian $\nabla^2 u$ in the spherical coordinate system is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}. \tag{2}$$

As you might imagine, problems involving (1) can be quite formidable. Consequently we shall consider only a few of the simpler problems that are independent of the azimuthal angle ϕ.

Our first example is the Dirichlet problem for a sphere.

EXAMPLE 1 Steady Temperatures in a Sphere

Find the steady-state temperature $u(r, \theta)$ in the sphere shown in **FIGURE 14.3.2**.

SOLUTION The temperature is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0,\ \ 0 < r < c, 0 < \theta < \pi$$

$$u(c, \theta) = f(\theta),\ 0 < \theta < \pi.$$

If $u = R(r)\Theta(\theta)$, the partial differential equation separates as

$$\frac{r^2R'' + 2rR'}{R} = -\frac{\Theta'' + \cot\theta\,\Theta'}{\Theta} = \lambda,$$

and so

$$r^2R'' + 2rR' - \lambda R = 0 \qquad (3)$$

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta' + \lambda\sin\theta\,\Theta = 0. \qquad (4)$$

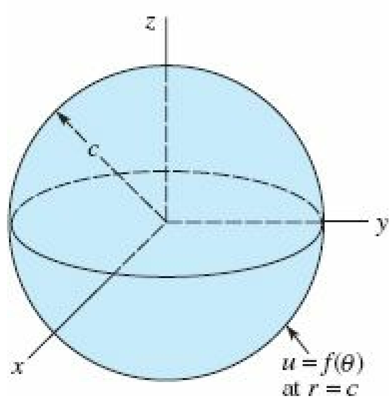

FIGURE 14.3.2 Dirichlet problem for a sphere in Example 1

After we substitute $x = \cos\theta$, $0 \le \theta \le \pi$, (4) becomes

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0, \quad -1 \le x \le 1. \qquad (5)$$

The latter equation is a form of Legendre's equation (see Problems 50 and 51 in Exercises 5.3). Now the only solutions of (5) that are continuous and have continuous derivatives on the closed interval [–1, 1] are the Legendre polynomials $P_n(x)$ corresponding to $\lambda = n(n + 1)$, $n = 0, 1, 2,\ldots$. Thus we take the solutions of (4) to be

$$\Theta(\theta) = P_n(\cos\theta).$$

Furthermore, when $\lambda = n(n + 1)$, the general solution of the Cauchy–Euler equation (3) is

$$R(r) = c_1 r^n + c_2 r_{-(n+1)}.$$

Since we again expect $u(r, \theta)$ to be bounded at $r = 0$, we define $c_2 = 0$. Hence $u_n = A_n r^n P_n(\cos\theta)$, and

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta).$$

At $r = c$,

$$f(\theta) = \sum_{n=0}^{\infty} A_n c^n P_n(\cos\theta).$$

Therefore $A_n c^n$ are the coefficients of the Fourier–Legendre series (23) of Section 12.6:

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta.$$

It follows that the solution is

$$u(r, \theta) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta\, d\theta\right)\left(\frac{r}{c}\right)^n P_n(\cos\theta).$$

14.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-33.

1. Solve the problem in Example 1 if

$$f(\theta) = \begin{cases} 50, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi. \end{cases}$$

Write out the first four nonzero terms of the series solution. [*Hint*: See Example 3, Section 12.6.]

2. The solution $u(r, \theta)$ in Example 1 could also be interpreted as the potential inside the sphere due to a charge distribution $f(\theta)$ on its surface. Find the potential outside the sphere.

3. Find the solution of the problem in Example 1 if $f(\theta) = \cos\theta$, $0 < \theta < \pi$. [*Hint*: $P_1(\cos\theta) = \cos\theta$. Use orthogonality.]

4. Find the solution of the problem in Example 1 if $f(\theta) = 1 - \cos 2\theta$, $0 < \theta < \pi$. [*Hint*: See Problem 18, Exercises 12.6.]

5. Find the steady-state temperature $u(r, \theta)$ within a hollow sphere $a < r < b$ if its inner surface $r = a$ is kept at temperature $f(\theta)$ and its outer surface $r = b$ is kept at temperature zero. The sphere in the first octant is shown in **FIGURE 14.3.3**.

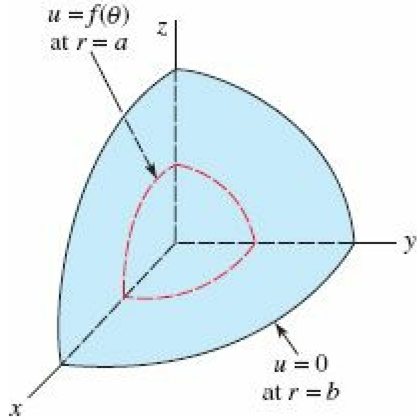

FIGURE 14.3.3 Hollow sphere in Problem 5

6. The steady-state temperature in a hemisphere of radius c is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, 0 < r < c, 0 < \theta < \pi/2$$

$$u(r, \pi/2) = 0, 0 < r < c$$

$$u(c, \theta) = f(\theta), 0 < \theta < \pi/2$$

Solve for $u(r, \theta)$. [Hint: $P_n(0) = 0$ only if n is odd. Also see Problem 20, Exercises 12.6.]

7. Solve Problem 6 when the base of the hemisphere is insulated; that is,

$$\left.\frac{\partial u}{\partial \theta}\right|_{\theta=\pi/2} = 0, \quad 0 < r < c.$$

8. Solve Problem 6 for $r > c$.

9. The time-dependent temperature within a sphere of radius 1 is determined from

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1, t > 0$$

$$u(1, t) = 100, t > 0$$

$$u(r, 0) = 0, 0 < r < 1.$$

Solve for $u(r, t)$. [*Hint*: Verify that the left side of the partial differential equation can be written as $\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$. Let $ru(r, t) = v(r, t) + \psi(r)$. Use only functions that are bounded as $r \to 0$.]

10. A uniform solid sphere of radius 1 at an initial constant temperature u_0 throughout is dropped into a large container of fluid that is kept at a constant temperature u_1 ($u_1 > u_0$) for all time. See FIGURE 14.3.4. Since there is heat transfer across the boundary $r = 1$, the temperature $u(r, t)$ in the sphere is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad 0 < r < 1, t > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = -h(u(1, t) - u_1), \quad 0 < h < 1$$

$$u(r, 0) = u_0, \quad 0 < r < 1.$$

Solve for $u(r, t)$. [Hint: Proceed as in Problem 9.]

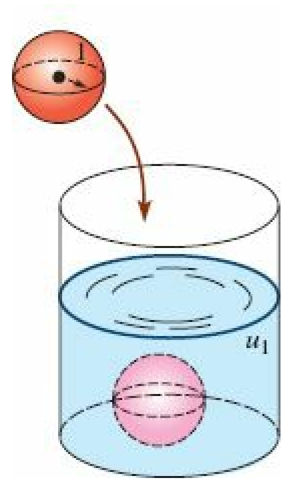

FIGURE 14.3.4 Container in Problem 10

11. Solve the boundary-value problem involving spherical vibrations:

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < c, t > 0$$

$$u(c, t) = 0, \quad t > 0$$

$$u(r, 0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < c.$$

[*Hint*: Write the left side of the partial differential equation as $a^2\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$. Let $v(r, t) = ru(r, t)$.]

12. A conducting sphere of radius c is grounded and placed in a uniform electric field that has intensity E in the z-direction. The potential $u(r, \theta)$ outside the sphere is determined from the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \quad r > c, 0 < \theta < \pi$$

$$u(c, \theta) = 0, \quad 0 < \theta < \pi$$

$$\lim_{r\to\infty} u(r, \theta) = -Ez = -Er\cos\theta.$$

Show that

$$u(r, \theta) = -Er\cos\theta + E\frac{c^3}{r^2}\cos\theta.$$

[*Hint*: Explain why $\int_0^\pi \cos\theta\, P_n(\cos\theta)\sin\theta\, d\theta = 0$ for all nonnegative integers except $n = 1$. See (24) of Section 12.6.]

In Problems 13 and 14, you are asked to find a product solution $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ of **Helmholtz's partial differential equation** $\nabla^2 u + k^2 u = 0$ where the Laplacian $\nabla^2 u$ is defined in (2).

13. **(a)** Proceed as in Example 1 but using $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ and the separation constant $n(n + 1)$ to show that the radial dependence of the solution u is defined by the equation

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0.$$

(b) Now use the second separation constant m^2 to show that the remaining separated equations are

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{d\Theta}{d\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0.$$

(c) Use the substitution $x = \cos\theta$ to show that the second differential equation in part (b) becomes

$$(1 - x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]\Theta = 0.$$

14. (a) Assume that m and n are nonnegative integers. Then find a product solution $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ of Helmholtz's PDE using the general solution of the ODE in part (a), the general solution of the first ODE in part (b), and a particular solution of the second ODE in part (b) of Problem 13. [*Hint:* See Problems 41, 42(c), and 52 in Exercises 5.3.]

(b) What product solution in part (a) would be bounded at the origin?

14 Chapter in Review

Answers to selected odd-numbered problems begin on page ANS-34.

In Problems 1 and 2, find the steady-state temperature $u(r, \theta)$ in a circular plate of radius c if the temperature on the circumference is as given.

1.
$$u(c, \theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ -u_0, & \pi < \theta < 2\pi \end{cases}$$

2.
$$u(c, \theta) = \begin{cases} 1, & 0 < \theta < \pi/2 \\ 0, & \pi/2 < \theta < 3\pi/2 \\ 1, & 3\pi/2 < \theta < 2\pi \end{cases}$$

In Problems 3 and 4, find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius 1 if boundary conditions are as given.

3. $u(1, \theta) = u_0(\pi\theta - \theta^2), 0 < \theta < \pi$
$u(r, 0) = 0, \quad u(r, \pi) = 0, 0 < r < 1$

4. $u(1, \theta) = \sin\theta, 0 < \theta < \pi$
$u(r, 0) = 0, \quad u(r, \pi) = 0, 0 < r < 1$

5. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius c if the boundaries $\theta = 0$ and $\theta = \pi$ are insulated and $u(c, \theta) = f(\theta)$, $0 < \theta < \pi$.

6. Find the steady-state temperature $u(r, \theta)$ in a semicircular plate of radius c if the boundary $\theta = 0$ is held at temperature zero, the boundary $\theta = \pi$ is insulated, and $u(c, \theta) = f(\theta)$, $0 < \theta < \pi$.

In Problems 7 and 8, find the steady-state temperature $u(r, \theta)$ in the plate shown in the figure.

7.

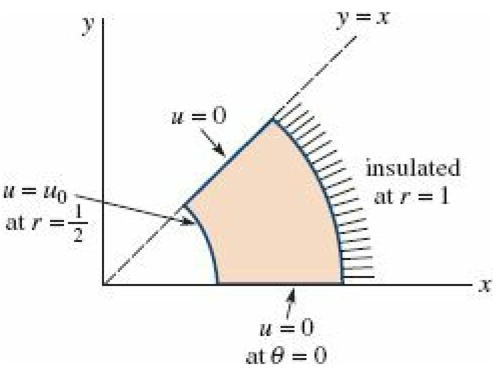

FIGURE 14.R.1 Plate in Problem 7

8.

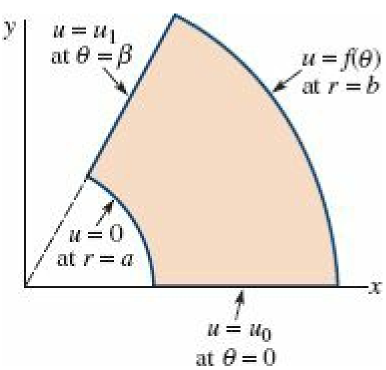

FIGURE 14.R.2 Plate in Problem 8

9. If the boundary conditions for an annular plate defined by $1 < r < 2$ are

$$u(1, \theta) = \sin^2\theta, \quad \left.\frac{\partial u}{\partial r}\right|_{r=2} = 0, 0 < \theta < 2\pi,$$

show that the steady-state temperature is

$$u(r, \theta) = \frac{1}{2} - \left(\frac{1}{34}r^2 + \frac{8}{17}r^{-2}\right)\cos 2\theta.$$

[*Hint*: See Figure 14.1.4. Also, use the identity $\sin^2\theta = ½(1 - \cos 2\theta)$.]

10. Find the steady-state temperature $u(r, \theta)$ in the infinite plate shown in **FIGURE 14.R.3**.

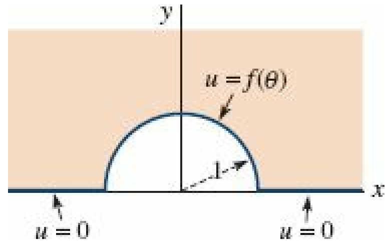

FIGURE 14.R.3 Infinite plate in Problem 10

11. Suppose heat is lost from the flat surfaces of a very thin circular plate of radius 1 into a surrounding medium at temperature zero. If the linear law of heat transfer applies, the heat equation assumes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - hu = \frac{\partial u}{\partial t}, \quad h > 0, 0 < r < 1, t > 0.$$

See **FIGURE 14.R.4** Find the temperature $u(r, t)$ if the edge $r = 1$ is kept at temperature zero and if initially the temperature of the plate is unity throughout.

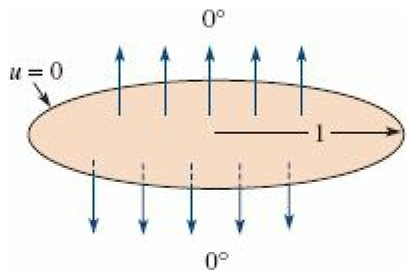

FIGURE 14.R.4 Circular plate in Problem 11

12. Suppose x_k is a positive zero of J_0. Show that a solution of the boundary-value problem

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, t > 0$$

$$u(1, t) = 0, \quad t > 0$$

$$u(r, 0) = u_0 J_0(x_k r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < r < 1$$

is $u(r, t) = u_0 J_0(x_k r) \cos ax_k t$.

13. Find the steady-state temperature $u(r, z)$ in the cylinder in Figure 14.2.5 if the lateral side is kept at temperature zero, the top $z = 4$ is kept at temperature 50, and the base $z = 0$ is insulated.

14. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, 0 < z < 1$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0, \quad 0 < z < 1$$

$$u(r, 0) = f(r), \quad u(r, 1) = g(r), \quad 0 < r < 1.$$

15. Find the steady-state temperature $u(r, \theta)$ in a sphere of unit radius if the surface is kept at

$$u(1, \theta) = \begin{cases} 100, & 0 < \theta < \pi/2 \\ -100, & \pi/2 < \theta < \pi. \end{cases}$$

[*Hint*: See Problem 22 in Exercises 12.6.]

16. Solve the boundary-value problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 1, t > 0$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = 0, \quad t > 0$$

$$u(r, 0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r), \quad 0 < r < 1.$$

[*Hint:* Proceed as in Problems 9 and 10 in Exercises 14.3, but let $v(r, t) = ru(r, t)$. See Section 13.7.]

17. The function $u(x) = Y_0(\alpha a)J_0(\alpha x) - J_0(\alpha a)Y_0(\alpha x)$, $a > 0$ is a solution of the parametric Bessel equation

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \alpha^2x^2u = 0$$

on the interval $[a, b]$. If the eigenvalues $\lambda_n = \alpha^2_n$ are defined by the positive roots of the equation

$$Y_0(\alpha a)J_0(\alpha x) - J_0(\alpha a)Y_0(\alpha x) = 0,$$

show that the functions

$$u_m(x) = Y_0(\alpha_m a)J_0(\alpha_m x) - J_0(\alpha_m a)Y_0(\alpha_m x)$$

$$u_n(x) = Y_0(\alpha_n a)J_0(\alpha_n x) - J_0(\alpha_n a)Y_0(\alpha_n x)$$

are orthogonal with respect to the weight function $p(x) = x$ on the interval $[a, b]$; that is,

$$\int_a^b x u_m(x) u_n(x)\, dx = 0, \; m \neq n.$$

[*Hint*: Follow the procedure on pages 676 and 677.]

18. Use the results of Problem 17 to solve the following boundary-value problem for the temperature $u(r, t)$ in an annular plate:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \quad a < r < b, t > 0$$

$$u(a, t) = 0, \quad u(b, t) = 0, \quad t > 0$$

$$u(r, 0) = f(r), \quad a < r < b.$$

19. Discuss how to solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < c, 0 < z < L$$

with the boundary conditions given in **FIGURE 14.R.5**.

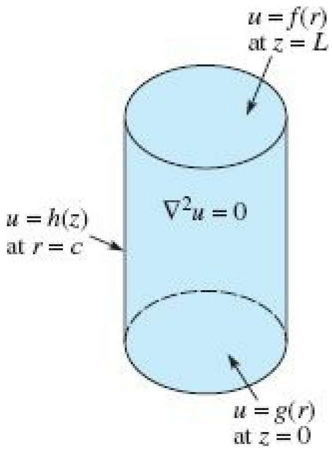

FIGURE 14.R.5 Cylinder in Problem 19

20. Carry out your ideas and find $u(r, z)$ in Problem 19. [*Hint*: Review (11) of Section 13.5.]

In Problems 21–24, solve the given boundary-value problem.

21.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, 0 < z < 1$$

$$u(1, z) = 100, \quad 0 < z < 1$$

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = 0, \quad 0 < r < 1$$

$$u(r, 1) = 200, \quad 0 < r < 1.$$

22.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 3, 0 < z < 1$$

$$u(3, z) = u_0, \quad 0 < z < 1$$

$$u(r, 0) = 0, \quad 0 < r < 3$$

$$u(r, 1) = 0, \quad 0 < r < 3.$$

23.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, z > 0$$

$$u(1, z) = 0, \quad z > 0$$

$$u(r, 0) = 100, \quad 0 < r < 1$$

24.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 < r < 1, z > 0$$

$$u(1, z) = 0, \quad z > 0$$

$$u(r, 0) = u_0(1 - r^2), \quad 0 < r < 1$$

*The problem in (18) is Example 2 of Section 3.9 with $L = \pi$.