Applications

Boundary-Value problems in Other Coordinate Systems

: Problems in Polar Coordinates

Laplacian in Polar Coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$.

these equations convert the two-dimensional Laplacian of the function u,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$ into polar coordinates

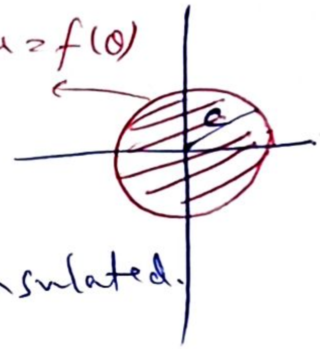
using the chain Rule (see the pdf-file), we get $\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0} \quad (*)$$

we wish to solve the Laplace's eq (*) for the steady-state temperature $u(r,\theta)$ in a circular disk or plate of radius c when the temperature of the circumference is

$u(c,\theta) = f(\theta), \quad 0 < \theta < 2\pi.$

It is assumed that the two faces of the plate are insulated.



ex. Find the temperature $u(r,\theta)$ in a semi circular plate by solving the following problem.

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < \theta < \pi, \; 0 < r < c. & (1) \\ u(c,\theta) = A, \quad 0 < \theta < \pi. & (2) \\ u(r,0) = 0, \; u(r,\pi) = 0, \quad 0 < r < c, & (3) \end{cases}$$

where A is constant.

Sol. Let $\boxed{u(r,\theta) = R(r)\,\Theta(\theta)}$

Substituting in (1), gives

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\,\Theta'' = 0.$$

Divide by $\frac{1}{r^2}R\Theta$ and seperating variables:

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

This implies

③

$$\boxed{r^2 R'' + r R' - \lambda R = 0} \quad ④$$

$$\boxed{\Theta'' + \lambda \Theta = 0} \quad ⑤$$

The homog. BC's give

$u(r,0) = 0 \Rightarrow \Theta(0) = 0$.

$u(r,\pi) = 0 \Rightarrow \Theta(\pi) = 0$

Now, consider the SLP:

$$\begin{cases} \Theta'' + \lambda \Theta = 0 \\ \Theta(0) = \Theta(\pi) = 0 . \end{cases} \quad ⑥$$

Case I: $\lambda = 0 \Rightarrow \Theta \equiv 0$ trivial أثبتها

Case II: $\lambda = -\alpha^2 < 0 \Rightarrow \Theta \equiv 0$ trivial

Case III. $\lambda = \alpha^2 > 0$

$$\Rightarrow \Theta(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta$$

$$\Theta(0) = c_1 = 0$$

$$\Theta(\pi) = c_2 \sin \alpha\pi = 0 .$$

let $c_2 \neq 0 \Rightarrow \alpha\pi = n\pi \; , n = 1, 2, \ldots$

$$\Rightarrow \boxed{\alpha_n = n} , \quad \boxed{\lambda_n = n^2}$$

$$\therefore \boxed{\Theta_n(\theta) = c_2 \sin(n\theta) , \quad n = 1, 2, \ldots}$$

Back to $\boxed{4}$ with $\lambda_n = n^2$:

$$\boxed{r^2 R'' + rR' - n^2 R = 0}$$ which is Cauch-Euler eq.

the aux. eq. is $m^2 + (1-1)m - n^2 = 0$.

$\Rightarrow m = \pm n$.

$$\boxed{R_n(r) = c_3 r^n + c_4 r^{-n}}$$

تذكر

$ax^2y'' + bxy' + cy = 0$.

the aux. eq.

$am^2 + (b-a)m + c = 0$

Since we expect a solution u of the problem to be bounded at $r=0$, then we must take $c_4 = 0$ since r^{-n} is unbdd at $r=0$.

$$\therefore \boxed{R_n(r) = c_3 r^n}$$

therefore, $\boxed{u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)}$.

Now, we use $u(c,\theta) = A$:

$$u(c,\theta) = \sum_{n=1}^{\infty} A_n c^n \sin(n\theta) = A.$$

is Fourier-Sine series

$$\Rightarrow A_n c^n = \frac{2}{\pi} \int_0^{\pi} A \cdot \sin(n\theta)\, d\theta.$$

$$\Rightarrow A_n = \frac{2A}{\pi c^n} \left[\frac{1-(-1)^n}{n}\right]$$

$$\Rightarrow u(r,\theta) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \left(\frac{r}{c}\right)^n \sin(n\theta).$$

Ex 2 (H.W) Find the temperature $u(r,\theta)$ in the circular plate by solving the following problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & 0 < \theta < 2\pi, \ 0 < r < 1 \\ u(1,\theta) = 5 , & 0 < \theta < 2\pi \\ u(r,0) = 0, \ u(r,2\pi) = 0, & 0 < r < 1. \end{cases}$$

Hint: See Example 1 in the pdf file. related to this section (14.1)

Problems in Cylindrical Coordinates

Radial Symmetry the two-dimensional heat and wave equations

$$\boxed{k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}}$$ and

$$\boxed{a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}}$$ expressed

in polar coordinates are, in turn,

$$\boxed{k\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial u}{\partial t}}$$ and

$$\boxed{a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)=\frac{\partial^2 u}{\partial t^2}},$$

where $u=u(r,\theta,t)$.

Rmk. To solve a BVP involving either of these equations by seperation of variables we must define $u=R(r)\,\Theta(\theta)\,T(t)$. this assumption leads to a multiple infinite series. We shall consider the simpler case

when u is independent of $\theta \Rightarrow \frac{\partial^2 u}{\partial \theta^2} = 0$
this is called radial symmetry and
this case still important. Hence in the
last wave & heat equations the solution

$$u = u(r,t).$$

Ex. Use seperation of variables method to
solve the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < r < 2, \; t > 0,$$

subject to the boundary conditions

$$\begin{cases} u(2,t) = 0, & t > 0 \\ u(r,0) = 0, & 0 < r < 2 \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{3}{2}, & 0 < r < 2 \end{cases}$$

Also, $u(r,t)$ is bounded at $r=0$.

Solution. Let $u(r,t) = R(r)T(t)$, we find

$$R''T + \frac{1}{r}R'T = RT'' \iff$$

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{T''}{T} = -\lambda \iff$$

$$\boxed{rR'' + R' + \lambda r R = 0}, \quad \boxed{T'' + \lambda T = 0}$$

$u(2,t) = 0 \Rightarrow R(2) = 0$

$u(r,0) = 0 \Rightarrow T(0) = 0$. Thus, we have

$$\boxed{r^2R'' + rR' + \lambda r^2 R = 0, \quad R(2) = 0}$$

$$\boxed{T'' + \lambda T = 0, \quad T(0) = 0}$$

Let $\lambda = \alpha^2$. then $\boxed{r^2R'' + rR' + (\alpha^2 r^2 - 0)R = 0}$

is parametric Bessel eq. with $\boxed{n = 0}$

$$\Rightarrow \boxed{R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)}$$

we know $Y_0(\alpha r) \to -\infty$ as $r \to 0^+$, so,

$\therefore$ we must take $c_2 = 0$ (since u is bdd).

$$\therefore \boxed{R(r) = c_1 J_0(\alpha r)}$$

and $R(2) = c_1 J_0(2\alpha) = 0$

put $c_1 \neq 0 \Rightarrow \boxed{J_0(2\alpha) = 0}$

$\Rightarrow \alpha_i$'s are the nonzero values such that

$J_0(2\alpha_i) = 0 \Rightarrow \boxed{R = J_0(\alpha_i r)}$

$\boxed{\lambda = \alpha_i^2}$. $T'' + \alpha_i^2 T = 0$

$\Rightarrow T_i = C_3 \cos(\alpha_i t) + C_4 \sin(\alpha_i t)$.

$0 = T(0) = C_3 \Rightarrow \boxed{C_3 = 0}$

$\therefore \boxed{T_i = C_4 \sin(\alpha_i t)}$

Thus, $\boxed{u(r,t) = \sum_{i=1}^{\infty} A_i \sin(\alpha_i t)\, J_0(\alpha_i r)}$

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{\infty} \alpha_i A_i \cos(\alpha_i t)\, J_0(\alpha_i r)$$

using $\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{3}{2} \Rightarrow \boxed{\frac{3}{2} = \sum_{i=1}^{\infty} \alpha_i A_i\, J_0(\alpha_i r)}$

here we have Bessel–Fourier Series

case (i) $J_0(2\alpha) = 0$

$$c_i = \alpha_i A_i = \frac{\cancel{2}}{2^2 J_1^2(2\alpha_i)} \int_0^2 \frac{3}{\cancel{2}} \cdot \underbrace{r\, J_0(\alpha_i r)\, dr}$$

let $x = \alpha_i r$

$dx = \alpha_i\, dr$

$$\alpha_i A_i = \frac{3}{4 J_1^2(2\alpha_i)} \int_0^{2\alpha_i} \frac{x}{\alpha_i} J_0(x) \frac{dx}{\alpha_i}$$

$$= \frac{3}{4\alpha_i^2 J_1^2(2\alpha_i)} \int_0^{2\alpha_i} x' J_0(x)\, dx .$$

$$= \frac{3}{4\alpha_i^2 J_1^2(2\alpha_i)} \; x J_1(x) \Big|_0^{2\alpha_i}$$

$$= \frac{3}{4\alpha_i^2 J_1^2(2\alpha_i)} \cdot 2\alpha_i J_1(2\alpha_i)$$

$$\Rightarrow \alpha_i A_i = \frac{3}{2\alpha_i J_1(2\alpha_i)}$$

or $$\boxed{A_i = \frac{3}{2\alpha_i^2 J_1(2\alpha_i)}}$$

Hence,

$$\boxed{u(r,t) = \frac{3}{2} \sum_{i=1}^{\infty} \frac{J_0(\alpha_i r)}{\alpha_i^2 J_1(2\alpha_i)} \sin(\alpha_i t)}.$$

$\square$

(H.w) Consider a vibrating circular membrance governed by the problem:

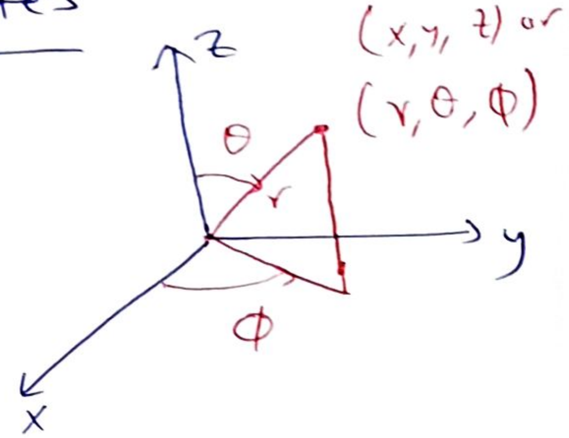
$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} = \dfrac{\partial^2 u}{\partial t^2}, \quad 0<r<1, \ t>0 \\ u(1,t)=0, \quad t>0 . \\ u(r,0)=0, \quad u_t(r,0)=1, \quad 0<r<1. \end{cases}$$

Use a seperation of variables to find an expression for $u(r,t)$.

Ans. $$\boxed{u(r,t) = 2\sum_{i=1}^{\infty} \frac{\sin(\alpha_i t)\, J_0(\alpha_i r)}{\alpha_i^2\, J_1(\alpha_i)}}$$

Problems in spherical coordinates

We are going to consider problems involving the heat, wave and Laplace's eq. in spherical coordinates



- Laplacian in spherical coordinates:

$\boxed{x = r\sin\theta\cos\phi}$, $\boxed{y = r\sin\theta\sin\phi}$, $\boxed{z = r\cos\theta}$

By using these eqs, you can show that the Laplacian $\nabla^2 u$ in spherical coordinate system is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}$$

In this section, we consider the problem where u is indep. of ϕ, i.e., $u = u(r, \theta)$.

Ex. Find the steady-state temperature $u(r,\theta)$ in a sphere of radius 2 by solving the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\cot\theta}{r^2}\dfrac{\partial u}{\partial \theta} = 0, \\ \qquad\qquad\qquad 0 < r < 2,\ 0 < \theta < \pi \\ u(2,\theta) = 1 - 2\cos\theta, \quad 0 < \theta < \pi. \end{cases}$$

Solution. Let $u(r,\theta) = R(r)\,\Theta(\theta)$. then

$$R''\Theta + \frac{2}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \frac{\cot\theta}{r^2}R\Theta' = 0$$

$$\Leftrightarrow \quad \frac{r^2R'' + 2rR'}{R} = -\frac{\Theta'' + \cot\theta\,\Theta'}{\Theta} = \lambda$$

$$\Rightarrow \boxed{\Theta'' + (\cot\theta)\,\Theta' + \lambda\Theta = 0} \text{ — } (1) \quad \text{and}$$

$$\boxed{r^2R'' + 2rR' - \lambda R = 0} \text{ — } (2)$$

let $x = \cos\theta$ in eq (1) and using the chain rule, we get

$$\boxed{(1-x^2)\,\Theta''(x) - 2x\,\Theta'(x) + \lambda\,\Theta(x) = 0}$$

this eq is Legendre eq. with $\lambda = n(n+1)$, $n = 0, 1, 2, \ldots$

$\Rightarrow \boxed{\Theta_n(x) = P_n(x)}$; $\boxed{\Theta_n(\theta) = P_n(\cos\theta)}$.

Back to (2): $r^2 R'' + 2rR' - n(n+1)R = 0$ is Cauchy-Euler eq. the aux. eq. is

$$m^2 + (2-1)m - n(n+1) = 0.$$

$$\Rightarrow m^2 + m - n(n+1) = 0.$$

$$(m-n)(m+(n+1)) = 0.$$

$$\Rightarrow \boxed{m_1 = n},\ \boxed{m_2 = -(n+1)}.$$

$$\boxed{R_n(r) = c_1 r^n + c_2 r^{-(n+1)}}$$

$u(r,\theta)$ is bounded $\Rightarrow c_2 = 0$ (r^{-n-1} is unbdd at $r=0$)

$$\therefore \boxed{R_n(r) = c_1 r^n}$$

$$\Rightarrow \boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)}.$$

$$u(2,\theta) = 1 - 2\cos\theta \Rightarrow$$

$$\underset{F(\theta)}{1 - 2\cos\theta} = \sum_{n=0}^{\infty} \overset{C_n}{A_n (2)^n} P_n(\cos\theta)$$

We use here the alternative form of Legendre-Fourier series

$$2^n A_n = \frac{2n+1}{2} \int_0^{\pi} (1 - 2\cos\theta) \, P_n(\cos\theta) \sin\theta \, d\theta$$

$$\Rightarrow A_n = \frac{2n+1}{2^{n+1}} \int_0^{\pi} P_0(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

$$- \frac{2n+1}{2^n} \int_0^{\pi} P_1(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta .$$

[here we use $1 = P_0(\cos\theta)$, $P_1(\cos\theta) = \cos\theta$]

By orthogonality $A_n = 0$, $n \neq 0, n \neq 1$.

$$A_0 = \frac{1}{2} \int_0^{\pi} P_0(\cos\theta) P_0(\cos\theta) \sin\theta \, d\theta - 0$$

$$= \frac{1}{2} \int_0^{\pi} 1^2 \cdot \sin\theta = -\frac{1}{2} \cos\theta \Big|_0^{\pi} = 1 .$$

$$A_1 = 0 - \frac{3}{2} \int_0^{\pi} P_1(\cos\theta) P_1(\cos\theta) \sin\theta \, d\theta$$

$$= -\frac{3}{2} \int_0^{\pi} \cos\theta \cdot \cos\theta \sin\theta \, d\theta .$$

$$= -\frac{3}{2} \int_1^{\pi} \cos^2\theta \sin\theta \, d\theta$$ Let $x = \cos\theta$

(16)

$$= \frac{3}{2}\int_{1}^{-1} x^2\,dx = \frac{3}{2}\,\frac{x^3}{3}\Big|_{1}^{-1} = \frac{3}{2}\left(-\frac{2}{3}\right)$$

$$= -1.$$

$$\therefore u(r,\theta) = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta) + \cancel{\sum_{n=2}^{\infty} A_n r^n P_n(\cos\theta)}^{\,0}$$

$$= 1\cdot 1 + -1\, r\cos\theta$$

$$\Rightarrow \boxed{u(r,\theta) = 1 - r\cos\theta}$$

(H.W)

Ex: Find the steady-state temperature $u(r,\theta)$ in a sphere of unit radius by solving the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\cot\theta}{r^2}\dfrac{\partial u}{\partial \theta} = 0, \\ \qquad 0 < r < 1,\quad 0 < \theta < \pi \\ u(1,\theta) = \cos^2\theta. \end{cases}$$

Ans: $u(r,\theta) = \frac{1}{3} + \frac{1}{3} r^2 (3\cos^2\theta - 1)$.