(1)

(b) Use a CAS to obtain the graph of $u(x, t)$ over the rectangular region defined by $0 \leq x \leq 10$, $0 \leq t \leq 15$. Assume $u_0 = 100$ and $k = 1$. Use 2D and 3D plots of $u(x, t)$ to verify your answer to part (a).

31. Humans gather most of their information on the outside world through sight and sound. But many creatures use chemical signals as their primary means of communication; for example, honeybees, when alarmed, emit a substance and fan their wings feverishly to relay the warning signal to the bees that attend to the queen. These molecular messages between members of the same species are called pheromones. The signals may be carried by moving air or water or by a diffusion process in which the random movement of gas molecules transports the chemical away from its source. **FIGURE 15.2.4** shows an ant emitting an alarm chemical into the still air of a tunnel. If $c(x, t)$ denotes the concentration of the chemical x centimeters from the source at time t, then $c(x, t)$ satisfies

$$k\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}, \quad x > 0,\ t > 0,$$

and k is a positive constant. The emission of pheromones as a discrete pulse gives rise to a boundary condition of the form

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

(a) Solve the boundary-value problem if it is further known that $c(x, 0) = 0$, $x > 0$, and $\lim_{x\to\infty} c(x, t) = 0$, $t > 0$.

(b) Use a CAS to plot the graph of the solution in part (a) for $x > 0$ at the fixed times $t = 0.1$, $t = 0.5$, $t = 1$, $t = 2$, $t = 5$.

(c) For a fixed time t, show that $\int_0^\infty c(x, t)\, dx = Ak$. Thus Ak represents the total amount of chemical discharged.

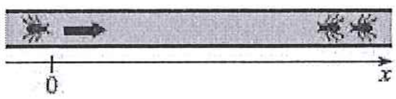


FIGURE 15.2.4 Ants in Problem 31

15.3 Fourier Integral

Introduction In preceding chapters, Fourier series were used to represent a function f defined on a *finite* interval $(-p, p)$ or $(0, L)$. When f and f' are piecewise continuous on such an interval, a Fourier series represents the function on the interval and converges to the periodic extension of f outside the interval. In this way we are justified in saying that Fourier series are associated only with *periodic* functions. We shall now derive, in a nonrigorous fashion, a means of representing certain kinds of *nonperiodic* functions that are defined on either an infinite interval $(-\infty, \infty)$ or a semi-infinite interval $(0, \infty)$.

□ **From Fourier Series to Fourier Integral**Suppose a function *f* is defined on $(-p, p)$. If we use the integral definitions of the coefficients (9), (10), and (11) of Section 12.2 in (8) of that section, then the Fourier series of *f* on the interval is

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right]. \quad (1)$$

If we let an $\alpha_n = n\pi/p$, $\nabla\alpha = \alpha_{n+1} - \alpha_n = \pi/p$, then (1) becomes

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{p} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha. \quad (2)$$

We now expand the interval $(-p, p)$ by letting $p \to \infty$. Since $p \to \infty$ implies that $\nabla\alpha \to 0$, the limit of (2) has the form $\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\Delta\alpha$, which is suggestive of the definition of the integral $\int_0^{\infty} F(\alpha)\,d\alpha$. Thus if $\int_{-\infty}^{\infty} f(t)\,dt$ exists, the limit of the first term in (2) is zero and the limit of the sum becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha. \quad (3)$$

The result given in (3) is called the **Fourier integral** of *f* on the interval $(-\infty, \infty)$. As the following summary shows, the basic structure of the Fourier integral is reminiscent of that of a Fourier series.

Definition 15.3.1 Fourier Integral

The **Fourier integral** of a function *f* defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha, \quad (4)$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \quad (5)$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx. \quad (6)$$

□ **Convergence of a Fourier Integral** Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but slightly more restrictive than, the conditions for a Fourier series.

Theorem 15.3.1 Conditions for Convergence

Let *f* and *f'* be piecewise continuous on every finite interval, and let *f* be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of *f* on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

EXAMPLE 1 Fourier Integral Representation

Find the Fourier integral representation of the piecewise-continuous function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

SOLUTION The function, whose graph is shown in **FIGURE 15.3.1** satisfies the hypotheses of Theorem 15.3.1. Hence from (5) and (6) we have at once

$$\begin{aligned} A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \\ &= \int_{-\infty}^{0} f(x)\cos\alpha x\,dx + \int_{0}^{2} f(x)\cos\alpha x\,dx + \int_{2}^{\infty} f(x)\cos\alpha x\,dx \\ &= \int_{0}^{2}\cos\alpha x\,dx = \frac{\sin 2\alpha}{\alpha} \end{aligned}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx = \int_{0}^{2}\sin\alpha x\,dx = \frac{1-\cos 2\alpha}{\alpha}.$$

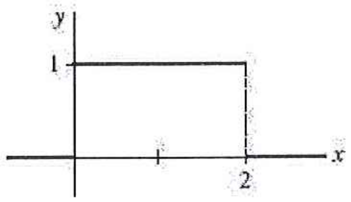


FIGURE 15.3.1 Function f in Example 1

Substituting these coefficients into (4) then gives

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\left(\frac{\sin 2\alpha}{\alpha}\right)\cos\alpha x + \left(\frac{1-\cos 2\alpha}{\alpha}\right)\sin\alpha x\right]d\alpha.$$

When we use trigonometric identities, the last integral simplifies to

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\frac{\sin\alpha\cos\alpha(x-1)}{\alpha}\,d\alpha. \tag{7}$$

≡

*This means that the integral $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges.

The Fourier integral can be used to evaluate integrals. For example, at $x = 1$ it follows from Theorem 15.3.1 that (7) converges to $f(1)$; that is,

$$\int_0^\infty \frac{\sin\alpha}{\alpha}\,d\alpha = \frac{\pi}{2}.$$

The latter result is worthy of special note since it cannot be obtained in the "usual" manner; the integrand $(\sin x)/x$ does not possess an antiderivative that is an elementary function.

□ **Cosine and Sine Integrals** When f is an even function on the interval $(-\infty, \infty)$, then the product $f(x)\cos\alpha x$ is also an even function, whereas $f(x)\sin\alpha x$ is an odd function. As a consequence of property (g) of Theorem 12.3.1, $B(\alpha) = 0$, and so (4) becomes

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha.$$

Here we have also used property (f) of Theorem 12.3.1 to write

$$\int_{-\infty}^\infty f(t)\cos\alpha t\,dt = 2\int_0^\infty f(t)\cos\alpha t\,dt.$$

Similarly, when f is an odd function on $(-\infty, \infty)$, products $f(x)\cos\alpha x$ and $f(x)\sin\alpha x$ are odd and even functions, respectively. Therefore $A(\alpha) = 0$ and

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha.$$

We summarize in the following definition.

Definition 15.3.2 Fourier Cosine and Sine Integrals

(*i*) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x\,d\alpha, \qquad (8)$$

where

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx. \qquad (9)$$

(*ii*) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin\alpha x\,d\alpha, \qquad (10)$$

where

$$B(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx. \qquad (11)$$

EXAMPLE 2 Cosine Integral Representation

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$$

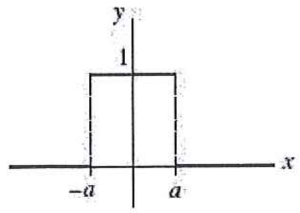


FIGURE 15.3.2 Function *f* in Example 2

SOLUTION It is apparent from **FIGURE 15.3.2** that *f* is an even function. Hence we represent *f* by the Fourier cosine integral (8). From (9) we obtain

$$\begin{aligned} A(\alpha) &= \int_0^\infty f(x)\cos \alpha x \, dx = \int_0^a f(x)\cos \alpha x \, dx + \int_a^\infty f(x)\cos \alpha x \, dx \\ &= \int_0^a \cos \alpha x \, dx = \frac{\sin a\alpha}{\alpha}, \end{aligned}$$

and so

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\alpha \cos \alpha x}{\alpha} \, d\alpha. \qquad (12)$$

The integrals (8) and (10) can be used when *f* is neither odd nor even and defined only on the half-line $(0, \infty)$. In this case (8) represents *f* on the interval $(0, \infty)$ and its even (but not periodic) extension to $(-\infty, 0)$, whereas (10) represents *f* on $(0, \infty)$ and its odd extension to the interval $(-\infty, 0)$. The next example illustrates this concept.

EXAMPLE 3 Cosine and Sine Integral Representations

Represent $f(x) = e^{-x}$, $x > 0$ **(a)** by a cosine integral; **(b)** by a sine integral.

SOLUTION The graph of the function is given in **FIGURE 15.3.3**.
(a) Using integration by parts, we find

$$A(\alpha) = \int_0^\infty e^{-x}\cos \alpha x \, dx = \frac{1}{1+\alpha^2}.$$

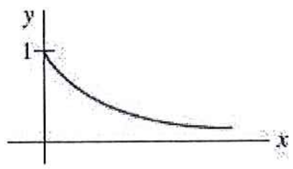


FIGURE 15.3.3 Function fin Example 3

Therefore from (8) the cosine integral of *f* is

(6)

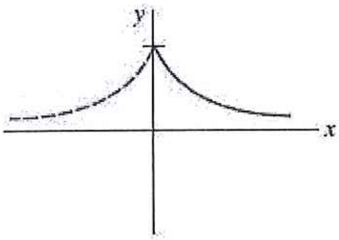
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos \alpha x}{1+\alpha^2}\,d\alpha. \tag{13}$$

(b) Similarly, we have

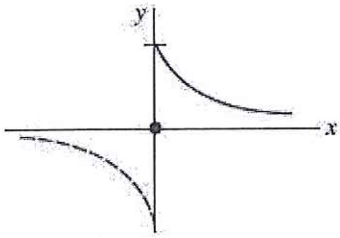
$$B(\alpha) = \int_0^\infty e^{-x}\sin \alpha x\,dx = \frac{\alpha}{1+\alpha^2}.$$

From (10) the sine integral of f is then

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2}\,d\alpha. \tag{14}$$



(a) Cosine integral



(b) Sine integral

FIGURE 15.3.4 In Example 3, (a) is the even extension of f; (b) is the odd extension of f

FIGURE 15.3.4 shows the graphs of the functions and their extensions represented by the two integrals. ≡

☐ **Complex Form** The Fourier integral (4) also possesses an equivalent **complex form**, or **exponential form**, that is analogous to the complex form of a Fourier series (see Section 12.4). If (5) and (6) are substituted into (4), then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x]\,dt\,d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos \alpha(t - x)\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos \alpha(t - x)\,dt\,d\alpha \tag{15}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos\alpha(t-x) + i\sin\alpha(t-x)]\,dt\,d\alpha \qquad (16)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt\right)e^{-i\alpha x}d\alpha. \qquad (17)$$

We note that (15) follows from the fact that the integrand is an even function of α. In (16) we have simply added zero to the integrand,

$$i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\,d\alpha = 0,$$

because the integrand is an odd function of α. The integral in (17) can be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{-i\alpha x}\,d\alpha, \qquad (18)$$

where

$$C(\alpha) = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx. \qquad (19)$$

This latter form of the Fourier integral will be put to use in the next section when we return to the solution of boundary-value problems.

□ **Use of Computers** The convergence of a Fourier integral can be examined in a manner that is similar to graphing partial sums of a Fourier series. To illustrate, let's use the results in parts (a) and (b) of Example 3. By definition of an improper integral, the Fourier cosine integral representation of $f(x) = e^{-x}$, $x > 0$ in (13) can be written as $f(x) = \lim_{b\to\infty} F_b(x)$, where

$$F_b(x) = \frac{2}{\pi}\int_0^b \frac{\cos\alpha x}{1+\alpha^2}\,d\alpha,$$

and x is treated as a parameter. Similarly, the Fourier sine integral representation of $f(x) = e^{-x}$ in (14) can be written as $f(x) = \lim_{b\to\infty} G_b(x)$, where

$$G_b(x) = \frac{2}{\pi}\int_0^b \frac{\alpha\sin\alpha x}{1+\alpha^2}\,d\alpha.$$

(8)

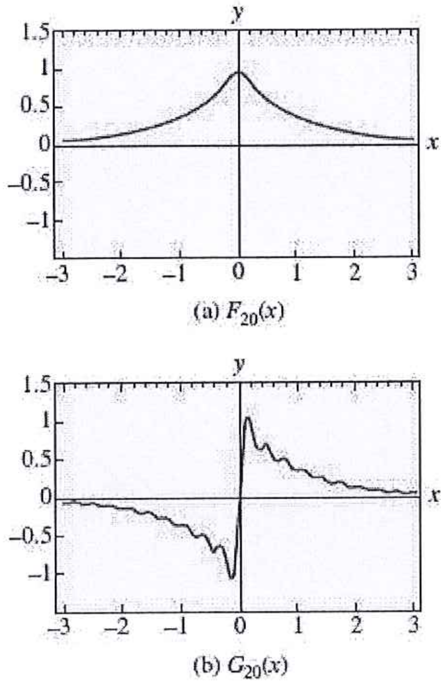


FIGURE 15.3.5 Graphs of partial integrals

Because the Fourier integrals (13) and (14) converge, the graphs of the partial integrals $F_b(x)$ and $G_b(x)$ for a specified value of $b > 0$ will be an approximation to the graph of f and its even and odd extensions shown in **Figure 15.3.4(a)** and **15.3.4(b)**, respectively. The graphs of $F_b(x)$ and $G_b(x)$ for $b = 20$ given in **FIGURE 15.3.5** were obtained using *Mathematica* and its **NIntegrate** application. See Problem 21 in Exercises 15.3.

1, 4, 9, 10, 12, 16, 19

15.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-35.

In Problems 1–6, find the Fourier integral representation of the given function.

1.
$$f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

2.
$$f(x) = \begin{cases} 0, & x < \pi \\ 4, & \pi < x < 2\pi \\ 0, & x > 2\pi \end{cases}$$

3.
$$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 3 \\ 0, & x > 3 \end{cases}$$

4.
$$f(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

(9)

5. $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

6. $f(x) = \begin{cases} e^{x}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

In Problems 7–12, represent the given function by an appropriate cosine or sine integral.

7. $f(x) = \begin{cases} 0, & x < -1 \\ -5, & -1 < x < 0 \\ 5, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

8. $f(x) = \begin{cases} 0, & |x| < 1 \\ \pi, & 1 < |x| < 2 \\ 0, & |x| > 2 \end{cases}$

9. $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

10. $f(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

11. $f(x) = e^{-|x|} \sin x$

12. $f(x) = xe^{-|x|}$

In Problems 13–16, find the cosine and sine integral representations of the given function.

13. $f(x) = e^{-kx}, k > 0, x > 0$

14. $f(x) = e^{-x} - e^{-3x}, x > 0$

15. $f(x) = xe^{-2x}, x > 0$

16. $f(x) = e^{-x} \cos x, x > 0$

In Problems 17 and 18, solve the given integral equation for the function f.

17. $\displaystyle\int_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$

18. $\displaystyle\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$

19. **(a)** Use (7) to show that $\displaystyle\int_0^\infty \frac{\sin 2x}{x} dx = \frac{\pi}{2}$.

[*Hint*: α is a dummy variable of integration.]

(b) Show in general that, for $k > 0$, $\displaystyle\int_0^\infty \frac{\sin kx}{x} dx = \frac{\pi}{2}$.

20. Use the complex form (19) to find the Fourier integral representation of $f(x) = e^{-|x|}$. Show that the result is the same as that obtained from (8) and (9).