$$(*)\begin{cases} u_{xx} + \sin x \cos t = u_{tt}, & 0<x<\pi,\ t>0 \quad (1) \\ u(0,t) = 0 = u(\pi,t), & t>0. \quad (2) \\ u(x,0) = 0 = u_t(x,0), & 0<x<\pi. \quad (3) \end{cases}$$

Sol. the eigenvalues and eigenfunctions of the Sturm-Liouville problem corresponding to the homog. problem of $(*)$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 = X(\pi). \end{cases}$$ are

$$\lambda_n = \alpha_n^2 = n^2 \quad \text{and} \quad X_n = \sin(nx), \quad n=1,2,\dots$$

We assume that $$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(nx)} \quad (4)$$

if we Substitute (4) into (1), we get

$$\sum_{n=1}^{\infty} -n^2 u_n(t)\sin(nx) - \sum_{n=1}^{\infty} u_n''(t)\sin(nx) = -\sin x \cos t$$

or $$\boxed{\sum_{n=1}^{\infty} \left(u_n''(t) + n^2 u_n(t)\right)\sin(nx) = \sin x \cos t}$$

$$\Rightarrow \boxed{u_1''(t) + u_1(t) = \cos t, \quad n=1} \quad (5)$$

$$\boxed{u_n''(t) + n^2 u_n(t) = 0, \quad \forall n \neq 1} \quad (6)$$

Scanned by TapScanner

Applying the IC's 3 :

$$0 = u(x,0) = \sum_{n=1}^{\infty} u_n(0)\sin(nx) \Rightarrow \boxed{u_n(0) = 0}, \forall n = 1, 2, \ldots$$

$$0 = u_t(x,0) = \sum_{n=1}^{\infty} u_n'(0)\sin(nx) \Rightarrow \boxed{u_n'(0) = 0}, \forall n = 1, 2, \ldots$$

Next, we solve 5 with $u_1(0) = 0 = u_1'(0)$.

$u_1^{(h)}$: $u_1''(t) + u_1(t) = 0$

$$\Rightarrow \boxed{u_1^{(h)} = A\cos t + B\sin t}$$

$$\boxed{u_1^{(p)} = Ct\sin t + Dt\cos t} - 7$$

~~Setting 7 into 5:~~

$$\cancel{-C\sin t - D\cos t + C\sin t + D\cos t = \cos t}$$

$$[u_1^{(p)}]' = C\sin t + Ct\cos t + D\cos t - Dt\sin t$$

$$[u_1^{(p)}]'' = C.\cos t + C.\cos t - Ct\sin t - D\sin t - D\sin t - Dt\cos t.$$

$$= 2C\cos t - Ct\sin t - 2D\sin t - Dt\cos t$$

Substitute $u_1^{(p)}$ into 5 :

$$2C\cos t - \cancel{Ct\sin t} - 2D\sin t - \cancel{Dt\cos t} + \cancel{Ct\sin t} + \cancel{Dt\cos t} = \cos t.$$

(3)

$\Rightarrow \boxed{2C = 1}$ and $-2D = 0$

$\Rightarrow \boxed{C = \frac{1}{2}}$, $\boxed{D = 0}$

$\therefore \boxed{U_1^{(P)} = \frac{1}{2} t \sin t}$

$\therefore U_1 = U_1^{(h)} + U_1^{(P)}$

$\boxed{U_1 = A \cos t + B \sin t + \frac{1}{2} t \sin t}$

$U_1(0) = A = 0 \Rightarrow \boxed{A = 0}$

$U_1'(t) = -A \sin t - B \cos t + \frac{1}{2} \sin t + \frac{1}{2} t \cos t$.

$U_1(0) = -B = 0 \Rightarrow \boxed{B = 0}$

$\therefore \boxed{U_1(t) = \frac{1}{2} t \sin t}$

Next, we solve (6) with $U_n(0) = U_n'(0) = 0$

i.e., $\begin{cases} U_n''(t) + n^2 U_n(t) = 0 \\ U_n(0) = U_n'(0) = 0 \end{cases}$, $n \neq 1$.

$\boxed{U_n(t) = E \cos(nt) + F \sin(nt)}$

$U_n(0) = E = 0 \Rightarrow \boxed{E = 0}$

$U_n'(t) = -nE \sin(nt) + nF \cos(nt)$

④

$0 = U_n'(0) = nF \Rightarrow F = 0$ (since $n=1,2,\dots$

$\therefore U_n(t) \equiv 0, \quad \forall n \neq 1.$

$\cancel{\therefore U_n t}$

$$\therefore \; u(x,t) = \sum_{n=1}^{\infty} U_n(t) \sin(nx)$$

$$= U_1(t)\sin x + \underbrace{\sum_{n=2}^{\infty} U_n(t) \sin(nx)}_{0}.$$

$$\boxed{u(x,t) = \frac{1}{2} t \sin t \sin x}$$