(1.)

Show that $\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0<x<L,\ t>0 \\ u(0,t) = \mu(t),\quad u(L,t) = \psi(t), & t>0 \\ u(x,0) = \varphi(x),\quad u_t(x,0) = \eta(x), & 0<x<L \end{cases}$ (*)

has at most one solution.

First method: Let u, v be two solution for above problem, need to show $u = v$ so let $w = u - v$ then $w_{tt} = u_{tt} - v_{tt} = a^2 u_{xx} + f(x,t) - a^2 v_{xx} - f(x,t) \Rightarrow w_{tt} = a^2[u_{xx} - v_{xx}] = a^2 w_{xx}$

Also
$$w(0,t) = u(0,t) - v(0,t) = \mu(t) - \mu(t) = 0$$
$$w(L,t) = u(L,t) - v(L,t) = \psi(t) - \psi(t) = 0$$
$$w(x,0) = u(x,0) - v(x,0) = \varphi(x) - \varphi(x) = 0$$
$$w_t(x,0) = u_t(x,0) - v_t(x,0) = \eta(x) - \eta(x) = 0$$

So w is solution for the following problem: $\begin{cases} w_{tt} = a^2 w_{xx}, & 0<x<L,\ t>0 \\ w(0,t) = w(L,t) = 0, & t>0 \\ w(x,0) = w_t(x,0) = 0, & 0<x<L \end{cases}$

Using sepration method to solve above problem, let $w = XT$
then: $X\ddot{T} = a^2 \ddot{X} T \Rightarrow \dfrac{\ddot{T}}{a^2 T} = \dfrac{\ddot{X}}{X} = -\lambda \Rightarrow \ddot{X} + \lambda X = 0$ — (1)
$\Rightarrow \ddot{T} + \lambda a^2 T = 0$ — (2)

Consider $\begin{cases} \ddot{X} + \lambda X = 0 \\ X(0) = 0,\ X(L) = 0 \end{cases}$

Case 1: $\lambda = 0 \Rightarrow X = C_1 + C_2 x \Rightarrow X(0) = C_1 = 0,\ X(L) = C_2 L = 0 \Rightarrow C_2 = 0$
$\Rightarrow X \equiv 0$ (trivial)

Case 2: $\lambda = -\alpha^2, \alpha > 0 \Rightarrow X = C_1 \cosh\alpha x + C_2 \sinh\alpha x \Rightarrow X(0) = C_1 = 0$
$\Rightarrow X(L) = C_2 \sinh(\alpha L) = 0$

But $\sinh(\alpha L) \neq 0$ so $C_2 = 0 \Rightarrow X \equiv 0$ trivial

Case 3: $\lambda = \alpha^2, \alpha > 0 \Rightarrow X = C_1 \cos\alpha x + C_2 \sin\alpha x \Rightarrow X(0) = C_1 = 0$

$\Rightarrow X(L) = C_2 \sin\alpha L = 0$

Let $C_2 \neq 0 \Rightarrow \sin\alpha L = 0 \Rightarrow \alpha L = n\pi \Rightarrow \alpha_n = \frac{n\pi}{L}, \; n = 1, 2, \ldots$

$\Rightarrow X_n = C_n \sin\left(\frac{n\pi}{L}\right)x, \; \lambda_n = \left(\frac{n\pi}{L}\right)^2$

Back to : $\ddot{T}_n + \left(\frac{an\pi}{L}\right)^2 T_n = 0 \Rightarrow T_n = C_1 \cos\left(\frac{an\pi}{L}\right)t + C_2 \sin\left(\frac{an\pi}{L}\right)t$

$\Rightarrow \cancel{u(x,t)}\; w(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{an\pi}{L}\right)t + B_n \sin\left(\frac{an\pi}{L}\right)t\right] \sin\left(\frac{n\pi}{L}\right)x.$

$\Rightarrow w(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)x = 0 \Rightarrow A_n = 0, \; \forall n$

$\Rightarrow w_t(x,0) = \sum_{n=1}^{\infty} \frac{an\pi}{L} B_n \sin\left(\frac{n\pi}{L}\right)x = 0 \Rightarrow B_n * \frac{a*n*\pi}{L} = 0 \Rightarrow B_n = 0$

$\Rightarrow w(x,t) = 0$, But $w(x,t) = u(x,t) - v(x,t) = 0 \Rightarrow u(x,t) = v(x,t)$ #

Method 2: Let u, v be solution for (*) so $w = u - v$ is solution for

$$\begin{cases} w_{tt} = a^2 w_{xx}, & 0<x<L, \; t>0 \\ w(0,t) = w(L,t) = 0, & t>0 \\ w(x,0) = w_t(x,0) = 0, & 0<x<L. \end{cases}$$

we have $w_{tt} = a^2 w_{xx}$ multiply both side By w_t and integrate both side with respect to x.

$\Rightarrow \int_0^L w_t w_{tt}\, dx = a^2 \int_0^L w_t w_{xx}\, dx \quad \longleftarrow (1)$

Solve $a^2 \int_0^L w_t w_{xx}\, dx$ by parts

$u = w_t \qquad dv = w_{xx}\,dx$

$du = w_{xt} \qquad v = w_x$

so we get : $w_t w_x \Big|_0^L - \int_0^L w_x w_{xt}\, dx$

$\Rightarrow w_t(L,t)\, w_x(L,t) - w_t(0,t)\, w_x(0,t) - \int_0^L w_x w_{xt}\, dx.$

But $w_t(L,t) = w_t(0,t) = 0$ Since $w(0,t) = 0$ $\forall t>0$ So $w_t(0,t) = 0$ $\forall t$
Also $w(L,t) = 0$ so $w_t(L,t) = 0$, $\forall t>0$.

Therefore ① become $\int_0^L w_t\, w_{tt}\, dx = -\int_0^L a^2 w_{xt}\, w_x\, dx$ —— ②

But $\int_0^L w_t\, w_{tt}\, dx = \frac{1}{2}\int_0^L \frac{d}{dt}(w_t)^2\, dx$ — (i) Also

$-a^2\int_0^L w_{xt}\, w_x\, dx = \frac{-a^2}{2}\int_0^L \frac{d}{dt}(w_x)^2\, dx$ — (ii)

Substitute (i) and (ii) in ②

$\Rightarrow \frac{d}{dt}\int_0^L \frac{1}{2}(w_t)^2\, dx = \frac{d}{dt}\int_0^L -\frac{a^2}{2}(w_x)^2\, dx$

$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\int_0^L \left[(w_t)^2 + a^2 (w_x)^2\right] dx\right] = 0$

By calling $E(t) = \frac{1}{2}\int_0^L a^2 w_x^2 + w_t^2\, dx$. then $\acute{E}(t) = 0$ so $E(t) =$ Constant, But $w(x,0) = 0$ for any $0<x<L$ So $w_x(x,0) = 0$ and given that $w_t(x,0) = 0$ therefore $E(0) = 0$ So $E(t) \equiv 0$.

Since $E(t) = 0$ then $\int_0^L a^2 w_x^2 + w_t^2 = 0$ hence $w_x \equiv 0$ and $w_t \equiv 0$ $\forall t>0$ and $0<x<L$ this is possible only if $w(x,t) =$ Constant say c but $w(x,0) = c = 0$ then $w(x,t) \equiv 0$ But $w(x,t) =$ $u(x,t) - v(x,t) = 0$ so $u(x,t) = v(x,t)$ # then ~~the~~ !