(b) Use a CAS to obtain the graph of $u(x, t)$ over the rectangular region defined by $0 \le x \le 10$, $0 \le t \le 15$. Assume $u_0 = 100$ and $k = 1$. Use 2D and 3D plots of $u(x, t)$ to verify your answer to part (a).

31. Humans gather most of their information on the outside world through sight and sound. But many creatures use chemical signals as their primary means of communication; for example, honeybees, when alarmed, emit a substance and fan their wings feverishly to relay the warning signal to the bees that attend to the queen. These molecular messages between members of the same species are called pheromones. The signals may be carried by moving air or water or by a diffusion process in which the random movement of gas molecules transports the chemical away from its source. **FIGURE 15.2.4** shows an ant emitting an alarm chemical into the still air of a tunnel. If $c(x, t)$ denotes the concentration of the chemical x centimeters from the source at time t, then $c(x, t)$ satisfies

$$k\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}, \quad x > 0,\ t > 0,$$

and k is a positive constant. The emission of pheromones as a discrete pulse gives rise to a boundary condition of the form

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = -A\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

(a) Solve the boundary-value problem if it is further known that $c(x, 0) = 0$, $x > 0$, and $\lim_{x\to\infty} c(x, t) = 0$, $t > 0$.

(b) Use a CAS to plot the graph of the solution in part (a) for $x > 0$ at the fixed times $t = 0.1$, $t = 0.5$, $t = 1$, $t = 2$, $t = 5$.

(c) For a fixed time t, show that $\int_0^\infty c(x, t)\, dx = Ak$. Thus Ak represents the total amount of chemical discharged.

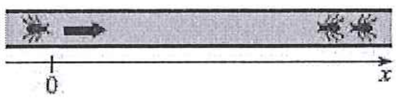


FIGURE 15.2.4 Ants in Problem 31

15.3 Fourier Integral

Introduction In preceding chapters, Fourier series were used to represent a function f defined on a *finite* interval $(-p, p)$ or $(0, L)$. When f and f' are piecewise continuous on such an interval, a Fourier series represents the function on the interval and converges to the periodic extension of f outside the interval. In this way we are justified in saying that Fourier series are associated only with *periodic* functions. We shall now derive, in a nonrigorous fashion, a means of representing certain kinds of *nonperiodic* functions that are defined on either an infinite interval $(-\infty, \infty)$ or a semi-infinite interval $(0, \infty)$.

(2)

□ **From Fourier Series to Fourier Integral** Suppose a function f is defined on $(-p, p)$. If we use the integral definitions of the coefficients (9), (10), and (11) of Section 12.2 in (8) of that section, then the Fourier series of f on the interval is

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{p} f(t)\sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right]. \quad (1)$$

If we let an $\alpha_n = n\pi/p$, $\nabla\alpha = \alpha_{n+1} - \alpha_n = \pi/p$, then (1) becomes

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{p} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha. \quad (2)$$

We now expand the interval $(-p, p)$ by letting $p \to \infty$. Since $p \to \infty$ implies that $\nabla\alpha \to 0$, the limit of (2) has the form $\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty}F(\alpha_n)\Delta\alpha$, which is suggestive of the definition of the integral $\int_0^{\infty}F(\alpha)\,d\alpha$. Thus if $\int_{-\infty}^{\infty} f(t)\,dt$ exists, the limit of the first term in (2) is zero and the limit of the sum becomes

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha. \quad (3)$$

The result given in (3) is called the **Fourier integral** of f on the interval $(-\infty, \infty)$. As the following summary shows, the basic structure of the Fourier integral is reminiscent of that of a Fourier series.

Definition 15.3.1 Fourier Integral

The **Fourier integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^{\infty}[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha, \quad (4)$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \quad (5)$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx. \quad (6)$$

□ **Convergence of a Fourier Integral** Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but slightly more restrictive than, the conditions for a Fourier series.

Theorem 15.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

EXAMPLE 1 Fourier Integral Representation

Find the Fourier integral representation of the piecewise-continuous function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$$

SOLUTION The function, whose graph is shown in **FIGURE 15.3.1** satisfies the hypotheses of Theorem 15.3.1. Hence from (5) and (6) we have at once

$$\begin{aligned} A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx \\ &= \int_{-\infty}^{0} f(x)\cos\alpha x\, dx + \int_{0}^{2} f(x)\cos\alpha x\, dx + \int_{2}^{\infty} f(x)\cos\alpha x\, dx \\ &= \int_{0}^{2} \cos\alpha x\, dx = \frac{\sin 2\alpha}{\alpha} \end{aligned}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx = \int_{0}^{2} \sin\alpha x\, dx = \frac{1 - \cos 2\alpha}{\alpha}.$$

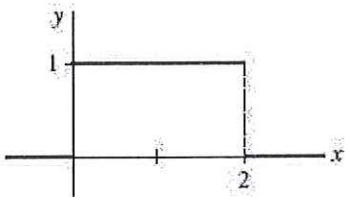


FIGURE 15.3.1 Function f in Example 1

Substituting these coefficients into (4) then gives

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\left(\frac{\sin 2\alpha}{\alpha}\right)\cos\alpha x + \left(\frac{1 - \cos 2\alpha}{\alpha}\right)\sin\alpha x\right] d\alpha.$$

When we use trigonometric identities, the last integral simplifies to

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\alpha\cos\alpha(x - 1)}{\alpha}\, d\alpha. \qquad (7)$$

≡

*This means that the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

The Fourier integral can be used to evaluate integrals. For example, at $x = 1$ it follows from Theorem 15.3.1 that (7) converges to $f(1)$; that is,

$$\int_0^{\infty} \frac{\sin \alpha}{\alpha} d\alpha = \frac{\pi}{2}.$$

The latter result is worthy of special note since it cannot be obtained in the "usual" manner; the integrand $(\sin x)/x$ does not possess an antiderivative that is an elementary function.

□ **Cosine and Sine Integrals** When f is an even function on the interval $(-\infty, \infty)$, then the product $f(x) \cos \alpha x$ is also an even function, whereas $f(x) \sin \alpha x$ is an odd function. As a consequence of property (g) of Theorem 12.3.1, $B(\alpha) = 0$, and so (4) becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cos \alpha t \, dt \right) \cos \alpha x \, d\alpha.$$

Here we have also used property (f) of Theorem 12.3.1 to write

$$\int_{-\infty}^{\infty} f(t) \cos \alpha t \, dt = 2 \int_0^{\infty} f(t) \cos \alpha t \, dt.$$

Similarly, when f is an odd function on $(-\infty, \infty)$, products $f(x) \cos \alpha x$ and $f(x) \sin \alpha x$ are odd and even functions, respectively. Therefore $A(\alpha) = 0$ and

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \sin \alpha t \, dt \right) \sin \alpha x \, d\alpha.$$

We summarize in the following definition.

Definition 15.3.2 Fourier Cosine and Sine Integrals

(*i*) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \cos \alpha x \, d\alpha, \tag{8}$$

where

$$A(\alpha) = \int_0^{\infty} f(x) \cos \alpha x \, dx. \tag{9}$$

(*ii*) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi} \int_0^{\infty} B(\alpha) \sin \alpha x \, d\alpha, \tag{10}$$

where

$$B(\alpha) = \int_0^{\infty} f(x) \sin \alpha x \, dx. \tag{11}$$

EXAMPLE 2 Cosine Integral Representation

Find the Fourier integral representation of the function

(5)

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a. \end{cases}$$

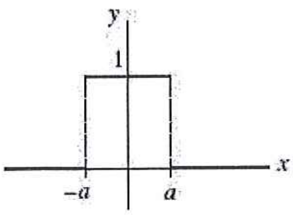


FIGURE 15.3.2 Function *f* in Example 2

SOLUTION It is apparent from **FIGURE 15.3.2** that *f* is an even function. Hence we represent *f* by the Fourier cosine integral (8). From (9) we obtain

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx = \int_0^a f(x)\cos\alpha x\,dx + \int_a^\infty f(x)\cos\alpha x\,dx$$
$$= \int_0^a \cos\alpha x\,dx = \frac{\sin a\alpha}{\alpha},$$

and so

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\alpha \cos\alpha x}{\alpha}\,d\alpha. \qquad (12)$$

≡

The integrals (8) and (10) can be used when *f* is neither odd nor even and defined only on the half-line $(0, \infty)$. In this case (8) represents *f* on the interval $(0, \infty)$ and its even (but not periodic) extension to $(-\infty, 0)$, whereas (10) represents *f* on $(0, \infty)$ and its odd extension to the interval $(-\infty, 0)$. The next example illustrates this concept.

EXAMPLE 3 Cosine and Sine Integral Representations

Represent $f(x) = e^{-x}$, $x > 0$ **(a)** by a cosine integral; **(b)** by a sine integral.

SOLUTION The graph of the function is given in **FIGURE 15.3.3**.

(a) Using integration by parts, we find

$$A(\alpha) = \int_0^\infty e^{-x}\cos\alpha x\,dx = \frac{1}{1+\alpha^2}.$$

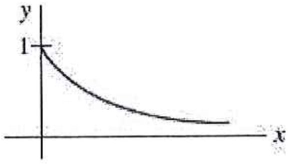


FIGURE 15.3.3 Function fin Example 3

Therefore from (8) the cosine integral of *f* is

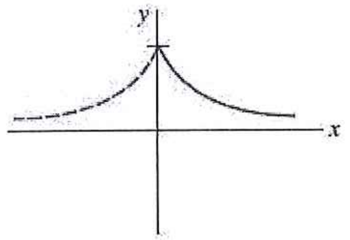
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos \alpha x}{1+\alpha^2}\,d\alpha. \tag{13}$$

(b) Similarly, we have

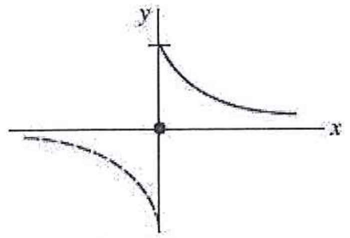
$$B(\alpha) = \int_0^\infty e^{-x}\sin \alpha x\,dx = \frac{\alpha}{1+\alpha^2}.$$

From (10) the sine integral of f is then

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2}\,d\alpha. \tag{14}$$



(a) Cosine integral



(b) Sine integral

FIGURE 15.3.4 In Example 3, (a) is the even extension of f; (b) is the odd extension of f

FIGURE 15.3.4 shows the graphs of the functions and their extensions represented by the two integrals.

Complex Form The Fourier integral (4) also possesses an equivalent **complex form**, or **exponential form**, that is analogous to the complex form of a Fourier series (see Section 12.4). If (5) and (6) are substituted into (4), then

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x]\,dt\,d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos \alpha(t-x)\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos \alpha(t-x)\,dt\,d\alpha \tag{15}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)[\cos\alpha(t-x) + i\sin\alpha(t-x)]\,dt\,d\alpha \tag{16}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\alpha(t-x)}\,dt\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt\right)e^{-i\alpha x}d\alpha. \tag{17}$$

We note that (15) follows from the fact that the integrand is an even function of α. In (16) we have simply added zero to the integrand,

$$i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(t-x)\,dt\,d\alpha = 0,$$

because the integrand is an odd function of α. The integral in (17) can be expressed as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\alpha)e^{-i\alpha x}\,d\alpha, \tag{18}$$

where

$$C(\alpha) = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx. \tag{19}$$

This latter form of the Fourier integral will be put to use in the next section when we return to the solution of boundary-value problems.

□ **Use of Computers** The convergence of a Fourier integral can be examined in a manner that is similar to graphing partial sums of a Fourier series. To illustrate, let's use the results in parts (a) and (b) of Example 3. By definition of an improper integral, the Fourier cosine integral representation of $f(x) = e^{-x}$, $x > 0$ in (13) can be written as $f(x) = \lim_{b\to\infty} F_b(x)$, where

$$F_b(x) = \frac{2}{\pi}\int_0^b \frac{\cos\alpha x}{1+\alpha^2}\,d\alpha,$$

and x is treated as a parameter. Similarly, the Fourier sine integral representation of $f(x) = e^{-x}$ in (14) can be written as $f(x) = \lim_{b\to\infty} G_b(x)$, where

$$G_b(x) = \frac{2}{\pi}\int_0^b \frac{\alpha\sin\alpha x}{1+\alpha^2}\,d\alpha.$$

(8)

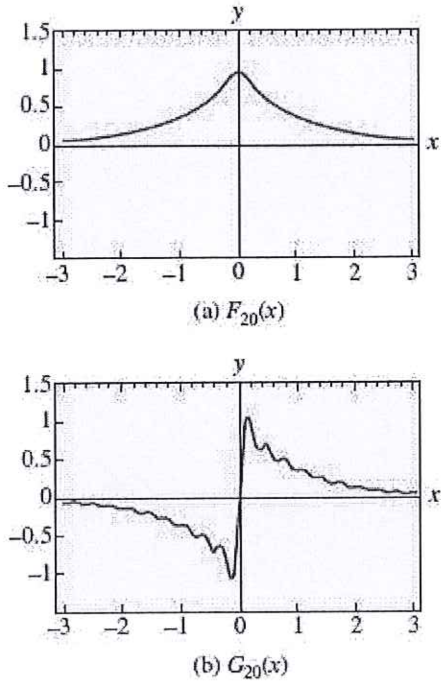


FIGURE 15.3.5 Graphs of partial integrals

Because the Fourier integrals (13) and (14) converge, the graphs of the partial integrals $F_b(x)$ and $G_b(x)$ for a specified value of $b > 0$ will be an approximation to the graph of f and its even and odd extensions shown in **Figure 15.3.4(a)** and **15.3.4(b)**, respectively. The graphs of $F_b(x)$ and $G_b(x)$ for $b = 20$ given in **FIGURE 15.3.5** were obtained using *Mathematica* and its **NIntegrate** application. See Problem 21 in Exercises 15.3.

1, 4, 9, 10, 12, 16, 19

15.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-35.

In Problems 1–6, find the Fourier integral representation of the given function.

1. $$f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

2. $$f(x) = \begin{cases} 0, & x < \pi \\ 4, & \pi < x < 2\pi \\ 0, & x > 2\pi \end{cases}$$

3. $$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 3 \\ 0, & x > 3 \end{cases}$$

4. $$f(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

5. $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

6. $f(x) = \begin{cases} e^{x}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

In Problems 7–12, represent the given function by an appropriate cosine or sine integral.

7. $f(x) = \begin{cases} 0, & x < -1 \\ -5, & -1 < x < 0 \\ 5, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

8. $f(x) = \begin{cases} 0, & |x| < 1 \\ \pi, & 1 < |x| < 2 \\ 0, & |x| > 2 \end{cases}$

9. $f(x) = \begin{cases} |x|, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

10. $f(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

11. $f(x) = e^{-|x|} \sin x$

12. $f(x) = xe^{-|x|}$

In Problems 13–16, find the cosine and sine integral representations of the given function.

13. $f(x) = e^{-kx}, k > 0, x > 0$

14. $f(x) = e^{-x} - e^{-3x}, x > 0$

15. $f(x) = xe^{-2x}, x > 0$

16. $f(x) = e^{-x} \cos x, x > 0$

In Problems 17 and 18, solve the given integral equation for the function f.

17. $\int_0^\infty f(x) \cos \alpha x \, dx = e^{-\alpha}$

18. $\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1, & 0 < \alpha < 1 \\ 0, & \alpha > 1 \end{cases}$

19. **(a)** Use (7) to show that $\int_0^\infty \frac{\sin 2x}{x} dx = \frac{\pi}{2}$.

[*Hint*: α is a dummy variable of integration.]

(b) Show in general that, for $k > 0$, $\int_0^\infty \frac{\sin kx}{x} dx = \frac{\pi}{2}$.

20. Use the complex form (19) to find the Fourier integral representation of $f(x) = e^{-|x|}$. Show that the result is the same as that obtained from (8) and (9).

Let $x=e^t$ and $t=\ln x$. Then

$$\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2}=\frac{1}{x^2}\left[\frac{d^2y}{dt^2}-\frac{dy}{dt}\right]$$

The new ODE with the independent variable t is

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0 \tag{1.15}$$

The characteristic equation

$$m^2+6m+9=0 \qquad (m+3)(m+3)=0$$

has a double root -3. Equation (1.15) has a general solution

$$y(t)=(c_1+c_2t)e^{-3t}$$

Using the transformation again, one obtains

$$y(x)=(c_1+c_2\ln x)x^{-3}$$

Euler equations appear in solutions of BVPs involving spherical geometry.

Exercises 1.2

1. Determine the general solution for the equation $y''-4y'+4y=0$.
2. Solve the differential equation $y''+2y'+2y=0$.
3. Find a general solution for $y'''-2y'-4y=0$.
   Hint: Show first that the characteristic equation has a root 2.
4. Solve the boundary value problem $y''-y=0$, $y(0)=0$, $y'(\pi)=1$.
5. Find a general solution for $y^{(4)}-y=0$.
6. Solve the differential equation $y'''-5y''+6y'=0$.
7. Determine a general solution for the equation $x^2y''-3xy'+3y=0$.
8. Solve the BVP $x^2y''-3xy'+4y=0$, $y(1)=0$, $y(e)=e^2$.
9. Find a general solution for $x^2y''-xy'+5y=0$.
10. Find a solution for the BVP $x^2y''+xy'+y=0$, $y(0)=1$, $y(\pi/2)=2$.

1.5. LINEAR PDEs

A PDE is called *linear* if L is a linear partial differential operator so that

$$Lu=f \tag{1.16}$$

The variable u is dependent and f is a function of the independent variables alone. If the equation is not linear it is described as *nonlinear*. Equation (1.16) is *homogeneous* if $f\equiv 0$; otherwise it is referred to as *nonhomogeneous*. A *solution* for the equation is a function of independent variables which satisfies (1.16). The order of a PDE is the order of its highest order derivative. The following are examples of PDEs.

$$Lu=u_x+u_y=x(x+2y) \tag{1.17}$$

$$Lu=u_{xy}+u_{yy}=0 \tag{1.18}$$

$$Lu=u_yu_{yy}+uu_x=0 \tag{1.19}$$

Equation (1.17) is linear, nonhomogeneous of order 1 with a solution $u=x^2y$. The second equation (1.18) is linear, homogeneous of order 2. One can verify that $u=\sin x$, $u=e^{y-x}$, $u=g(x)$ and $u=h(y-x)$ are all solutions of (1.18). The functions g and h are arbitrary. The last equation (1.19) is nonlinear, homogeneous of order 2. It has a solution $u=\sin(x+y)$.

For ODEs of nth order, general solutions are families of functions with n arbitrary constants. Instead of arbitrary constants, general solutions for PDEs are arbitrary functions of definite functions. The last two solutions mentioned for (1.18) were arbitrary functions $g(x)$ and $h(y-x)$. This implies that functions e^x, $\cos x$, $\sin(y-x)$, $(y-x)^2$, $\ln(y-x)$, and all others that are appropriately differentiable functions of x alone or $y-x$ are solutions of (1.18). Finding a particular solution from a general solution satisfying a constraint may be a difficult task. It may be preferable to find a particular solution satisfying specified conditions directly.

1.6. CLASSIFICATION OF A LINEAR PDE OF SECOND ORDER

A second order linear PDE with two independent variables has the form

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \tag{1.20}$$

where coefficients $A,\ldots,G$ are functions of x and y alone. The equation is *hyperbolic*, *elliptic*, or *parabolic* at a specific point in a domain as

$$B^2-4AC \tag{1.21}$$

is positive, negative, or zero. The classification is analogous to the geometry classification of conic sections. It can be shown by proper co...

transformation that the nature of (1.20) is invariant and the sign of (1.21) is unaltered. Equation (1.20) can be classified different at different points. Should the coefficients $A,\ldots,G$ be constants, then the equation is a single type for all points of the domain. For details of the classification, and information on canonical forms and characteristic equations, the reader may refer to Sommerfeld [31, pp. 36–43]. Illustrations of the classification follow:

(a) $u_{xx}-u_{yy}=0$ is hyperbolic with $B^2-4AC=4$.
(b) $u_{xx}+u_{yy}+u=xy$ is elliptic with $B^2-4AC=-4$.
(c) $u_{xx}+u_x-u_y+u=0$ is parabolic with $B^2-4AC=0$.
(d) $u_{xx}+xu_{yy}=0$ is elliptic, parabolic, or hyperbolic as $x>0$, $x=0$, or $x<0$ since $B^2-4AC=-4x$.

1.7. BOUNDARY VALUE PROBLEMS WITH PDEs

A mathematical problem composed of a PDE and certain constraints on the boundary of the domain is called *a boundary value problem*. If u is the dependent variable of the PDE it must satisfy the PDE in a domain of its independent variables and also constraint equations involving u and appropriate partial derivatives of u.

Problems involving time t as one of the independent variables of the PDE may have a condition given at one specified time, frequently when $t=0$. Such a constraint is referred to as an initial condition. If all the supplementary conditions are initial conditions then the problem is an *initial value problem*. A problem that has both initial and boundary conditions is properly called an *initial-boundary value problem*. In the literature one often finds the use of the terminology *boundary value problem* to include the initial-boundary value problem or mixed problem. In the problem

$$u_t(x,t)=a^2u_{xx}(x,t),\qquad (0<x<1,\ t>0) \tag{1.22}$$

$$u(0,t)=u(1,t)=0,\qquad (t\geq 0) \tag{1.23}$$

$$u(x,0)=f(x),\qquad (0\leq x\leq 1) \tag{1.24}$$

the condition (1.24) is an initial condition, while (1.23) are boundary conditions. The problem (1.22)–(1.24) is an initial-boundary value problem or simply a boundary value problem depending on one's preference.

Existence and uniqueness are important topics for boundary or initial value problems of PDEs. At this time we indicate only a Cauchy-Kovalevsky theorem for the second order PDE with initial conditions. For details see Zachmanoglon and Thoe [39, pp. 100–109].

Theorem.* Let

$$u_{tt}=F(t,x,u,u_t,u_x,u_{tx}u_{xx}) \tag{1.25}$$

be the PDE with initial conditions

$$u(0,x)=f(x)$$
$$u_t(0,x)=g(x) \tag{1.26}$$

Functions $f(x)$ and $g(x)$ are defined on an interval of the x axis containing the origin. Assume that $f(x)$ and $g(x)$ are analytic in a neighborhood of the origin and F is analytic in a neighborhood of the point $(0,0,f(0),g(0),f'(0),g'(0),f''(0))$. Then the problem (1.25), (1.26) has a unique analytic solution $u(x,t)$ in a neighborhood of the origin.

The Cauchy-Kovalevsky theorem serves as an example of an existence-uniqueness theorem for an IVP with a PDE. At a later time we will investigate properties of existence and uniqueness for a few problems of mathematical physics.

A mathematical problem is *well posed* if it has a unique solution that depends continuously on initial or boundary data. The last requirement implied above is sometimes referred to as *stability*. For a mathematical model to describe a specified phenomenon, a small modification in the original data should result only in a small variation of the solution. Even though most of our problems are well posed, it is important to know that there are problems that fail to meet these conditions. From a family of examples attributed to Hadamard [16, p. 33–34] the elliptic equation

$$u_{xx}+u_{yy}=0,\qquad -\infty<x<\infty,\qquad y>0$$

with the initial conditions on the x axis

$$u(x,0)=0,\qquad -\infty<x<\infty$$
$$u_y(x,0)=e^{-\sqrt{n}}\sin nx,\qquad -\infty<x<\infty$$

has the solution

$$u(x,y)=\frac{e^{-\sqrt{n}}}{n}\sin nx\sinh ny \tag{1.27}$$

As $n\to\infty$, $e^{-\sqrt{n}}\sin nx\to 0$, but for $x\neq 0$ the solution $e^{-\sqrt{n}}/n\sin nx\sinh ny\to\infty$ for any $y\neq 0$. The solution (1.27) fails to depend continuously on the initial data, and therefore is unstable.

*From Zach... [illegible]

1.8. SECOND ORDER LINEAR PDEs WITH CONSTANT COEFFICIENTS

One of the simplest equations in this category is a second order partial derivative equal to a function of the independent variables. Illustrations of this type follow.

Example 1.6. Find a solution for the PDE

$$u_{xy}=xy^2$$

First integrate relative to y with x fixed. Then

$$u_x=\frac{xy^3}{3}+f'(x)$$

where $f'(x)$ is an arbitrary function of x only. A second integration relative to x with y fixed produces the solution

$$u=\frac{x^2y^3}{6}+f(x)+g(y)$$

where $g(y)$ is an arbitrary function of y alone. Anticipating an integration relative to x, we select an arbitrary function $f'(x)$ in derivative form in the first step.

Example 1.7. Solve the PDE

$$u_{yy}=e^y$$

with the supplementary conditions

$$u_y(x,0)=x^3 \quad \text{and} \quad u(x,0)=e^x$$

Integrating the PDE relative to y, one obtains

$$u_y=e^y+f(x)$$

Due to the nature of the first supplementary condition we determine $f(x)$ before finding u.

$$u_y(x,0)=x^3=1+f(x)$$

This implies that

$$f(x)=x^3-1$$

Therefore,

$$u_y=e^y+x^3-1$$

Integrating a second time relative to y, one finds

$$u=e^y+x^3y-y+g(x)$$

To determine $g(x)$ we use the second condition,

$$u(x,0)=e^x=1+g(x)$$

It follows that

$$g(x)=e^x-1$$

The solution for the problem is

$$u=e^y+x^3y-y+e^x-1$$

For a second type, we consider the equation with second partial derivatives only

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0 \qquad (1.28)$$

where A, B, and C are real constants. Let

$$u=f(y+mx) \qquad (1.29)$$

be a proposed solution. We attempt to find m so that (1.29) satisfies (1.28). If f is a solution of (1.28) it must be twice differentiable. Substituting (1.29) into (1.28), we obtain

$$Am^2f''(y+mx)+Bmf''(y+mx)+Cf''(y+mx)=0$$

If $f''(y+mx)\neq 0$,

$$Am^2+Bm+C=0 \qquad (1.30)$$

The polynomial equation (1.30) is a characteristic equation. If it has distinct roots $m=m_1$ and $m=m_2$ then $u=f(y+m_1x)$ and $u=g(y+m_2x)$ are solutions of (1.28). The linear combination

$$u=f(y+m_1x)+g(y+m_2x) \qquad (1.31)$$

is a general solution of (1.28)

If m_1 and m_2 are distinct and new variables

$$r=y+m_1x \quad \text{and} \quad s=y+m_2x \tag{1.32}$$

are introduced in (1.28), the new equation is

$$A\left[m_1^2u_{rr}+2m_1m_2u_{rs}+m_2^2u_{ss}\right]+B\left[m_1u_{rr}+(m_1+m_2)u_{rs}+m_2u_{ss}\right]$$
$$+C\left[u_{rr}+2u_{rs}+u_{ss}\right]=0 \tag{1.33}$$

assuming $u_{rs}=u_{sr}$. Equation (1.33) can be simplified so that the coefficients of u_{rr} and u_{ss} are both zero, and

$$u_{rs}=0 \tag{1.34}$$

Equation (1.34) is a special type solvable by integration. It has the solution

$$u=f(r)+g(s)$$

Replacing r and s as given in (1.32) one obtains the solution (1.31).

The d'Alembert solution of the wave equation

$$u_{tt}=c^2u_{xx}, c>0 \tag{1.35}$$

is a good illustration of the transformation described in (1.32). Equation (1.35) is hyperbolic. The auxiliary equation is

$$m^2-c^2=0 \tag{1.36}$$

The transformation (1.32) becomes

$$r=x+ct \quad \text{and} \quad s=x-ct \tag{1.37}$$

Using (1.37) as described above, we obtain

$$u=f(x+ct)+g(x-ct)$$

for the solution of the wave equation.

The solutions of the characteristic equation (1.30) may be (a) real and distinct, (b) double, or (c) conjugate (imaginary part nonzero) complex numbers. The discriminant for the quadratic equation (1.30) is the same as the discriminant for (1.28). Therefore, a hyperbolic PDE (1.28) is matched by real and distinct roots in (1.30); an elliptic equation (1.28) is paired with conjugate complex roots in (1.30); and a parabolic equation (1.28) is associated with a double root in (1.30).

If $m_1=m_2$ in (1.30), then $B^2-4AC=0$. The two roots are $m_1=-B/2A$. A second solution for (1.28) is

$$u=xg(y+m_1x)$$

This result can be verified if $m_1=m_2=-B/2A$ is employed. In this case

$$u=f(y+m_1x)+xg(y+m_1x) \tag{1.38}$$

is a general solution for (1.28). One can show that

$$u=f(y+m_1x)+yg(y+m_1x) \tag{1.39}$$

is a general solution of (1.28) also.

Example 1.8. Find a general solution for $u_{xx}+4u_{xy}+4u_{yy}=0$.

This equation is parabolic. The characteristic equation has a double root -2. A general solution using (1.38) is

$$u=f(y-2x)+xg(y-2x)$$

If (1.39) is used

$$u=f(y-2x)+yg(y-2x)$$

is a general solution.

Example 1.9. Determine a solution for $u_{xx}+4u_{yy}=0$.

The discriminant $B^2-4AC<0$. Therefore, the equation is elliptic. The characteristic equation has roots $\pm 2i$. The general solution is written in the same form as (1.31). For this PDE

$$u=f(y-2ix)+g(y+2ix)$$

is a general solution.

By comparison with an ODE one may suspect the existence of an exponential solution for the homogeneous PDE

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0 \tag{1.40}$$

where the coefficients $A,\ldots,F$ are real constants. Let

$$u=e^{\alpha x+\beta y} \tag{1.41}$$

where α and β are real, be a proposed solution. Substituting (1.41) in (1.40), one obtains the condition

$$A\alpha^2+B\alpha\beta+C\beta^2+D\alpha+E\beta+F=0 \qquad (1.42)$$

In the quadratic equation (1.42), one may solve for β as a function of α or α as a function of β. Assume that we solve for β and obtain $\beta_1(\alpha)$ and $\beta_2(\alpha)$. A particular solution

$$u=K_1e^{\alpha x+\beta_1(\alpha)y}+K_2e^{\alpha x+\beta_2(\alpha)y}$$

is the result.

Example 1.10. Determine a solution for the PDE

$$u_{xx}-u_{yy}-2u_x+u=0 \qquad (1.43)$$

Substitute the exponential function

$$u=e^{\alpha x+\beta y}$$

in (1.43). The characteristic equation

$$\alpha^2-\beta^2-2\alpha+1=0$$

has solutions

$$\beta=\alpha-1 \quad \text{and} \quad \beta=-\alpha+1$$

Using superposition of the two solutions one finds the particular solution

$$u=K_1e^{\alpha x+(\alpha-1)y}+K_2e^{\alpha x+(-\alpha+1)y}$$

This solution may be written

$$u=K_1e^{-y}e^{\alpha(x+y)}+K_2e^{y}e^{\alpha(x-y)}$$

We may conjecture that a general solution has the form

$$u=e^{-y}f(x+y)+e^{y}g(x-y) \qquad (1.44)$$

where f and g are twice differentiable arbitrary functions. By substituting (1.44) into (1.43), we confirm that (1.44) is a solution.

When the left member of (1.42) has distinct linear factors, the type of simplification discussed is possible. The case of a repeated linear factor may be considered by using a result comparable to (1.38) or (1.39).

Example 1.11. Examine

$$u_{xx}-2u_{xy}+u_{yy}-2u_y+2u_x+u=0$$

for a general solution.

Let $u=e^{\alpha x+\beta y}$ and obtain a characteristic equation

$$\alpha^2-2\alpha\beta+\beta^2-2\beta+2\alpha+1=0$$

The double root is

$$\beta=\alpha+1$$

An exponential form of a solution is

$$u=e^{y}\left[K_1e^{\alpha(x+y)}+K_2xe^{\alpha(x+y)}\right]$$

A general solution

$$u=e^{y}\left[f(x+y)+xg(x+y)\right]$$

can be verified.

Certain cases may arise in (1.42) where linear factors with imaginary elements appear.

Example 1.12. Investigate a solution for the equation

$$u_{xx}+u_{yy}-2u_y+u=0 \qquad (1.45)$$

Let

$$u=e^{\alpha x+\beta y}$$

be a proposed solution. The characteristic equation

$$\alpha^2+\beta^2-2\beta+1=0$$

has two linear factors with imaginary elements for which

$$\beta=1\pm i\alpha$$

An exponential solution is

$$u=e^{y}\left[e^{\alpha(x+iy)}+e^{\alpha(x-iy)}\right] \qquad (1.46)$$

A general solution for (1.45) is suggested by (1.46)

$$u=e^{y}\left[f(x+iy)+g(x-iy)\right] \qquad (1.47)$$

It is easy to verify that (1.47) is a solution of (1.45).

In some situations the exponential procedure may produce a set of useful particular solutions, but fail to suggest a general solution.

Example 1.13. Determine a solution for the equation

$$u_{xx}+u_{yy}+4u=0$$

One obtains a characteristic equation

$$\alpha^2+\beta^2+4=0$$

with

$$\beta=\pm i\sqrt{\alpha^2+4}$$

If the exponential substitution is followed then

$$u=e^{\alpha x}\left[K_1e^{i\sqrt{\alpha^2+4}\,y}+K_2e^{-i\sqrt{\alpha^2+4}\,y}\right]$$

This solution can be expressed

$$u=e^{\alpha x}\left[M_1\cos\sqrt{\alpha^2+4}\,y+M_2\sin\sqrt{\alpha^2+4}\,y\right]$$

if K_1 and K_2 are properly related to M_1 and M_2 using Euler's identity.

Equation (1.40) can be solved almost like an ODE if only partial derivatives with respect to one variable appear. Arbitrary constants of the ODE solution become arbitrary functions of the remaining variable.

Example 1.14. Solve the PDE

$$u_{yy}-4u_y+3u=0$$

The dependent variable u is a function of x and y, but the only derivatives involved are relative to y alone. The corresponding ODE, with u as a function of y,

$$\frac{d^2u}{dy^2}-4\frac{du}{dy}+3u=0$$

has a solution

$$u=c_1e^{3y}+c_2e^{y}$$

Arbitrary constants c_1 and c_2 are replaced by arbitrary functions of x alone. The general solution becomes

$$u=e^{3y}f(x)+e^{y}g(x)$$

Other PDEs may be solved by using comparable solutions of ODEs.

Example 1.15. Find a solution for the PDE

$$xu_{xy}+2u_y=y^2$$

We observe that the equation may be written

$$\frac{\partial}{\partial y}\left[xu_x+2u\right]=y^2$$

By integrating, we obtain

$$xu_x+2u=\frac{y^3}{3}+f(x)$$

Dividing by x, with y fixed, one recognizes a linear differential equation of first order

$$u_x+\frac{2}{x}u=\frac{y^3}{3x}+\frac{f(x)}{x}$$

The integrating factor is x^2. This equation may be displayed

$$\frac{\partial}{\partial x}(x^2u)=\frac{xy^3}{3}+xf(x)$$

Integrating the most recent equation, we obtain

$$x^2u=\frac{x^2y^3}{6}+f^*(x)+G(y)$$

An explicit form of the solution is

$$u=\frac{y^3}{6}+F(x)+\frac{1}{x^2}G(y)$$

For more information regarding Section 1.8, the reader may consult Hildebrand [18, Chapter 8].

Exercises 1.3

1. Solve the boundary value problem

$$u_{xy}=0, \quad u_x(x,0)=\cos x, \quad u\left(\frac{\pi}{2}, y\right)=\sin y$$

2. Find the solution for

$$u_{yx}=x^2y, \quad u_y(0, y)=y^2, \quad u(x,1)=\cos x$$

3. Determine a solution for $u_{xx}=\cos x$ if

$$u(0, y)=y^2 \quad \text{and} \quad u(\pi, y)=\pi \sin y.$$

4. Classify the following PDEs as hyperbolic, parabolic or elliptic:
   (a) $yu_{xx}+xu_{yy}=0.$
   (b) $x^2u_{xx}+2xyu_{xy}+y^2u_{yy}+u_x+u_y=0.$
   (c) $u_{xx}+2u_{xy}-3u_{yy}=0.$
   (d) $u_{xx}-2u_{xy}+u_{yy}=0.$
   (e) $u_{xx}+a^2u_{yy}=0,\ a>0.$
   (f) $u_{xx}-2u_{xy}+2u_{yy}=0.$
   Solve the equations (c)–(f).

5. The d'Alembert solution of the wave equation (1.35) is

$$u=f(x+ct)+g(x-ct)$$

Solve the wave equation if $u(x,0)=0$ and $u_t(x,0)=\phi(x)$.

6. (a) Determine a general solution for equation 4(c) by using the transformation $s=y-3x,\ r=y+x$.
   (b) If $u(0, y)=0$ and $u_x(0, y)=\phi(y)$ in (a), show that

$$u=\frac{1}{4}\int_{y-3x}^{y+x}\phi(\alpha)\,d\alpha$$

7. Determine a solution for $u_{xx}+2u_{xy}+u_{yy}+u_x+u_y=0$ by letting $u=e^{\alpha x+\beta y}$. After finding β as a function of α, propose a general solution. Verify the general solution.

8. Using the substitution $u=e^{\alpha x+\beta y}$ (a) find an exponential solution for $4u_{xx}-u_{yy}-2u_x+4u_y=0$; (b) propose and verify a general solution for the equation.

9. Solve the PDE $xu_{xy}+3u_y=y^3$.

10. If $Au_{xx}+Bu_{xy}+Cu_{yy}=F(x, y)$, A, B, and C are constants, then the equation has a general solution

$$u=u_c(x, y)+u_p(x, y)$$

where $u_c(x, y)$ is a general solution of $Au_{xx}+Bu_{xy}+Cu_{yy}=0$ and $u_p(x, y)$ is a particular solution of the original equation. Find a general solution for the following equations:
   (a) $u_{xx}-2u_{xy}+3u_{yy}=e^x$
   (b) $u_{xx}-u_{xy}-2u_{yy}=\sin y.$

1.9. SEPARATION OF VARIABLES

It is assumed in this method that the solution of a PDE can be expressed in the form of a product of functions of single independent variables. Using this procedure we produce an equation with one member a function of a single variable and the other member a function of the remaining variables. Each member can be a constant but not a function of all the original independent variables. This process is illustrated in the following examples.

Example 1.16. Find a solution for the PDE

$$u_t=4u_{xx} \tag{1.48}$$

using the separation of variables.

We assume that the solution of (1.48) has the form

$$u(x,t)=X(x)T(t) \tag{1.49}$$

where X is a function of x alone and T is a function of t alone. Inserting (1.49) into (1.48) we obtain

$$XT'=4X''T$$

After dividing by $4XT$, one has the variables separated in the form

$$\frac{T'}{4T}=\frac{X''}{X} \tag{1.50}$$

If (1.50) is differentiated partially relative to t, one attains the result

$$\frac{\partial}{\partial t}\left(\frac{T'}{4T}\right)=0 \tag{1.51}$$

Assuming ϕ is an arbitrary function of x alone, the solution of (1.51) is

$$\frac{T'}{4T}=\phi(x)$$

This violates the condition that T is a function of t alone unless $\phi(x)$ is a constant. A similar partial differentiation of (1.50) relative to x leads to a PDE which has a solution

$$\frac{X''}{X}=\psi(t)$$

valid only if $\psi(t)$ is constant. Therefore both members of (1.50) must be equal to the same constant, say α^2 or $-\alpha^2$.

If α^2 is used (1.50) becomes

$$\frac{T'}{4T}=\frac{X''}{X}=\alpha^2 \tag{1.52}$$

Result (1.52) is equivalent to two ODEs

$$T'-4\alpha^2T=0$$

$$X''-\alpha^2X=0 \tag{1.53}$$

The solutions of the two ODEs of (1.53) are respectively,

$$T=Ae^{4\alpha^2t}$$

$$X=B_1e^{\alpha x}+B_2e^{-\alpha x} \tag{1.54}$$

Inserting the solutions of (1.54) in (1.49) we find a solution

$$u(x,t)=e^{4\alpha^2t}[C_1e^{\alpha x}+C_2e^{-\alpha x}]$$

Where $C_1=AB_1$ and $C_2=AB_2$.

If $-\alpha^2$ is used instead of α^2 in (1.52) the two ODEs are

$$T'+4\alpha^2T=0$$

$$X''+\alpha^2X=0 \tag{1.55}$$

The solutions of (1.55) are

$$T=A^*e^{-4\alpha^2t}$$

$$X=B_1^*\cos\alpha x+B_2^*\sin\alpha x \tag{1.56}$$

Using the solutions of (1.56) in (1.49) we have

$$u=e^{-4\alpha^2t}[C_1^*\cos\alpha x+C_2^*\sin\alpha x]$$

In most of our BVPs a bounded solution will be necessary. The constants α^2 or $-\alpha^2$ must be selected to satisfy this requirement.

Example 1.17. Determine a solution for

$$u_t=a^2(u_{xx}+u_{yy}) \tag{1.57}$$

Since three independent variables appear in (1.57), we let

$$u(x,y,t)=T(t)X(x)Y(y) \tag{1.58}$$

Equation (1.57) has the form

$$T'XY=a^2(TX''Y+TXY'') \tag{1.59}$$

after substituting (1.58) in the PDE. Equation (1.59) has another form

$$\frac{T'}{a^2T}=\frac{X''}{X}+\frac{Y''}{Y} \tag{1.60}$$

Partially differentiating (1.60) relative to x, then y, and finally t, we have respectively

$$\frac{\partial}{\partial x}\left(\frac{X''}{X}\right)=0$$

$$\frac{\partial}{\partial y}\left(\frac{Y''}{Y}\right)=0 \tag{1.61}$$

$$\frac{\partial}{\partial t}\left(\frac{T'}{a^2T}\right)=0$$

Solutions of the three PDEs of (1.61) are

$$\frac{X''}{X}=-\alpha^2$$

$$\frac{Y''}{Y}=-\beta^2 \tag{1.62}$$

$$\frac{T'}{a^2T}=-(\alpha^2+\beta^2)$$

... that (1.60) be satisfied we select $-(\alpha^2+\beta^2)$ as the constant in the ... of the T-equation.
... three associated ODEs

$$X''+\alpha^2X=0$$

$$Y''+\beta^2Y=0$$

$$T'+(\alpha^2+\beta^2)a^2T=0$$

... solutions

$$X=B_1\cos\alpha x+B_2\sin\alpha x$$

$$Y=C_1\cos\beta y+C_2\sin\beta y$$

$$T=A\exp\left[-(\alpha^2+\beta^2)a^2t\right]$$

Therefore,

$$u=\exp\left[-(\alpha^2+\beta^2)a^2t\right]\left[B_1^*\cos\alpha x+B_2^*\sin\alpha x\right]\left[C_1\cos\beta y+C_2\sin\beta y\right]$$

is a solution of (1.57). Other forms for the solution are available. The one displayed is a bounded solution.

The method of separation of variables is valuable for solving a number of important problems of mathematical physics, yet it fails for many PDEs and BVPs. Myint-U [25, pp. 128–129] shows that the second order PDE* with variable coefficients in x and y

$$A(x,y)u_{xx}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=0 \tag{1.63}$$

is separable when a functional multiplier $1/[\phi(x,y)]$ converts the new equation

$$A(x,y)X''Y+C(x,y)XY''+D(x,y)X'Y+E(x,y)XY'+F(x,y)XY=0$$

into the form

$$A_1(x)X''Y+B_1(y)XY''+A_2(x)X'Y+B_2(y)XY'+\left[A_3(x)+B_3(y)\right]XY=0$$

Explicit rules for the workability of this method are a bit elusive. Types of differential equations, kinds of coordinate systems, and forms of boundary conditions are all important items for the success of the procedure.

*The example that follows is from Myint-U [25], by permission of Elsevier North Holland, Inc.

Exercises 1.4

1(b, c, n, i, k), 2(b, c)

1. Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.
   (a) $u_{xy}-u=0.$
   (b) $u_{tt}-u_{xx}=0.$
   (c) $u_{xx}-u_{yy}-2u_y=0.$
   (d) $u_{xx}-u_{yy}+2u_x-2u_y+u=0.$
   (e) $t^2u_{tt}-x^2u_{xx}=0.$
   (f) $(t^2+x^2)u_{tt}+u_{xx}=0.$
   (g) $u_{xx}-y^2u_{yy}-yu_y=0.$
   (h) $u_{xy}=0.$
   (i) $u_{xx}-u_{xy}+u_{yy}=2x.$
   (j) $u_{xx}=u_{yy}-u_y=0.$
   (k) $u_t=u_{xx}.$

2. Find a solution for the boundary (or initial) value problems:
   (a) $u_{tt}-u_{xx}=0$, $u(x,0)=u(0,t)=0.$
   (b) $u_{xx}-u_{yy}-2u_y=0$, $u_x(0,y)=u(x,0)=0.$
   (c) $u_t=u_{xx}$, $u_x(0,t)=0.$

3. (a) Show that the equation with constant coefficients

$$Au_{xx}+Bu_{xy}+Cu_{yy}=0$$

is separable if the coefficients meet proper conditions. Determine appropriate conditions. *Note*: Let $u(x,y)=X(x)Y(y)$ and show that a result

$$\left(\frac{X''}{X}\right)'+\frac{B}{A}\left(\frac{X'}{X}\right)'\left(\frac{Y'}{Y}\right)=0$$

is obtained from

$$\frac{X''}{X}+\frac{B}{A}\frac{X'}{X}\frac{Y'}{Y}+\frac{C}{A}\frac{Y''}{Y}=0$$

Finally, show that

$$Y'+\lambda Y=0 \quad \text{and} \quad X''-\lambda\frac{B}{A}X'+\lambda^2\frac{C}{A}X=0$$

are related ODEs.

(b) Find a solution for $u_{xx}-u_{xy}+u_{yy}=0$ by separating variables.

CHAPTER 13

CHAPTER CONTENTS

In this and the next two chapters, the emphasis will be on two procedures that are frequently used in solving problems involving temperatures, oscillatory displacements, and potentials. These problems, called **boundary-value problems** (BVPs) are described by relatively simple linear second-order partial differential equations (PDEs). The thrust of both procedures is to find particular solutions of a PDE by reducing it to two or more ordinary differential equations (ODEs).

We begin with the method of **separation of variables** for linear PDEs. This method applied to a boundary-value problem leads naturally back to the important topics of Chapter 12; namely, Sturm–Liouville problems, eigenvalues, eigenfunctions, and the expansion of a function in a series of orthogonal functions.

13.1 Separable Partial Differential Equations

Introduction Partial differential equations (PDEs), like ordinary differential equations (ODEs) are classified as *linear or nonlinear*. Analogous to a linear ODE (see (6) of Section 1.1), the dependent variable and its partial derivatives appear only to the first power in a linear PDE. In this and the chapters that follow, we are concerned only with linear partial differential equations.

Linear Partial Differential Equation If we let u denote the dependent variable and x and y the independent variables, then the general form of a **linear second-order partial differential equation** is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G, \tag{1}$$

where the coefficients A, B, C, …, G are constants or functions of x and y. When $G(x, y) = 0$, equation (1) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

EXAMPLE 1 Linear Second-Order PDEs

The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = xy$$

are examples of linear second-order PDEs. The first equation is homogeneous and the second is nonhomogeneous.

Solution of a PDE A **solution** of a linear partial differential equation (1) is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

It is not our intention to examine procedures for finding *general solutions* of linear partial differential equations. Not only is it often difficult to obtain a general solution of a linear second-order PDE, but a general solution is usually not all that useful in applications. Thus our focus throughout will be on finding *particular solutions* of some of the important linear PDEs, that is, equations that appear in many applications.

We are interested only in particular solutions of PDEs.

Separation of Variables Although there are several methods that can be tried to find particular solutions of a linear PDE, in the **method of separation of variables** we seek to find a particular solution of the form of a *product* of a function of x and a function of y,

$$u(x, y) = X(x)Y(y).$$

With this assumption, it is sometimes possible to reduce a linear PDE in two variables to two ODEs. To this end we observe that

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY'',$$

where the primes denote ordinary differentiation.

EXAMPLE 2 Using Separation of Variables

Find product solutions of $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$.

SOLUTION Substituting $u(x, y) = X(x)Y(y)$ into the partial differential equation yields

$$X''Y = 4XY'.$$

After dividing both sides by $4XY$, we have separated the variables:

$$\frac{X''}{4X} = \frac{Y'}{Y}.$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x and y. In other words, each side of the equation must be a constant. As a practical matter it is convenient to write this real **separation constant** as $-\lambda$. From the two equalities,

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda$$

we obtain the two linear ordinary differential equations

$$X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0. \tag{2}$$

▶ See Example 2, Section 3.9 and Example 1, Section 12.5.

For the three cases for λ: zero, negative, or positive; that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$, the ODEs in (2) are, in turn,

$$X'' = 0 \quad \text{and} \quad Y' = 0, \tag{3}$$
$$X'' - 4\alpha^2 X = 0 \quad \text{and} \quad Y' - \alpha^2 Y = 0, \tag{4}$$
$$X'' + 4\alpha^2 X = 0 \quad \text{and} \quad Y' + \alpha^2 Y = 0. \tag{5}$$

Case I ($\lambda = 0$): The DEs in (3) can be solved by integration. The solutions are $X = c_1 + c_2 x$ and $Y = c_3$. Thus a particular product solution of the given PDE is

$$u = XY = (c_1 + c_2 x)c_3 = A_1 + B_1 x, \tag{6}$$

where we have replaced c_1c_3 and c_2c_3 by A_1 and B_1, respectively.

Case II ($\lambda = -\alpha^2$): The general solutions of the DEs in (4) are

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \quad \text{and} \quad Y = c_6 e^{\alpha^2 y},$$

respectively. Thus, another particular product solution of the PDE is

$$u = XY = (c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x)c_6 e^{\alpha^2 y}$$

or

$$u = A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x, \tag{7}$$

where $A_2 = c_4c_6$ and $B_2 = c_5c_6$.

Case III ($\lambda = \alpha^2$): Finally, the general solutions of the DEs in (5) are

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \quad \text{and} \quad Y = c_9 e^{-\alpha^2 y},$$

respectively. These results give yet another particular solution

$$u = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_3 e^{-\alpha^2 y} \sin 2\alpha x, \tag{8}$$

where $A_3 = c_7c_9$ and $B_3 = c_8c_9$.

≡

It is left as an exercise to verify that (6), (7), and (8) satisfy the given partial differential equation $u_{xx} = 4u_y$. See Problem 29 in Exercises 13.1.

Separation of variables is not a general method for finding particular solutions; some linear partial differential equations are simply not separable. You should verify that the assumption $u = XY$ does not lead to a solution for $\partial^2 u/\partial x^2 - \partial u/\partial y = x$.

Superposition Principle The following theorem is analogous to Theorem 3.1.2 and is known as the superposition principle.

Theorem 13.1.1 Superposition Principle

If $u_1, u_2, \ldots, u_k$ are solutions of a homogeneous linear partial differential equation, then the linear combination

$$u = c_1u_1 + c_2u_2 \ldots + c_ku_k,$$

where the c_i, $i = 1, 2, \ldots, k$ are constants, is also a solution.

Throughout the remainder of the chapter we shall assume that whenever we have an infinite set u_1, u_2, u_3, … of solutions of a homogeneous linear equation, we can construct yet another solution u by forming the infinite series

$$u = \sum_{k=1}^{\infty} c_k u_k$$

where the c_k, $k = 1, 2, \ldots$, are constants.

□ **Classification of Equations** A linear second-order partial differential equation in two independent variables with constant coefficients can be classified as one of three types. This classification depends only on the coefficients of the second-order derivatives. Of course, we assume that at least one of the coefficients A, B, and C is not zero.

Definition 13.1.1 Classification of Equations

The linear second-order partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G,$$

where A, B, C, D, E, F, and G are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$,

parabolic if $B^2 - 4AC = 0$,

elliptic if $B^2 - 4AC < 0$.

EXAMPLE 3 Classifying Linear Second-Order PDEs

Classify the following equations:

$$\text{(a)}\ 3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} \qquad \text{(b)}\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad \text{(c)}\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

SOLUTION **(a)** By rewriting the given equation as

$$3\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

we can make the identifications $A = 3$, $B = 0$, and $C = 0$. Since $B^2 - 4AC = 0$, the equation is **parabolic**.

(b) By rewriting the equation as

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

we see that $A = 1, B = 0, C = -1$, and $B^2 - 4AC = -4(1)(-1) > 0$. The equation is **hyperbolic**.

(c) With $A = 1, B = 0, C = 1$, and $B^2 - 4AC = -4(1)(1) < 0$, the equation is **elliptic**.

A detailed explanation of why we would want to classify a second-order partial differential equation is beyond the scope of this text. But the answer lies in the fact that we wish to solve partial differential equations subject to certain side conditions known as boundary and initial conditions. The kinds of side conditions appropriate for a given equation depend on whether the equation is hyperbolic, parabolic, or elliptic.

13.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-30.

In Problems 1–16, use separation of variables to find, if possible, product solutions for the given partial differential equation.

1. $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y}$

2. $\dfrac{\partial u}{\partial x} + 3\dfrac{\partial u}{\partial y} = 0$

3. $u_x + u_y = u$

4. $u_x = u_y + u$

5. $x\dfrac{\partial u}{\partial x} = y\dfrac{\partial u}{\partial y}$

6. $y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0$

7. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

8. $y\dfrac{\partial^2 u}{\partial x \partial y} + u = 0$

9. $k\dfrac{\partial^2 u}{\partial x^2} - u = \dfrac{\partial u}{\partial t}, \quad k > 0$

10. $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}, \quad k > 0$

11. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$

12. $a^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

13. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + 2k\dfrac{\partial u}{\partial t}, \quad k > 0$

14. $x^2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

15. $u_{xx} + u_{yy} = u$

16. $a^2 u_{xx} - g = u_{tt}, \quad g$ a constant

In Problems 17–26, classify the given partial differential equation as hyperbolic, parabolic, or elliptic.

17. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

18. $3\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

19. $\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} = 0$

20. $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0$

21. $\frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 u}{\partial x \partial y}$

22. $\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0$

23. $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - 6\frac{\partial u}{\partial y} = 0$

24. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$

25. $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

26. $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \ k > 0$

In Problems 27 and 28, show that the given partial differential equation possesses the indicated product solution.

27. $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t};$

$u = e^{-k\alpha^2 t}(c_1 J_0(\alpha r) + c_2 Y_0(\alpha r))$

28. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0;$

$u = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha})$

29. Verify that each of the products $u = X(x)Y(y)$ in (6), (7), and (8) satisfies the second-order PDE in Example 2.

30. Definition 13.1.1 generalizes to linear PDEs with coefficients that are functions of x and y. Determine the regions in the xy-plane for which the equation

$$(xy + 1)\frac{\partial^2 u}{\partial x^2} + (x + 2y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + xy^2 u = 0$$

is hyperbolic, parabolic, or elliptic.

Discussion Problems

In Problems 31 and 32, discuss whether product solutions $u = X(x)Y(y)$ can be found for the given partial differential equation. [*Hint*: Use the superposition principle.]

31. $\frac{\partial^2 u}{\partial x^2} - u = 0$

32. $\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} = 0$

13.2 Classical PDEs and Boundary-Value Problems

Introduction For the remainder of this and the next chapter we shall be concerned with finding product solutions of the second-order partial differential equations

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad k > 0 \tag{1}$$

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

or slight variations of these equations. These classical equations of mathematical physics are known, respectively, as the **one-dimensional heat equation**, the **one-dimensional wave equation**, and **Laplace's equation in two dimensions**. "One-dimensional" refers to the fact that x denotes a spatial dimension whereas t represents time; "two dimensional" in (3) means that x and y are both spatial dimensions. Laplace's equation is abbreviated $\nabla^2 u = 0$, where

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is called the **two-dimensional Laplacian** of the function u. In three dimensions the **Laplacian** of u is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

By comparing equations (1)–(3) with the linear second-order PDE given in Definition 13.1.1, with t playing the part of y, we see that the heat equation (1) is parabolic, the wave equation (2) is hyperbolic, and Laplace's equation (3) is elliptic. This classification is important in Chapter 16.

 Heat Equation Equation (1) occurs in the theory of heat flow—that is, heat transferred by conduction in a rod or thin wire. The function $u(x, t)$ is temperature. Problems in mechanical vibrations often lead to the wave equation (2). For purposes of discussion, a solution $u(x, t)$ of (2) will represent the displacement of an idealized string. Finally, a solution $u(x, y)$ of Laplace's equation (3) can be interpreted as the steady-state (that is, time-independent) temperature distribution throughout a thin, two-dimensional plate.

Even though we have to make many simplifying assumptions, it is worthwhile to see how equations such as (1) and (2) arise.

Suppose a thin circular rod of length L has a cross-sectional area A and coincides with the x-axis on the interval $[0, L]$. See **FIGURE 13.2.1**. Let us suppose:

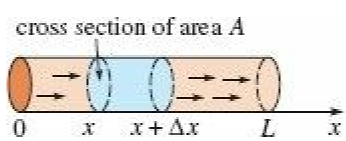


FIGURE 13.2.1 One-dimensional flow of heat

- The flow of heat within the rod takes place only in the x-direction.
- The lateral, or curved, surface of the rod is insulated; that is, no heat escapes from this surface.
- No heat is being generated within the rod.
- The rod is homogeneous; that is, its mass per unit volume ρ is a constant.
- The specific heat γ and thermal conductivity K of the material of the rod are constants.

To derive the partial differential equation satisfied by the temperature $u(x, t)$, we need two empirical laws of heat conduction:

(*i*) *The quantity of heat Q in an element of mass m is*

$$Q = \gamma m u, \tag{4}$$

where u is the temperature of the element.

(ii) *The rate of heat flow Q_t through the cross section indicated in Figure 13.2.1 is proportional to the area A of the cross section and the partial derivative with respect to x of the temperature:*

$$Q_t = -KAu_x. \tag{5}$$

Since heat flows in the direction of decreasing temperature, the minus sign in (5) is used to ensure that Q_t is positive for $u_x < 0$ (heat flow to the right) and negative for $u_x > 0$ (heat flow to the left). If the circular slice of the rod shown in Figure 13.2.1 between x and $x + \Delta x$ is very thin, then $u(x, t)$ can be taken as the approximate temperature at each point in the interval. Now the mass of the slice is $m = \rho(A\,\Delta x)$, and so it follows from (4) that the quantity of heat in it is

$$Q = \gamma \rho A\,\Delta x\, u. \tag{6}$$

Furthermore, when heat flows in the positive x-direction, we see from (5) that heat builds up in the slice at the net rate

$$-KAu_x(x, t) - [-KAu_x(x + \Delta x, t)] = KA[u_x(x + \Delta x, t) - u_x(x, t)]. \tag{7}$$

By differentiating (6) with respect to t we see that this net rate is also given by

$$Q_t = \gamma \rho A\,\Delta x\, u_t. \tag{8}$$

Equating (7) and (8) gives

$$\frac{K}{\gamma\rho}\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} = u_t. \tag{9}$$

Taking the limit of (9) as $\Delta x \to 0$ finally yields (1) in the form*

$$\frac{K}{\gamma\rho} u_{xx} = u_t.$$

It is customary to let $k = K/\gamma\rho$ and call this positive constant the **thermal diffusivity**.

□ **Wave Equation** Consider a string of length L, such as a guitar string, stretched taut between two points on the x-axis—say, $x = 0$ and $x = L$. When the string starts to vibrate, assume that the motion takes place in the xy-plane in such a manner that each point on the string moves in a direction perpendicular to the x-axis (transverse vibrations). As shown in **FIGURE 13.2.2(a)**, let $u(x, t)$ denote the vertical displacement of any point on the string measured from the x-axis for $t > 0$. We further assume:

- The string is perfectly flexible.
- The string is homogeneous; that is, its mass per unit length ρ is a constant.
- The displacements u are small compared to the length of the string.

- The slope of the curve is small at all points.
- The tension **T** acts tangent to the string, and its magnitude T is the same at all points.
- The tension is large compared with the force of gravity.
- No other external forces act on the string.

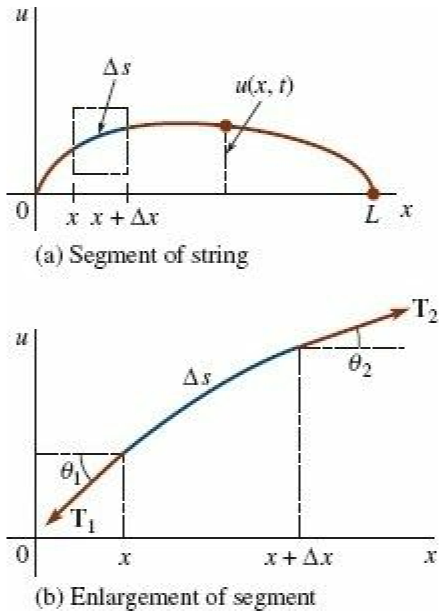


FIGURE 13.2.2 Taut string anchored at two points on the x-axis

Now in Figure 13.2.2(b) the tensions $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent to the ends of the curve on the interval $[x, x+\Delta x]$. For small values of θ_1 and θ_2 the net vertical force acting on the corresponding element Δs of the string is then

$$T\sin\theta_2 - T\sin\theta_1 \approx T\tan\theta_2 - T\tan\theta_1$$
$$= T[u_x(x+\Delta x, t) - u_x(x, t)],^{\dagger}$$

where $T = |\mathbf{T}_1| = |\mathbf{T}_2|$. Now $\rho\,\Delta s \approx \rho\,\Delta x$ is the mass of the string on $[x, x+\Delta x]$, and so Newton's second law gives

$$T[u_x(x+\Delta x, t) - u_x(x, t)] = \rho\,\Delta x\, u_{tt}$$

or

$$\frac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x} = \frac{\rho}{T} u_{tt}.$$

If the limit is taken as $\Delta x \to 0$, the last equation becomes $u_{xx} = (\rho/T)u_{tt}$. This of course is (2) with $a^2 = T/\rho$.

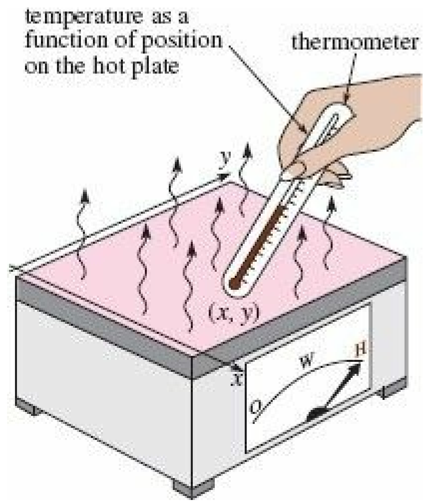


FIGURE 13.2.3 Steady-state temperatures in a rectangular plate

□ **Laplace's Equation** Although we shall not present its derivation, Laplace's equation in two and three dimensions occurs in time-independent problems involving potentials such as electrostatic, gravitational, and velocity in fluid mechanics. Moreover, a solution of Laplace's equation can also be interpreted as a steady-state temperature distribution. As illustrated in **FIGURE 13.2.3** a solution $u(x, y)$ of (3) could represent the temperature that varies from point to point—but not with time—of a rectangular plate.

We often wish to find solutions of equations (1), (2), and (3) that satisfy certain side conditions.

□ **Initial Conditions** Since solutions of (1) and (2) depend on time t, we can prescribe what happens at $t = 0$; that is, we can give **initial conditions (IC)**. If $f(x)$ denotes the initial temperature distribution throughout the rod in Figure 13.2.1, then a solution $u(x, t)$ of (1) must satisfy the single initial condition $u(x, 0) = f(x), 0 < x < L$. On the other hand, for a vibrating string, we can specify its initial displacement (or shape) $f(x)$ as well as its initial velocity $g(x)$. In mathematical terms we seek a function $u(x, t)$ satisfying (2) and the two initial conditions:

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{10}$$

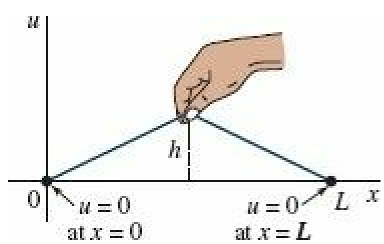


FIGURE 13.2.4 Plucked string

For example, the string could be plucked, as shown in **FIGURE 13.2.4**, and released from rest ($g(x) = 0$).

□ **Boundary Conditions** The string in Figure 13.2.4 is secured to the x-axis at $x = 0$ and $x = L$ for all time. We interpret this by the two **boundary conditions (BC)**:

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t > 0.$$

Note that in this context the function f in (10) is continuous, and consequently $f(0) = 0$ and $f(L) = 0$. In general, there are three types of boundary conditions associated with equations (1), (2), and (3). On a boundary we can specify the values of one of the following:

$$(i)\ u, \qquad (ii)\ \frac{\partial u}{\partial n}, \qquad \text{or} \qquad (iii)\ \frac{\partial u}{\partial n} + hu, \quad h \text{ a constant.}$$

Here $\partial u/\partial n$ denotes the normal derivative of u (the directional derivative of u in the direction perpendicular to the boundary). A boundary condition of the first type (i) is called a **Dirichlet condition**; a boundary condition of the second type (ii) is called a **Neumann condition**; and a boundary condition of the third type (iii) is known as a **Robin condition**. For example, for $t > 0$ a typical condition at the right-hand end of the rod in Figure 13.2.1 can be

$$(i)' \quad u(L, t) = u_0, \quad u_0 \text{ a constant,}$$

$$(ii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \quad \text{or}$$

$$(iii)' \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = -h(u(L, t) - u_m), \quad h > 0 \text{ and } u_m \text{ constants.}$$

Condition $(i)'$ simply states that the boundary $x = L$ is held by some means at a constant *temperature* u_0 for all time $t > 0$. Condition $(ii)'$ indicates that the boundary $x = L$ is *insulated*. From the empirical law of heat transfer, the flux of heat across a boundary (that is, the amount of heat per unit area per unit time conducted across the boundary) is proportional to the value of the normal derivative $\partial u/\partial n$ of the temperature u. Thus when the boundary $x = L$ is thermally insulated, no heat flows into or out of the rod and so

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

We can interpret $(iii)'$ to mean that *heat is lost* from the right-hand end of the rod by being in contact with a medium, such as air or water, that is held at a constant temperature. From Newton's law of cooling, the outward flux of heat from the rod is proportional to the difference between the temperature $u(L, t)$ at the boundary and the temperature u_m of the surrounding medium. We note that if heat is lost from the left-hand end of the rod, the boundary condition is

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = h(u(0, t) - u_m).$$

The change in algebraic sign is consistent with the assumption that the rod is at a higher temperature than the medium surrounding the ends so that $u(0, t) > u_m$ and $u(L, t) > u_m$. At $x = 0$ and $x = L$, the slopes $u_x(0, t)$ and $u_x(L, t)$ must be positive and negative, respectively.

Of course, at the ends of the rod we can specify different conditions at the same time. For example, we could have

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad u(L, t) = u_0, \quad t > 0.$$

We note that the boundary condition in $(i)'$ is homogeneous if $u_0 = 0$; if $u_0 \neq 0$, the boundary condition is nonhomogeneous. The boundary condition $(ii)'$ is homogeneous; $(iii)'$ is homogeneous if $u_m = 0$ and nonhomogeneous if $u_m \neq 0$.

□ **Boundary-Value Problems** Problems such as

$$\begin{aligned}
&\textit{Solve:} && a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ 0<x<L,\ t>0 \\
&\textit{Subject to:} && \text{(BC)}\ u(0,t)=0,\ u(L,t)=0,\ t>0 \\
& && \text{(IC)}\ u(x,0)=f(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),\ 0<x<L
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\textit{Solve:} && \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0<x<a,\ 0<y<b \\
&\textit{Subject to:} && \text{(BC)} \begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, & \left.\dfrac{\partial u}{\partial x}\right|_{x=a} = 0,\ 0<y<b \\ u(x,0)=0, & u(x,b)=f(x),\ 0<x<a \end{cases}
\end{aligned} \tag{12}$$

are called **boundary-value problems**. The problems in (11) and (12) are classified as **homogeneous BVPs** since the partial differential equation and the boundary conditions are homogeneous.

□ **Variations** The partial differential equations (1), (2), and (3) must be modified to take into consideration internal or external influences acting on the physical system. More general forms of the one-dimensional heat and wave equations are, respectively,

$$k\frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_x) = \frac{\partial u}{\partial t} \tag{13}$$

and

$$a^2\frac{\partial^2 u}{\partial x^2} + F(x,t,u,u_t) = \frac{\partial^2 u}{\partial t^2}. \tag{14}$$

For example, if there is heat transfer from the lateral surface of a rod into a surrounding medium that is held at a constant temperature u_m, then the heat equation (13) is

$$k\frac{\partial^2 u}{\partial x^2} - h(u-u_m) = \frac{\partial u}{\partial t},$$

where h is a constant. In (14) the function F could represent the various forces acting on the string. For example, when external, damping, and elastic restoring forces are taken into account, (14) assumes the form
external force

$$a^2\frac{\partial^2 u}{\partial x^2} + \underbrace{\overset{\text{external force}}{f(x,t)} - c\overset{\text{damping}}{\frac{\partial u}{\partial t}} - \overset{\text{restoring force}}{ku}}_{F(x,t,u,u_t)} = \frac{\partial^2 u}{\partial t^2}. \tag{15}$$

Remarks

The analysis of a wide variety of diverse phenomena yields the mathematical models (1), (2), or (3) or their generalizations involving a greater number of spatial variables. For example, (1) is sometimes called the **diffusion equation** since the diffusion of dissolved substances in solution is analogous to the flow of heat in a solid. The function $c(x, t)$ satisfying the partial differential equation in this case represents the concentration of the dissolved substance. Similarly, equation (2) and its generalization (15) arise in the analysis of the flow of electricity in a long cable or transmission line. In this setting (2) is known as the **telegraph equation**. It can be shown that under certain assumptions the current $i(x, t)$ and the voltage $v(x, t)$ in the line satisfy two partial differential equations identical to (2) (or (15)). The wave equation (2) also appears in fluid mechanics, acoustics, and elasticity. Laplace's equation (3) is encountered in determining the static displacement of membranes.

13.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1–6, a rod of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the temperature $u(x, t)$.

1. The left end is held at temperature zero, and the right end is insulated. The initial temperature is $f(x)$ throughout.

2. The left end is held at temperature u_0, and the right end is held at temperature u_1. The initial temperature is zero throughout.

3. The left end is held at temperature 100°, and there is heat transfer from the right end into the surrounding medium at temperature zero. The initial temperature is $f(x)$ throughout.

4. There is heat transfer from the left end into a surrounding medium at temperature 20°, and the right end is insulated. The initial temperature is $f(x)$ throughout.

5. The left end is at temperature $\sin(\pi t/L)$, the right end is held at zero, and there is heat transfer from the lateral surface of the rod into the surrounding medium held at temperature zero. The initial temperature is $f(x)$ throughout.

6. The ends are insulated, and there is heat transfer from the lateral surface of the rod into the surrounding medium held at temperature 50°. The initial temperature is 100° throughout.

In Problems 7–10, a string of length L coincides with the interval $[0, L]$ on the x-axis. Set up the boundary-value problem for the displacement $u(x, t)$.

7. The ends are secured to the x-axis. The string is released from rest from the initial displacement $x(L - x)$.

8. The ends are secured to the x-axis. Initially the string is undisplaced but has the initial velocity $\sin(\pi x/L)$.

9. The left end is secured to the x-axis, but the right end moves in a transverse manner according to $\sin \pi t$. The string is released from rest from the initial displacement $f(x)$. For $t > 0$ the transverse vibrations are damped with a force proportional to the instantaneous velocity.

10. The ends are secured to the x-axis, and the string is initially at rest on that axis. An external vertical force proportional to the horizontal distance from the left end acts on the string for $t > 0$.

In Problems 11 and 12, set up the boundary-value problem for the steady-state temperature $u(x, y)$.

11. A thin rectangular plate coincides with the region in the xy-plane defined by $0 \le x \le 4$, $0 \le y \le 2$. The left end and the bottom of the plate are insulated. The top of the plate is held at temperature zero, and the right end of the plate is held at temperature $f(y)$.

12. A semi-infinite plate coincides with the region defined by $0 \le x \le \pi$, $y \ge 0$. The left end is held at temperature e^{-y}, and the right end is held at temperature 100° for $0 < y \le 1$ and temperature zero for $y > 1$. The bottom of the plate is held at temperature $f(x)$.

13.3 Heat Equation

Introduction Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. If the rod shown in **FIGURE 13.3.1** satisfies the assumptions given on page 693, then the temperature $u(x, t)$ in the rod is determined from the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \ t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad 0 < x < L. \tag{3}$$

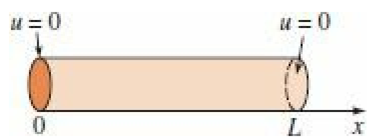


FIGURE 13.3.1 Find the temperature u in a finite rod

In the discussion that follows next we show how to solve this BVP using the method of separation of variables introduced in Section 13.1.

Solution of the BVP Using the product $u(x, t) = X(x)T(t)$, and $-\lambda$ as the separation constant, leads to

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \tag{4}$$

and

$$X'' + \lambda X = 0 \tag{5}$$

$$T' + k\lambda T = 0. \tag{6}$$

Now the boundary conditions in (2) become $u(0, t) = X(0)T(t) = 0$ and $u(L, t) = X(L)T(t) = 0$. Since the last equalities must hold for all time t, we must have $X(0) = 0$ and $X(L) = 0$. These homogeneous boundary conditions together with the homogeneous ODE (5) constitute a regular Sturm–Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{7}$$

The solution of this BVP was discussed in detail in Example 2 of Section 3.9 and on page 675 of Section 12.5. In that example, we considered three possible cases for the parameter λ: zero, negative, and positive. The corresponding general solutions of the DEs are

$$X(x) = c_1 + c_2x, \qquad \lambda = 0 \tag{8}$$

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \quad \lambda = -\alpha^2 < 0 \tag{9}$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \qquad \lambda = \alpha^2 > 0. \tag{10}$$

Recall, when the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to (8) and (9) these solutions yield only $X(x) = 0$ and so we are left with the unusable result $u = 0$. Applying the first boundary condition $X(0) = 0$ to the solution in (10) gives $c_1 = 0$. Therefore $X(x) = c_2 \sin \alpha x$. The second boundary condition $X(L) = 0$ now implies

$$X(L) = c_2 \sin \alpha L = 0. \tag{11}$$

If $c_2 = 0$, then $X = 0$ so that $u = 0$. But (11) can be satisfied for $c_2 \neq 0$ when $\sin \alpha L = 0$. This last equation implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$, where $n = 1, 2, 3, \ldots$. Hence (7) possesses nontrivial solutions when $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2, n = 1, 2, 3, \ldots$. The values λ_n and the corresponding solutions

$$X(x) = c_2 \sin \frac{n\pi}{L}x, \quad n = 1, 2, 3, \ldots \tag{12}$$

are the **eigenvalues** and **eigenfunctions**, respectively, of the problem in (7).

The general solution of (6) is $T(t) = c_3e^{-k(n^2\pi^2/L^2)t}$, and so

$$u_n = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L}x, \tag{13}$$

where we have replaced the constant c_2c_3 by A_n. The products $u_n(x, t)$ given in (13) satisfy the partial differential equation (1) as well as the boundary conditions (2) for each value of the positive integer n. However, in order for the functions in (13) to satisfy the initial condition (3), we would have to choose the coefficient A_n in such a manner that

$$u_n(x, 0) = f(x) = A_n \sin \frac{n\pi}{L}x. \tag{14}$$

In general, we would not expect condition (14) to be satisfied for an arbitrary, but reasonable, choice of f. Therefore we are forced to admit that $u_n(x, t)$ *is not a solution of the problem given in* (1)–(3). Now by the superposition principle the function

$$u(x, t) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L}x \tag{15}$$

must also, although formally, satisfy equation (1) and the conditions in (2). If we substitute $t = 0$ into (15), then

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x.$$

This last expression is recognized as the half-range expansion of f in a sine series. If we make the identification $A_n = b_n$, $n = 1, 2, 3, \ldots$, it follows from (5) of Section 12.3 that

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x \, dx. \tag{16}$$

We conclude that a solution of the boundary-value problem described in (1), (2), and (3) is given by the infinite series

$$u(x, t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x. \tag{17}$$

In the special case when the initial temperature is $u(x, 0) = 100$, $L = \pi$, and $k = 1$, you should verify that the coefficients (16) are given by

$$A_n = \frac{200}{\pi}\left[\frac{1 - (-1)^n}{n}\right],$$

and that the series (17) is

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1 - (-1)^n}{n}\right]e^{-n^2 t}\sin nx. \tag{18}$$

Use of Computers The solution u in (18) is a function of two variables and as such its graph is a surface in 3-space. We could use the 3D-plot application of a computer algebra system to approximate this surface by graphing partial sums $S_n(x, t)$ over a rectangular region defined by $0 \le x \le \pi, 0 \le t \le T$. Alternatively, with the aid of the 2D-plot application of a CAS we plot the solution $u(x, t)$ on the x-interval $[0, \pi]$ for increasing values of time t. See **FIGURE 13.3.2(a)** In Figure 13.3.2(b) the solution $u(x, t)$ is graphed on the t-interval [0, 6] for increasing values of x ($x = 0$ is the left end and $x = \pi/2$ is the midpoint of the rod of length $L = \pi$). Both sets of graphs verify that which is apparent in (18)—namely, $u(x, t) \to 0$ as $t \to \infty$.

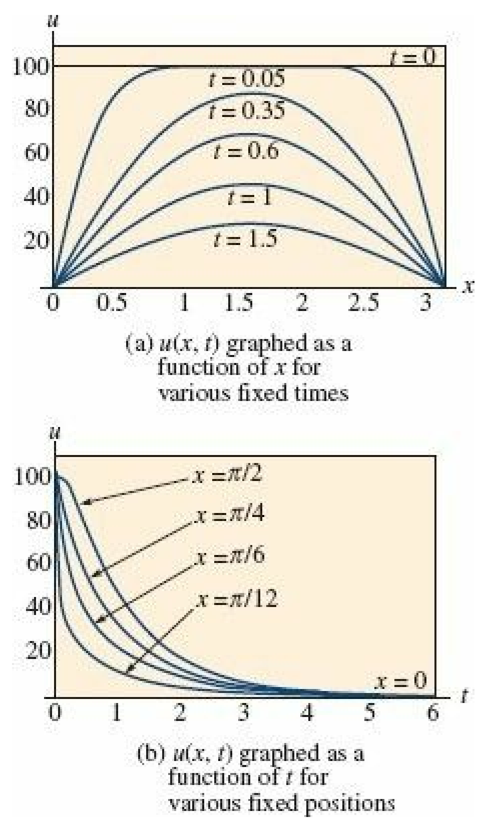


FIGURE 13.3.2 Graphs obtained using partial sums of (18)

13.3	**Exercises** Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1 and 2, solve the heat equation (1) subject to the given conditions. Assume a rod of length L.

1. $u(0, t) = 0, \quad u(L, t) = 0$
$$u(x, 0) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$
2. $u(0, t) = 0, \quad u(L, t) = 0$
$u(x, 0) = x(L - x)$
3. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.
4. Solve Problem 3 if $L = 2$ and
$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$
5. Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form
$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \quad 0 < x < L,\ t > 0,$$
h a constant. Find the temperature $u(x, t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated. See **FIGURE 13.3.3**.

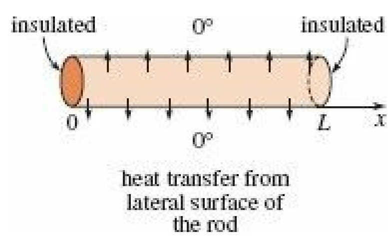


FIGURE 13.3.3 Rod in Problem 5

6. Solve Problem 5 if the ends $x = 0$ and $x = L$ are held at temperature zero.
7. A thin wire coinciding with the x-axis on the interval $[-L, L]$ is bent into the shape of a circle so that the ends $x = -L$ and $x = L$ are joined. Under certain conditions the temperature $u(x, t)$ in the wire satisfies the boundary-value problem
$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -L < x < L,\ t > 0,$$
$$u(-L, t) = u(L, t),\ t > 0$$
$$\left.\frac{\partial u}{\partial x}\right|_{x=-L} = \left.\frac{\partial u}{\partial x}\right|_{x=L},\ t > 0$$
$$u(x, 0) = f(x),\ -L < x < L,$$
Find the temperature $u(x, t)$.
8. Find the temperature $u(x, t)$ for the boundary-value problem (1)–(3) when $L = 1$ and $f(x) = 100 \sin 6\pi x$. [*Hint*: Look closely at (13) and (14).]

Computer Lab Assignments

9. (a) Solve the heat equation (1) subject to

$$u(0, t) = 0, \quad u(100, t) = 0, \quad t > 0$$

$$u(x, 0) = \begin{cases} 0.8x, & 0 \le x \le 50 \\ 0.8(100 - x), & 50 < x \le 100. \end{cases}$$

(b) Use the 3D-plot application of your CAS to graph the partial sum $S_5(x, t)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le 100$, $0 \le t \le 200$. Assume that $k = 1.6352$. Experiment with various three-dimensional viewing perspectives of the surface (called the **ViewPoint** option in *Mathematica*).

Discussion Problems

10. In Figure 13.3.2(b) we have the graphs of $u(x, t)$ on the interval $[0, 6]$ for $x = 0$, $x = \pi/12$, $x = \pi/6$, $x = \pi/4$, and $x = \pi/2$. Describe or sketch the graphs of $u(x, t)$ on the same time interval but for the fixed values $x = 3\pi/4$, $x = 5\pi/6$, $x = 11\pi/12$, and $x = \pi$.

13.4 Wave Equation

Introduction We are now in a position to solve the boundary-value problem (11) discussed in Section 13.2. The vertical displacement $u(x, t)$ of a string of length L that is freely vibrating in the vertical plane shown in Figure 13.2.2(a) is determined from

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < L, \ t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{3}$$

Solution of the BVP With the usual assumption that $u(x, t) = X(x)T(t)$, separating variables in (1) gives

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

so that

$$X'' + \lambda X = 0 \tag{4}$$

$$T'' + a^2 \lambda T = 0. \tag{5}$$

As in Section 13.3, the boundary conditions (2) translate into $X(0) = 0$ and $X(L) = 0$. The ODE in (4) along with these boundary-conditions is the regular Sturm–Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{6}$$

Of the usual three possibilities for the parameter λ: $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, only the last choice leads to nontrivial solutions. Corresponding to $\lambda = \alpha^2$, $\alpha > 0$, the general solution of (4) is

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

$X(0) = 0$ and $X(L) = 0$ indicate that $c_1 = 0$ and $c_2 \sin \alpha L = 0$. The last equation again implies that $\alpha L = n\pi$ or $\alpha = n\pi/L$. The eigenvalues and corresponding eigenfunctions of (6) are $\lambda_n = n^2\pi^2/L^2$ and $X(x) = c_2 \sin \frac{n\pi}{L}x$, $n = 1, 2, 3, \ldots$. The general solution of the second-order equation (5) is then

$$T(t) = c_3 \cos \frac{n\pi a}{L}t + c_4 \sin \frac{n\pi a}{L}t.$$

By rewriting c_2c_3 as A_n and c_2c_4 as B_n, solutions that satisfy both the wave equation (1) and boundary conditions (2) are

$$u_n = \left(A_n \cos \frac{n\pi a}{L}t + B_n \sin \frac{n\pi a}{L}t\right) \sin \frac{n\pi}{L}x \tag{7}$$

and

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos \frac{n\pi a}{L}t + B_n \sin \frac{n\pi a}{L}t\right) \sin \frac{n\pi}{L}x. \tag{8}$$

Setting $t = 0$ in (8) and using the initial condition $u(x, 0) = f(x)$ gives

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L}x.$$

Since the last series is a half-range expansion for f in a sine series, we can write $A_n = b_n$:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}x\, dx. \tag{9}$$

To determine B_n we differentiate (8) with respect to t and then set $t = 0$:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-A_n \frac{n\pi a}{L} \sin \frac{n\pi a}{L}t + B_n \frac{n\pi a}{L} \cos \frac{n\pi a}{L}t\right) \sin \frac{n\pi}{L}x$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = \sum_{n=1}^{\infty}\left(B_n \frac{n\pi a}{L}\right) \sin \frac{n\pi}{L}x.$$

In order for this last series to be the half-range sine expansion of the initial velocity g on the interval, the *total* coefficient $B_n n\pi a/L$ must be given by the form b_n in (5) of Section 12.3—that is,

$$B_n \frac{n\pi a}{L} = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi}{L}x\, dx$$

from which we obtain

$$B_n = \frac{2}{n\pi a}\int_0^L g(x) \sin \frac{n\pi}{L}x\, dx. \tag{10}$$

The solution of the boundary-value problem (1)–(3) consists of the series (8) with coefficients A_n and B_n defined by (9) and (10), respectively.

We note that when the string is released from rest, then $g(x) = 0$ for every x in the interval $[0, L]$ and consequently $B_n = 0$.

□ **Plucked String** A special case of the boundary-value problem in (1)–(3) when $g(x) = 0$ is a model of a **plucked string**. We can see the motion of the string by plotting the solution or displacement $u(x, t)$ for increasing values of time t and using the animation feature of a CAS. Some frames of a movie generated in this manner are given in **FIGURE 13.4.1** You are asked to emulate the results given in the figure by plotting a sequence of partial sums of (8). See Problems 7, 8, and 27 in Exercises 13.4.

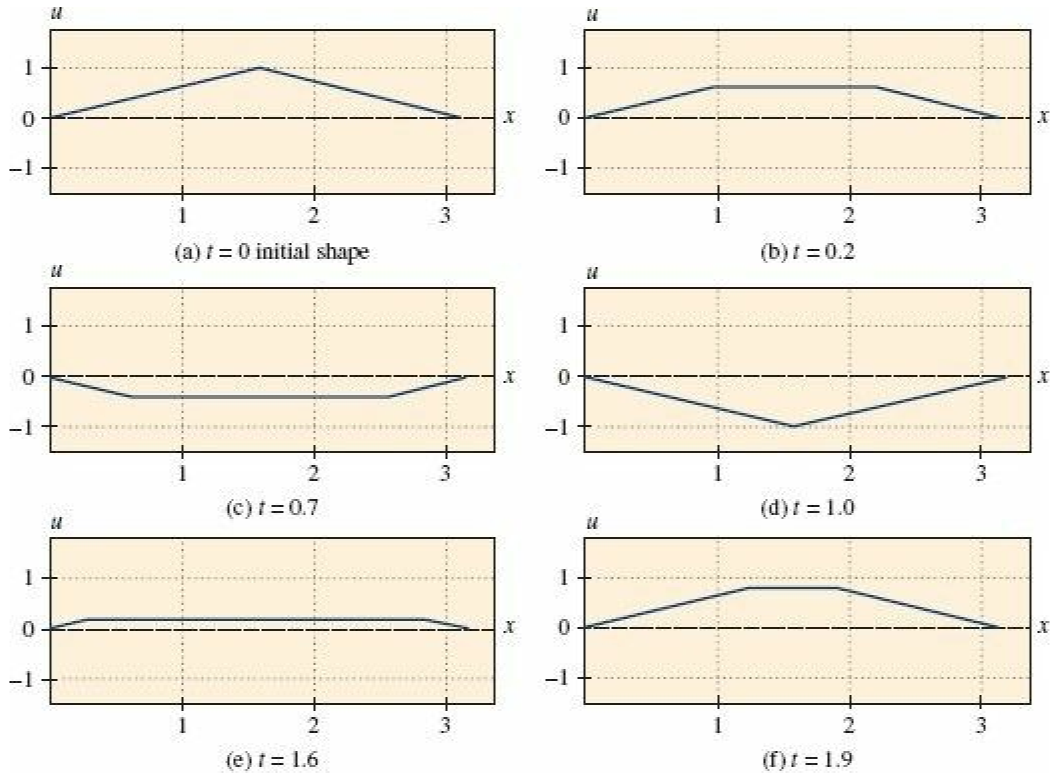


FIGURE 13.4.1 Frames of plucked-string movie

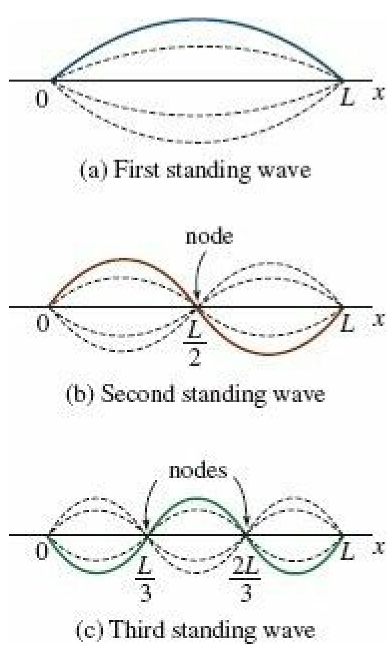


FIGURE 13.4.2 First three standing waves

□ **Standing Waves** Recall from the derivation of the wave equation in Section 13.2 that the constant a appearing in the solution of the boundary-value problem in (1)–(3) is given by $\sqrt{T/\rho}$, where ρ is mass per unit length and T is the magnitude of the tension in the string. When T is large enough, the vibrating string produces a musical sound. This sound is the result of standing waves. The solution (8) is a superposition of product solutions called **standing waves** or **normal modes**:

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots.$$

In view of (6) and (7) of Section 3.8, the product solutions (7) can be written as

$$u_n(x, t) = C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right)\sin\frac{n\pi}{L}x, \tag{11}$$

where $C_n = \sqrt{A_n^2 + B_n^2}$ and ϕ_n is defined by $\sin\phi_n = A_n/C_n$ and $\cos\phi_n = B_n/C_n$. For $n = 1, 2, 3, \ldots$ the standing waves are essentially the graphs of $\sin(n\pi x/L)$, with a time-varying amplitude given by

$$C_n \sin\left(\frac{n\pi a}{L}t + \phi_n\right).$$

Alternatively, we see from (11) that at a fixed value of x each product function $u_n(x, t)$ represents simple harmonic motion with amplitude $C_n|\sin(n\pi x/L)|$ and frequency $f_n = na/2L$. In other words, each point on a standing wave vibrates with a different amplitude but with the same frequency. When $n = 1$,

$$u_1(x, t) = C_1 \sin\left(\frac{\pi a}{L}t + \phi_1\right)\sin\frac{\pi}{L}x$$

is called the **first standing wave, the first normal mode**, or the **fundamental mode of vibration**. The first three standing waves, or normal modes, are shown in **FIGURE 13.4.2** The dashed graphs represent the standing waves at various values of time. The points in the interval $(0, L)$, for which $\sin(n\pi/L)x = 0$, correspond to points on a standing wave where there is no motion. These points are called **nodes**. For example, in Figures 13.4.2(b) and (c) we see that the second standing wave has one node at $L/2$ and the third standing wave has two nodes at $L/3$ and $2L/3$. In general, the nth normal mode of vibration has $n - 1$ nodes.

The frequency

$$f_1 = \frac{a}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

of the first normal mode is called the **fundamental frequency** or **first harmonic** and is directly related to the pitch produced by a stringed instrument. It is apparent that the greater the tension on the string, the higher the pitch of the sound. The frequencies f_n of the other normal modes, which are integer multiples of the fundamental frequency, are called overtones. The second harmonic is the first **overtone**, and so on.

□ **Superposition Principle** The superposition principle, Theorem 13.1.1, is the key in making the method of separation of variables an effective means of solving certain kinds of boundary-value problems involving linear partial differential equations. Sometimes a problem can also be solved by using a superposition of solutions of two easier problems. If we can solve each of the problems, Problem 1 | Problem 2

$$a^2\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}, \quad 0 < x < L, \quad t > 0 \qquad a^2\frac{\partial^2 u_2}{\partial x^2} = \frac{\partial^2 u_2}{\partial t^2}, \quad 0 < x < L, \quad t > 0$$

$$u_1(0, t) = 0, \quad u_1(L, t) = 0, \quad t > 0 \qquad u_2(0, t) = 0, \quad u_2(L, t) = 0, \quad t > 0 \tag{12}$$

$$u_1(x, 0) = f(x), \quad \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L \qquad u_2(x, 0) = 0, \quad \left.\frac{\partial u_2}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L$$

then a solution of (1)–(3) is given by $u(x, t) = u_1(x, t) + u_2(x, t)$. To see this we know that $u(x, t) = u_1(x, t) + u_2(x, t)$ is a solution of the homogeneous equation in (1) because of Theorem 13.1.1. Moreover, $u(x, t)$ satisfies the boundary condition (2) and the initial conditions (3) because, in turn,

$$\text{BC}\begin{cases} u(0, t) = u_1(0, t) + u_2(0, t) = 0 + 0 = 0 \\ u(L, t) = u_1(L, t) + u_2(L, t) = 0 + 0 = 0, \end{cases}$$

and

$$\text{IC}\begin{cases} u(x, 0) = u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x) \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \left.\dfrac{\partial u_1}{\partial t}\right|_{t=0} + \left.\dfrac{\partial u_2}{\partial t}\right|_{t=0} = 0 + g(x) = g(x). \end{cases}$$

You are encouraged to try this method to obtain (8), (9), and (10). See Problems 5 and 14 in Exercises 13.4.

13.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1–6, solve the wave equation (1) subject to the given conditions.

1. $u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$

   $u(x, 0) = \frac{1}{4}x(L - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < L$

2. $u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$

   $u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(L - x), \quad 0 < x < L$

3. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin x, \quad 0 < x < \pi$

4. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = \frac{1}{6}x(\pi^2 - x^2), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$

5. $u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$

   $u(x, 0) = x(1 - x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(1 - x), \quad 0 < x < 1$

6. $u(0, t) = 0, \quad u(\pi, t) = 0, \quad t > 0$

   $u(x, 0) = 0.01 \sin 3\pi x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$

In Problems 7–10, a string is tied to the x-axis at $x = 0$ and at $x = L$ and its initial displacement

$u(x, 0) = f(x)$, $0 < x < L$, is shown in the figure. Find $u(x, t)$ if the string is released from rest.

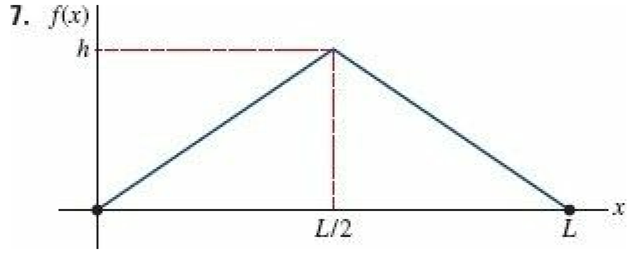


FIGURE 13.4.3 Initial displacement for Problem 7

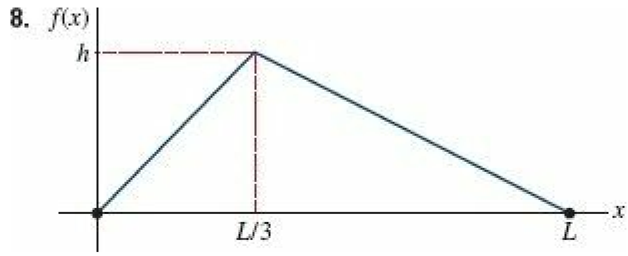


FIGURE 13.4.4 Initial displacement for Problem 8

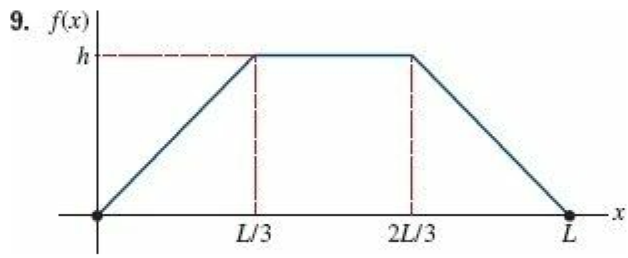


FIGURE 13.4.5 Initial displacement for Problem 9

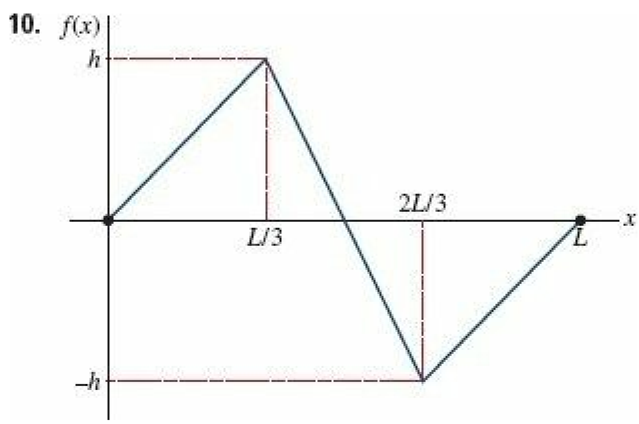


FIGURE 13.4.6 Initial displacement for Problem 10

11. The longitudinal displacement of a vibrating elastic bar shown in **FIGURE 13.4.7** satisfies the wave equation (1) and the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \; t > 0$$

$$u(x, 0) = x, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \; 0 < x < L.$$

The boundary conditions at $x = 0$ and $x = L$ are called **free-end conditions**. Find the displacement $u(x, t)$.

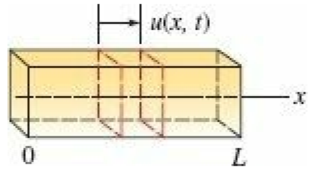


FIGURE 13.4.7 Elastic bar in Problem 11

12. A model for the motion of a vibrating string whose ends are allowed to slide on frictionless sleeves attached to the vertical axes $x = 0$ and $x = L$ is given by the wave equation (1) and the conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0, \ t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \ 0 < x < L.$$

See **FIGURE 13.4.8** The boundary conditions indicate that the motion is such that the slope of the curve is zero at its ends for $t > 0$. Find the displacement $u(x, t)$.

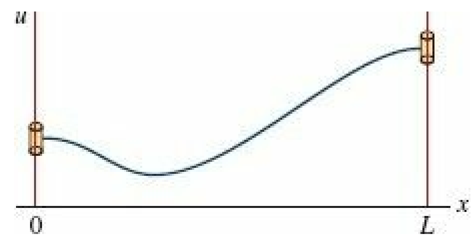


FIGURE 13.4.8 String whose ends are attached to frictionless sleeves in Problem 12

13. In Problem 10, determine the value of $u(L/2, t)$ for $t \geq 0$.

14. Rederive the results given in (8), (9), and (10), but this time use the superposition principle discussed on page 703.

15. A string is stretched and secured on the x-axis at $x = 0$ and $x = \pi$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the wave equation takes on the form

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + 2\beta\frac{\partial u}{\partial t}, \quad 0 < \beta < 1, \quad t > 0.$$

Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.

16. Show that a solution of the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + u, \quad 0 < x < \pi, \ t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \ t > 0$$

$$u(x, 0) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad 0 < x < \pi$$

is

$$u(x, t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin(2k-1)x\cos\sqrt{(2k-1)^2+1}t.$$

17. Consider the boundary-value problem given in (1)–(3) of this section. If $g(x) = 0$ on $0 < x < L$, show that the solution of the problem can be written as

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)].$$

[*Hint*: Use the identity

$$2 \sin \theta_1 \cos \theta_2 = \sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2).]$$

18. The vertical displacement $u(x, t)$ of an infinitely long string is determined from the initial-value problem

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \ t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x). \qquad (13)$$

This problem can be solved without separating variables.

(a) Show that the wave equation can be put into the form $\partial^2 u/\partial\eta\,\partial\xi = 0$ by means of the substitutions $\xi = x + at$ and $\eta = x - at$.

(b) Integrate the partial differential equation in part (a), first with respect to η and then with respect to ξ, to show that $u(x, t) = F(x + at) + G(x - at)$, where F and G are arbitrary twice differentiable functions, is a solution of the wave equation. Use this solution and the given initial conditions to show that

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds + c$$

and $$G(x) = \frac{1}{2}f(x) - \frac{1}{2a}\int_{x_0}^{x} g(s)\, ds - c,$$

where x_0 is arbitrary and c is a constant of integration.

(c) Use the results in part (b) to show that

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\, ds. \quad (14)$$

Note that when the initial velocity $g(x) = 0$ we obtain

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)], -\infty < x < \infty.$$

The last solution can be interpreted as a superposition of two **traveling waves**, one moving to the right (that is, $\frac{1}{2}f(x - at)$) and one moving to the left ($\frac{1}{2}f(x + at)$). Both waves travel with speed a and have the same basic shape as the initial displacement $f(x)$. The form of $u(x, t)$ given in (14) is called **d'Alembert's solution**.

In Problems 19–21, use d'Alembert's solution (14) to solve the initial-value problem in Problem 18 subject to the given initial conditions.

19. $f(x) = \sin x,\ g(x) = 1$

20. $f(x) = \sin x,\ g(x) = \cos x$

21. $f(x) = 0,\ g(x) = \sin 2x$

22. Suppose $f(x) = 1/(1 + x^2), g(x) = 0$, and $a = 1$ for the initial-value problem given in Problem 18. Graph d'Alembert's solution in this case at the time $t = 0$, $t = 1$, and $t = 3$.

23. The transverse displacement $u(x, t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

$$a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0,\ 0 < x < L,\ t > 0.$$

If the beam is **simply supported**, as shown in **FIGURE 13.4.9** the boundary and initial conditions are

$$u(0, t) = 0, \qquad u(L, t) = 0,\ t > 0$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=L} = 0,\ t > 0$$

$$u(x, 0) = f(x), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),\ 0 < x < L.$$

Solve for $u(x, t)$. [*Hint*: For convenience use $\lambda = \alpha^4$ when separating variables.]

FIGURE 13.4.9 Simply supported beam in Problem 23

Computer Lab Assignments

24. If the ends of the beam in Problem 23 are **embedded** at $x = 0$ and $x = L$, the boundary conditions become, for $t > 0$,

$$u(0, t) = 0, \qquad u(L, t) = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0.$$

(a) Show that the eigenvalues of the problem are $\lambda = x_n^2/L^2$ where x_n, $n = 1, 2, 3, \ldots$, are the positive roots of the equation $\cosh x \cos x = 1$.

(b) Show graphically that the equation in part (a) has an infinite number of roots.

(c) Use a CAS to find approximations to the first four eigenvalues. Use four decimal places.

25. A model for an infinitely long string that is initially held at the three points (–1, 0), (1, 0), and (0, 1) and then simultaneously released at all three points at time $t = 0$ is given by (13) with

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = 0.$$

(a) Plot the initial position of the string on the interval [–6, 6].

(b) Use a CAS to plot d'Alembert's solution (14) on [–6, 6] for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(c) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

26. An infinitely long string coinciding with the x-axis is struck at the origin with a hammer whose head is 0.2 inch in diameter. A model for the motion of the string is given by (13) with

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1, & |x| \le 0.1 \\ 0, & |x| > 0.1. \end{cases}$$

(a) Use a CAS to plot d'Alembert's solution (14) on [–6, 6] for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

27. The model of the vibrating string in Problem 7 is called a **plucked string**.

(a) Use a CAS to plot the partial sum $S_6(x, t)$; that is, the first six nonzero terms of your solution $u(x, t)$, for $t = 0.1k$, $k = 0, 1, 2, \ldots, 20$. Assume that $a = 1$, $h = 1$, and $L = \pi$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution to Problem 7.

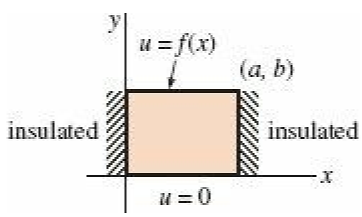


FIGURE 13.5.1 Find the temperature u in a rectangular plate

13.5 Laplace's Equation

Introduction Suppose we wish to find the steady-state temperature $u(x, y)$ in a rectangular plate whose vertical edges $x = 0$ and $x = a$ are insulated, and whose upper and lower edges $y = b$ and $y = 0$ are maintained at temperatures $f(x)$ and 0, respectively. See **FIGURE 13.5.1** When no heat escapes from the lateral faces of the plate, we solve the following boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; 0 < x < a, \; 0 < y < b \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \; 0 < y < b \tag{2}$$

$$u(x, 0) = 0, \quad u(x, b) = f(x), \; 0 < x < a. \tag{3}$$

Solution of the BVP With $u(x, y) = X(x)Y(y)$, separation of variables in (1) leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \tag{4}$$

$$Y'' - \lambda Y = 0. \tag{5}$$

The three homogeneous boundary conditions in (2) and (3) translate into $X'(0) = 0$, $X'(a) = 0$, and $Y(0) = 0$. The Sturm–Liouville problem associated with the equation in (4) is then

$$X'' + \lambda X = 0, \quad X'(0) = 0, X'(a) = 0. \tag{6}$$

Examination of the cases corresponding to $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$, has already been carried out in Example 1 in Section 12.5. For convenience a shortened version of that analysis follows.

For $\lambda = 0$, (6) becomes

$$X'' = 0, \quad X'(0) = 0, X'(a) = 0.$$

The solution of the ODE is $X = c_1 + c_2x$. The boundary condition $X'(0) = 0$ then implies $c_2 = 0$, and so $X = c_1$. Note that for any c_1, this constant solution satisfies the second boundary condition $X'(a) = 0$. By imposing $c_1 \neq 0$, $X = c_1$ is a nontrivial solution of the BVP (6). For $\lambda = -\alpha^2 < 0$, (6) possesses no nontrivial solution. For $\lambda = \alpha^2 > 0$, (6) becomes

$$X'' + \alpha^2 X = 0, \quad X'(0) = 0, X'(a) = 0.$$

Applying the boundary condition $X'(0) = 0$ the solution $X = c_1 \cos \alpha x + c_2 \sin \alpha x$ implies $c_2 = 0$ and so $X = c_1 \cos \alpha x$. The second boundary condition $X'(a) = 0$ applied to this last expression then gives $-c_1 \alpha \sin \alpha a = 0$. Because $\alpha > 0$, the last equation is satisfied when $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2, \ldots$. The eigenvalues of (6) are then λ_0 and $\lambda_n = \alpha_n^2 = n^2\pi^2/a^2$, $n = 1, 2, \ldots$.
By corresponding $\lambda_0 = 0$ with $n = 0$, the eigenfunctions of (6) are

$$X = c_1, \; n = 0, \quad \text{and} \quad X = c_1 \cos \frac{n\pi}{a} x, \quad n = 1, 2, \ldots.$$

We must now solve equation (5) subject to the single homogeneous boundary condition $Y(0) = 0$. First, for $\lambda_0 = 0$ the DE in (5) is simply $Y'' = 0$, and thus its solution is $Y = c_3 + c_4y$. But $Y(0) = 0$ implies $c_3 = 0$ so $Y = c_4y$. Second, for $\lambda_n = n^2\pi^2/a^2$, the DE in (5) is $Y'' - \frac{n^2\pi^2}{a^2} Y = 0$. Because $0 < y < b$ is a finite interval, we write the general solution in terms of hyperbolic functions:

$$Y(y) = c_3 \cosh(n\pi y/a) + c_4 \sinh(n\pi y/a).$$

Why hyperbolic functions? See page 675.

From this solution we see $Y(0) = 0$ again implies $c_3 = 0$ so $Y = c_4 \sinh(n\pi y/a)$.

Thus product solutions $u_n = X(x)Y(y)$ that satisfy the Laplace's equation (1) and the three homogeneous boundary conditions in (2) and (3) are

$$A_0 y, \quad n = 0, \quad \text{and} \quad A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x, \quad n = 1, 2, \ldots,$$

where we have rewritten c_1c_4 as A_0 for $n = 0$ and as A_n for $n = 1, 2, \ldots$.

The superposition principle yields another solution

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x. \qquad (7)$$

Finally, by substituting $y = b$ in (7) we see

$$u(x, b) = f(x) = A_0 b + \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} b \right) \cos \frac{n\pi}{a} x,$$

is a half-range expansion of f in a Fourier cosine series. If we make the identifications $A_0 b = a_0/2$ and $A_n \sinh(n\pi b/a) = a_n, n = 1, 2, \ldots,$ it follows from (2) and (3) of Section 12.3 that

$$2A_0 b = \frac{2}{a} \int_0^a f(x)\, dx$$

$$A_0 = \frac{1}{ab} \int_0^a f(x)\, dx \qquad (8)$$

and

$$A_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi}{a} x\, dx$$

$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi}{a} x\, dx. \qquad (9)$$

The solution of the boundary-value problem (1)–(3) consists of the series in (7), with coefficients A_0 and An defined in (8) and (9), respectively.

□ **Dirichlet Problem** A boundary-value problem in which we seek a solution to an elliptic partial differential equation such as Laplace's equation $\nabla^2 u = 0$ within a region R (in the plane or 3-space) such that u takes on prescribed values on the entire boundary of the region is called a **Dirichlet problem**. In Problem 1 in Exercises 13.5 you are asked to show that the solution of the Dirichle problem for a rectangular region

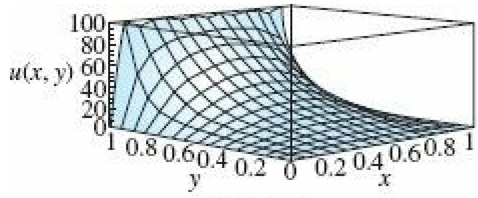
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = 0, \quad u(a, y) = 0$$

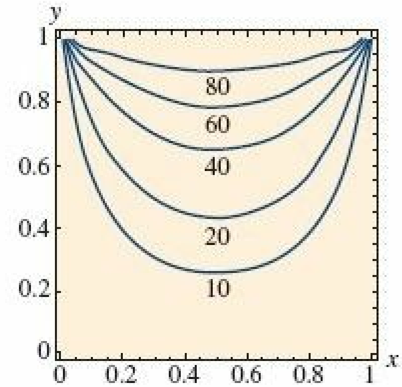
$$u(x, 0) = 0, \quad u(x, b) = f(x)$$

is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x \quad \text{where} \quad A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi}{a} x\, dx. \qquad (10)$$



(a) Surface



(b) Isotherms

FIGURE 13.5.2 Surface is graph of partial sums when $f(x) = 100$ and $a = b = 1$ in (10)

In the special case when $f(x) = 100$, $a = 1$, $b = 1$, the coefficients A_n are given by $A_n = 200\,\dfrac{1-(-1)^n}{n\pi \sinh n\pi}$. With the help of a CAS the plot of the surface defined by $u(x, y)$ over the region R: $0 \le x \le 1, 0 \le y \le 1$ is given in **FIGURE 13.5.2(a)** You can see in the figure that boundary conditions are satisfied; especially note that along $y = 1$, $u = 100$ for $0 \le x \le 1$. The isotherms, or curves, in the rectangular region along which the temperature $u(x, y)$ is constant can be obtained using the contour plotting capabilities of a CAS and are illustrated in Figure 13.5.2(b). The isotherms can also be visualized as the curves of intersection (projected into the xy-plane) of horizontal planes $u = 80$, $u = 60$, and so on, with the surface in Figure 13.5.2(a). Notice that throughout the region the maximum temperature is $u = 100$ and occurs on the portion of the boundary corresponding to $y = 1$. This is no coincidence. There is a **maximum principle** that states a solution u of Laplace's equation within a bounded region R with boundary B (such as a rectangle, circle, sphere, and so on) takes on its maximum and minimum values on B. In addition, it can be proved that u can have no relative extrema (maxima or minima) in the interior of R. This last statement is clearly borne out by the surface shown in Figure 13.5.2(a).

□ **Superposition Principle** A Dirichlet problem for a rectangle can be readily solved by separation of variables when homogeneous boundary conditions are specified on two *parallel* boundaries. However, the method of separation of variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are nonhomogeneous. To get around this difficulty we break the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a,\ 0 < y < b$$

$$u(0, y) = F(y), \qquad u(a, y) = G(y),\ 0 < y < b \tag{11}$$

$$u(x, 0) = f(x), \qquad u(x, b) = g(x),\ 0 < x < a$$

into two problems, each of which has homogeneous boundary conditions on parallel boundaries, as shown.

Problem 1

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, \ 0 < y < b$$

$$u_1(0, y) = 0, \qquad u_1(a, y) = 0, \ 0 < y < b$$

$$u_1(x, 0) = f(x), \qquad u_1(x, b) = g(x), \ 0 < x < a$$

Problem 2

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, \ 0 < y < b$$

$$u_2(0, y) = F(y), \qquad u_2(a, y) = G(y), \ 0 < y < b$$

$$u_2(x, 0) = 0, \qquad u_2(x, b) = 0, \ 0 < x < a$$

Suppose u_1 and u_2 are the solutions of Problems 1 and 2, respectively. If we define $u(x, y) = u_1(x, y) + u_2(x, y)$, it is seen that u satisfies all boundary conditions in the original problem (11). For example,

$$u(0, y) = u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y)$$

$$u(x, b) = u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x)$$

and so on. Furthermore, u is a solution of Laplace's equation by Theorem 13.1.1. In other words, by solving Problems 1 and 2 and adding their solutions we have solved the original problem. This additive property of solutions is known as the superposition principle. See **FIGURE 13.5.3**.

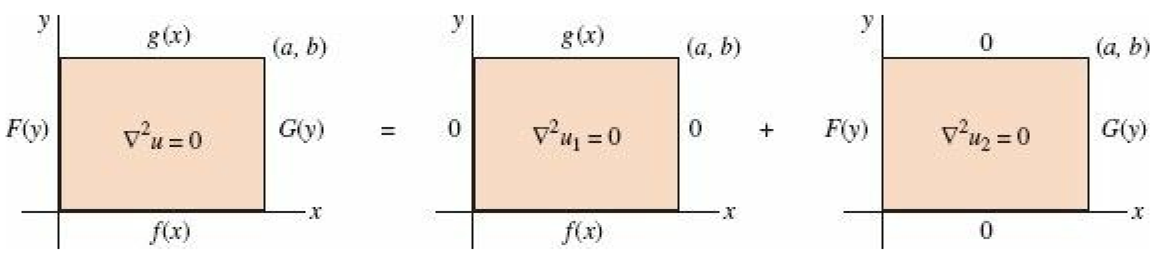


FIGURE 13.5.3 Solution u = Solution u_1 of Problem 1 + Solution u_2 of Problem 2

We leave as exercises (see Problems 13 and 14 in Exercises 13.5) to show that a solution of Problem 1 is

$$u_1(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y \right\} \sin \frac{n\pi}{a} x,$$

where

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b \right),$$

and that a solution of Problem 2 is

$$u_2(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x \right\} \sin \frac{n\pi}{b} y,$$

where

$$A_n = \frac{2}{b}\int_0^b F(y)\sin\frac{n\pi}{b}y\,dy$$

$$B_n = \frac{1}{\sinh\frac{n\pi}{b}a}\left(\frac{2}{b}\int_0^b G(y)\sin\frac{n\pi}{b}y\,dy - A_n\cosh\frac{n\pi}{b}a\right).$$

13.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-31.

In Problems 1–10, solve Laplace's equation (1) for a rectangular plate subject to the given boundary conditions.

1. $u(0, y) = 0,\ u(a, y) = 0$
   $u(x, 0) = 0,\ u(x, b) = f(x)$
2. $u(0, y) = 0,\ u(a, y) = 0$
   $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ u(x, b) = f(x)$
3. $u(0, y) = 0,\ u(a, y) = 0$
   $u(x, 0) = f(x),\ u(x, b) = 0$
4. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$
   $u(x, 0) = x,\ u(x, b) = 0$
5. $u(0, y) = 0,\ u(1, y) = 1 - y$
   $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0$
6. $u(0, y) = g(y),\ \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$
   $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ \left.\frac{\partial u}{\partial y}\right|_{y=\pi} = 0$
7. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = u(0, y),\ u(\pi, y) = 1$
   $u(x, 0) = 0,\ u(x, \pi) = 0$
8. $u(0, y) = 0,\ u(1, y) = 0$
   $\left.\frac{\partial u}{\partial y}\right|_{y=0} = u(x, 0),\ u(x, 1) = f(x)$
9. $u(0, y) = 0,\ u(1, y) = 0$
   $u(x, 0) = 100,\ u(x, 1) = 200$
10. $u(0, y) = 10y,\ \left.\frac{\partial u}{\partial x}\right|_{x=1} = -1$
    $u(x, 0) = 0,\ u(x, 1) = 0$

In Problems 11 and 12, solve Laplace's equation (1) for the semi-infinite plate extending in the positive y-direction. In each case assume that $u(x, y)$ is bounded at $y \to \infty$.

11.

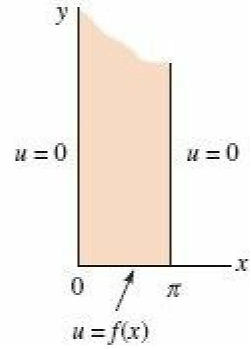


FIGURE 13.5.4 Semi-infinite Plate in Problem 11

12.

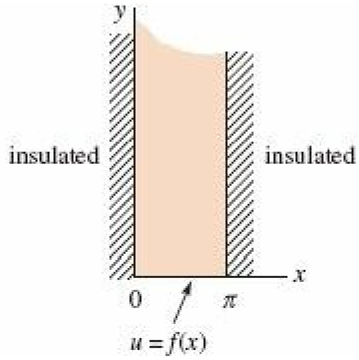


FIGURE 13.5.5 Semi-infinite Plate in Problem 12

In Problems 13 and 14, solve Laplace's equation (1) for a rectangular plate subject to the given boundary conditions.

13. $u(0, y) = 0, \; u(a, y) = 0$

$u(x, 0) = f(x), \; u(x, b) = g(x)$

14. $u(0, y) = F(y), \; u(a, y) = G(y)$

$u(x, 0) = 0, \; u(x, b) = 0$

In Problems 15 and 16, use the superposition principle to solve Laplace's equation (1) for a square plate subject to the given boundary conditions.

15. $u(0, y) = 1, \; u(\pi, y) = 1$

$u(x, 0) = 0, \; u(x, \pi) = 1$

16. $u(0, y) = 0, \; u(2, y) = y(2 - y)$

$$u(x, 0) = 0, \; u(x, 2) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$$

17. In Problem 16, what is the maximum value of the temperature u for $0 \le x \le 2, 0 \le y \le 2$?

Computer Lab Assignments

18. (a) In Problem 1 suppose $a = b = \pi$ and $f(x) = 100x(\pi - x)$. Without using the solution $u(x, y)$ sketch, by hand, what the surface would look like over the rectangular region defined by $0 \le x \le \pi, 0 \le y \le \pi$.

(b) What is the maximum value of the temperature u for $0 \le x \le \pi, 0 \le y \le \pi$?

(c) Use the information in part (a) to compute the coefficients for your answer in Problem 1. Then use the 3D-plot application of your CAS to graph the partial sum $S_5(x, y)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le \pi,\ 0 \le y \le \pi$. Use different perspectives and then compare with part (a).

19. (a) Use the contour-plot application of your CAS to graph the isotherms $u = 170, 140, 110, 80, 60, 30$ for the solution of Problem 9. Use the partial sum $S_5(x, y)$ consisting of the first five nonzero terms of the solution.

(b) Use the 3D-plot application of your CAS to graph the partial sum $S5(x, y)$.

20. Use the contour-plot application of your CAS to graph the isotherms $u = 2, 1, 0.5, 0.2, 0.1, 0.05, 0, -0.05$ for the solution of Problem 10. Use the partial sum $S5(x, y)$ consisting of the first five nonzero terms of the solution.

Discussion Problems

21. Solve the Neumann problem for a rectangle:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < a,\ 0 < y < b$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ \left.\frac{\partial u}{\partial y}\right|_{y=b} = 0,\ 0 < x < a$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=a} = g(y),\ 0 < y < b.$$

(a) Explain why a necessary condition for a solution u to exist is that g satisfy

$$\int_0^b g(y)\,dy = 0.$$

This is sometimes called a **compatibility condition**. Do some extra reading and explain the compatibility condition on physical grounds.

(b) If u is a solution of the BVP, explain why $u + c$, where c is an arbitrary constant, is also a solution.

22. Consider the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 1,\ 0 < y < \pi$$

$$u(0, y) = u_0 \cos y,\ u(1, y) = u_0(1 + \cos 2y)$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\ \left.\frac{\partial u}{\partial y}\right|_{y=\pi} = 0.$$

Discuss how the following answer was obtained

$$u(x, y) = u_0 x + u_0 \frac{\sinh(1 - x)}{\sinh 1} \cos y + \frac{u_0}{\sinh 2} \sinh 2x \cos 2y.$$

Carry out your ideas.

13.6 Nonhomogeneous BVPs

Introduction A boundary-value problem is said to be **nonhomogeneous** when either the partial differential equation or the boundary conditions are nonhomogeneous. For example, a typical nonhomogeneous BVP for the heat equation is

$$k\frac{\partial^2 u}{\partial x^2} + F(x, t) = \frac{\partial u}{\partial t}, \; 0 < x < L, \; t > 0$$
$$u(0, t) = u_0(t), \quad u(L, t) = u_1(t), \; t > 0 \qquad (1)$$
$$u(x, 0) = f(x), \; 0 < x < L.$$

We can interpret this problem as a model for the temperature distribution u within a rod of length L when heat is being generated internally at rate $F(x, t)$; the temperatures at the ends of the rod vary with time t. The method of separation of variables may not be applicable to a boundary-value problem when the partial differential equation or boundary conditions are nonhomogeneous. For example, when heat is generated at a constant rate r within the rod, the heat equation in (1) takes on the form

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}. \qquad (2)$$

Equation (2) is readily shown not to be separable. On the other hand, suppose we wish to solve the usual heat equation $ku_{xx} = u_t$ when the boundaries $x = 0$ and $x = L$ are held at nonzero temperatures u_0 and u_1. Even though the substitution $u(x, t) = X(x)r(t)$ separates the PDE, we quickly find ourselves at an impasse in determining eigenvalues and eigenfunctions since no conclusion about $X(0)$ and $X(L)$ can be drawn from $u(0, t) = X(0)T(t) = u_0$ and $u(L, t) = X(L)r(t) = u_1$.

Change of Dependent Variable In this section we consider certain types of nonhomogeneous boundary-value problems that can be solved by changing the dependent variable u to a new dependent variable v by means of the substitution $u = v + \psi$, where ψ is a function to be determined.

Time-Independent PDE and BCs We first consider a nonhomogeneous boundary-value problem such as (1) where the heat source term F and the boundary-conditions are time independent:

$$k\frac{\partial^2 u}{\partial x^2} + F(x) = \frac{\partial u}{\partial t}, \; 0 < x < L, \; t > 0$$
$$u(0, t) = u_0, \quad u(L, t) = u_1, \; t > 0 \qquad (3)$$
$$u(x, 0) = f(x), \; 0 < x < L.$$

In (3), u_0 and u_1 denote constants. By changing the dependent variable u to a new dependent variable v by the substitution $u(x, t) = v(x, t) + \psi(x)$, (3) can be reduced to two problems:

Problem 1: $\{k\psi'' + F(x) = 0,\ \psi(0) = u_0,\ \psi(L) = u_1$

$$\text{Problem 2:}\quad \begin{cases} k\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial v}{\partial t}, \\ v(0, t) = 0,\quad v(L, t) = 0 \\ v(x, 0) = f(x) - \psi(x). \end{cases}$$

Notice that the ODE in Problem 1 can be solved by integration. Moreover, Problem 2 is a homogeneous BVP that can be solved straightaway by separation of variables. A solution of the original problem is then

$$\text{Solution } u = \text{Solution } \psi \text{ of Problem 1} + \text{Solution } v \text{ of Problem 2}.$$

There is nothing given above in the two problems that should be memorized, but work through the substitution $u(x, t) = v(x, t) + \psi(x)$ each time as outlined in the next example.

EXAMPLE 1 Time-Independent PDE and BCs

Solve equation (2) subject to

$$u(0, t) = 0, \qquad u(1, t) = u_0,\ t > 0$$
$$u(x, 0) = f(x),\ 0 < x < 1.$$

SOLUTION Both the partial differential equation and the condition at the right boundary $x = 1$ are nonhomogeneous. If we let $u(x, t) = v(x, t) + \psi(x)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \psi'' \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} \tag{4}$$

since $\psi_t = 0$. Substituting these results in (4) into (3) gives

$$k\frac{\partial^2 v}{\partial x^2} + k\psi'' + r = \frac{\partial v}{\partial t}. \tag{5}$$

Equation (5) reduces to a homogeneous PDE if we demand that ψ be a function that satisfies the ODE

$$k\psi'' + r = 0 \quad \text{or} \quad \psi'' = -\frac{r}{k}.$$

Integrating the last equation twice reveals that

$$\psi(x) = -\frac{r}{2k}x^2 + c_1 x + c_2. \tag{6}$$

Furthermore,

$$u(0, t) = v(0, t) + \psi(0) = 0$$
$$u(1, t) = v(1, t) + \psi(1) = u_0.$$

We have $v(0, t) = 0$ and $v(1, t) = 0$, provided we choose

$$\psi(0) = 0 \quad \text{and} \quad \psi(1) = u_0.$$

Applying the latter two conditions to (6) gives, in turn, $c_2 = 0$ and $c_1 = r/2k + u_0$. Consequently

$$\psi(x) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x.$$

Finally, the initial condition $u(x, 0) = v(x, 0) + \psi(x)$ implies $v(x, 0) = u(x, 0) - \psi(x) = f(x) - \psi(x)$. Thus to determine $v(x, t)$ we solve the new homogeneous boundary-value problem

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \ t > 0$$
$$v(0, t) = 0, \qquad v(1, t) = 0, \ t > 0$$
$$v(x, 0) = f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x, \ 0 < x < 1$$

by separation of variables. In the usual manner we find

$$v(x, t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x,$$

where the initial condition $v(x, 0)$ determines the Fourier sine coefficients:

$$A_n = 2\int_0^1 \left[f(x) + \frac{r}{2k}x^2 - \left(\frac{r}{2k} + u_0\right)x\right] \sin n\pi x \, dx. \tag{7}$$

A solution of the original problem is obtained by adding $\psi(x)$ and $v(x, t)$:

$$u(x, t) = -\frac{r}{2k}x^2 + \left(\frac{r}{2k} + u_0\right)x + \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t} \sin n\pi x, \tag{8}$$

where the coefficients A_n are defined in (7). ≡

Observe in (8) that $u(x, t) \to \psi(x)$ as $t \to \infty$. In the context of the given boundary-value problem, ψ is called a **steady-state solution**. Since $v(x, t) \to 0$ as $t \to \infty$, v is called a transient solution.

Time-Dependent PDE and BCs We now return to the problem given in (1), where the heat source term F and the boundary-conditions are time dependent. Intuitively one might expect that the line of attack for this problem would be a natural extension of the procedure that worked in Example 1; namely, seek a solution of the form $u(x, t) = v(x, t) + \psi(x, t)$. While the latter form of the solution is correct, it is usually not possible to find a function of two variables $\psi(x, t)$ that reduces the problem in $v(x, t)$ to a homogeneous one. To understand why this is so, let's see what happens when $u(x, t) = v(x, t) + \psi(x, t)$ is substituted in (1). Since

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t}, \tag{9}$$

(1) becomes

$$k\frac{\partial^2 v}{\partial x^2} + k\frac{\partial^2 \psi}{\partial x^2} + F(x, t) = \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t}$$
$$v(0, t) + \psi(0, t) = u_0(t), \quad v(L, t) + \psi(L, t) = u_0(t) \tag{10}$$
$$v(x, 0) = f(x) - \psi(x, 0).$$

The boundary conditions on v in (10) will be homogeneous if we demand that

$$\psi(0, t) = u_0(t), \quad \psi(L, t) = u_0(t). \tag{11}$$

Were we, at this point, to follow the same steps in the method used in Example 1, we would try to force the problem in (10) to be homogeneous by solving $k\psi_{xx} + F(x, t) = \psi_t$ and then imposing the conditions in (11) on the solution ψ. In view of the fact that the defining equation for ψ is itself a nonhomogeneous PDE, this is an unrealistic expectation. We try an entirely different tack by simply constructing a function ψ that satisfies both conditions in (11). One such a function is given by

$$\psi(x, t) = u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)]. \tag{12}$$

Reinspection of (10) shows that we have gained some additional simplification with this choice of ψ since $\psi_{xx} = 0$. We now start over. This time if we substitute

$$u(x, t) = v(x, t) + u_0(t) + \frac{x}{L}[u_1(t) - u_0(t)] \tag{13}$$

the problem in (1) becomes

$$\begin{aligned} &k\frac{\partial^2 v}{\partial x^2} + G(x, t) = \frac{\partial v}{\partial t}, \quad 0 < x < L, \ t > 0 \\ &v(0, t) = 0, \quad v(L, t) = 0, \ t > 0 \\ &v(x, 0) = f(x) - \psi(x, 0), \ 0 < x < L, \end{aligned} \tag{14}$$

where $G(x, t) = F(x, t) - \psi_t$. While the problem in (14) is still nonhomogeneous (the boundary conditions are homogeneous but the partial differential equation is nonhomogeneous) it is a problem that we can solve.

Basic Strategy The solution method for (14) is a bit involved, so before illustrating with a specific example, we first outline the basic strategy:

Make the assumption that time-dependent coefficients $v_n(t)$ and $G_n(t)$ can be found such that both $v(x, t)$ and $G(x, t)$ in (14) can be expanded in the series

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi}{L}x \quad \text{and} \quad G(x, t) = \sum_{n=1}^{\infty} G_n(t) \sin\frac{n\pi}{L}x, \tag{15}$$

where $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$ are the eigenfunctions of $X'' + \lambda X = 0, X(0) = 0, X(L) = 0$ corresponding to the eigenvalues $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$. This Sturm–Liouville problem would have been obtained had separation of variables been applied to the associated homogeneous BVP of (14). In (15), observe that the assumed series for $v(x, t)$ already satisfies the boundary conditions in (14). Now substitute the first series in (15) into the nonhomogeneous PDE in (14), collect terms, and equate the resulting series with the actual series expansion found for $G(x, t)$.

The next example illustrates this method.

EXAMPLE 2 Time-Dependent PDE and BCs

Solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$$

$$u(0, t) = \cos t,\quad u(1, t) = 0,\ t > 0$$

$$u(x, 0) = 0,\ 0 < x < 1.$$

SOLUTION We match this problem with (1) by identifying $k = 1, L = 1, F(x, t) = 0$, $u_0(t) = \cos t, u_1(t) = 0$, and $f(x) = 0$. We begin with the construction of ψ. From (12) we get

$$\psi(x, t) = \cos t + x[0 - \cos t] = (1 - x)\cos t,$$

and then as indicated in (13), we use the substitution

$$u(x, t) = v(x, t) + (1 - x)\cos t \tag{16}$$

to obtain the BVP for $v(x, t)$:

$$\frac{\partial^2 v}{\partial x^2} + (1 - x)\sin t = \frac{\partial v}{\partial t},\ 0 < x < 1,\ t > 0$$

$$v(0, t) = 0,\quad v(1, t) = 0,\ t > 0 \tag{17}$$

$$v(x, 0) = x - 1,\ 0 < x < 1.$$

The eigenvalues and eigenfunctions of the Sturm–Liouville problem

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0$$

are found to be $\lambda_n = \alpha_n^2 = n^2\pi^2$ and $\sin n\pi x$, $n = 1, 2, 3, \ldots$. With $G(x, t) = (1 - x)\sin t$ we assume from (15) that for fixed t, v and G can be written as Fourier sine series:

$$v(x, t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x, \tag{18}$$

and

$$(1 - x)\sin t = \sum_{n=1}^{\infty} G_n(t) \sin n\pi x. \tag{19}$$

By treating t as a parameter, the coefficients G_n in (19) can be computed:

$$G_n(t) = \frac{2}{1}\int_0^1 (1 - x)\sin t \sin n\pi x\, dx = 2\sin t \int_0^1 (1 - x)\sin n\pi x\, dx = \frac{2}{n\pi}\sin t.$$

Hence,

$$(1 - x)\sin t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin t \sin n\pi x. \tag{20}$$

We can determine the coefficients $v_n(t)$ by substituting (19) and (20) back into the PDE in (17). To that end, the partial derivatives of v are

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} v_n(t)(-n^2\pi^2)\sin n\pi x \quad \text{and} \quad \frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} v_n'(t)\sin n\pi x. \tag{21}$$

Writing the PDE as $v_t - v_{xx} = (1 - x)\sin t$ and using (20) and (21) we get

$$\sum_{n=1}^{\infty}[v_n'(t) + n^2\pi^2 v_n(t)]\sin n\pi x = \sum_{n=1}^{\infty}\frac{2\sin t}{n\pi}\sin n\pi x.$$

We then equate the coefficients of $\sin n\pi x$ on each side of the equality to get

$$v_n'(t) + n^2\pi^2 v_n(t) = \frac{2\sin t}{n\pi}.$$

For each n, the last equation is a linear first-order ODE whose general solution is

$$v_n(t) = \frac{2}{n\pi}\left(\frac{n^2\pi^2\sin t - \cos t}{n^4\pi^4 + 1}\right) + C_n e^{-n^2\pi^2 t},$$

where C_n denotes the arbitrary constant. Therefore the assumed form of $v(x, t)$ in (18) can be written

$$v(x, t) = \sum_{n=1}^{\infty}\left\{2\frac{n^2\pi^2\sin t - \cos t}{n\pi(n^4\pi^4 + 1)} + C_n e^{-n^2\pi^2 t}\right\}\sin n\pi x. \tag{22}$$

The C_n can be found by applying the initial condition $v(x, 0)$ to (22). From the Fourier sine series,

$$x - 1 = \sum_{n=1}^{\infty}\left\{\frac{-2}{n\pi(n^4\pi^4 + 1)} + C_n\right\}\sin n\pi x$$

we see that the quantity in the brackets represents the Fourier sine coefficients b_n for $x - 1$. That is,

$$\frac{-2}{n\pi(n^4\pi^4 + 1)} + C_n = 2\int_0^1 (x - 1)\sin n\pi x\,dx \quad \text{or} \quad \frac{-2}{n\pi(n^4\pi^4 + 1)} + C_n = \frac{-2}{n\pi}.$$

Therefore, $C_n = \dfrac{2}{n\pi(n^4\pi^4 + 1)} - \dfrac{2}{n\pi}$.

By substituting the last result into (22) we obtain a solution of (17),

$$v(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{n^2\pi^2\sin t - \cos t + e^{-n^2\pi^2 t}}{n(n^4\pi^4 + 1)} - \frac{e^{-n^2\pi^2 t}}{n}\right\}\sin n\pi x.$$

At long last, then, it follows from (16) that the desired solution $u(x, t)$ is

$$u(x, t) = (1 - x)\cos t + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{n^2\pi^2\sin t - \cos t + e^{-n^2\pi^2 t}}{n(n^4\pi^4 + 1)} - \frac{e^{-n^2\pi^2 t}}{n}\right\}\sin n\pi x. \quad \equiv$$

Remarks

(i) If the boundary-value problem has homogeneous boundary conditions and a time-dependent term $F(x, t)$ in the PDE, then there is no need to change the dependent variable by substituting $u(x, t) = v(x, t) + \psi(x, t)$. For example, if u_0 and u_1 are 0 in a problem such as (1), then it follows from (12) that $\psi(x, t) = 0$. The method of solution is basically a frontal attack on the PDE by assuming appropriate orthogonal series expansions for $u(x, t)$ and $F(x, t)$. Again, if u_0 and u_1 are 0 in (1), the solution begins with the assumptions in (15), where the symbols v and G are naturally replaced by u and F, respectively. See Problems 13–16 in Exercises 13.6. In Problems 17 and 18 of Exercises 13.6 you will have to construct $\psi(x, t)$ as illustrated in Example 2. See also Problem 20 in Exercises 13.6.

(ii) Don't put any special emphasis on the fact that we used the heat equation throughout the foregoing discussion. The method outlined in Example 1 can be applied to the wave equation and Laplace's equation as well. See Problems 1–12 in Exercises 13.6. The method outlined in Example 2 is predicated on time dependence in the problem and so is not applicable to BVPs involving Laplace's equation.

13.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-32.

Time-Independent PDE and BCs

In Problems 1 and 2, solve the heat equation $ku_{xx} = u_t, 0 < x < 1$, $t > 0$ subject to the given conditions.

1. $u(0, t) = u_0, \quad u(1, t) = 0$
   $u(x, 0) = f(x)$
2. $u(0, t) = u_0, \quad u(1, t) = 0$
   $u(x, 0) = f(x)$

In Problems 3 and 4, solve the heat equation (2) subject to the given conditions.

3. $u(0, t) = u_0, \quad u(1, t) = u_0$
   $u(x, 0) = 0$
4. $u(0, t) = u_0, \quad u(1, t) = u_1$
   $u(x, 0) = f(x)$
5. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} + Ae^{-\beta x} = \frac{\partial u}{\partial t}, \ \beta > 0, 0 < x < 1, \ t > 0$$

$$u(0, t) = 0, \quad u(1, t) = 0, \ t > 0$$

$$u(x, 0) = f(x), \ 0 < x < 1,$$

   where A is a constant. The PDE is a form of the heat equation when heat is generated within a thin rod due to radioactive decay of the material.
6. Solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, 0 < x < \pi, \ t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = u_0, \ t > 0$$

$$u(x, 0) = 0, \ 0 < x < \pi.$$

   The PDE is a form of the heat equation when heat is lost by radiation from the lateral surface of a thin rod into a medium at temperature zero.
7. Find a steady-state solution $\psi(x)$ of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} - h(u - u_0) = \frac{\partial u}{\partial t}, 0 < x < 1, \ t > 0$$

$$u(0, t) = u_0, \quad u(1, t) = 0, \ t > 0$$

$$u(x, 0) = f(x), \ 0 < x < 1.$$

8. Find a steady-state solution $\psi(x)$ if the rod in Problem 7 is semi-infinite extending in the positive x-direction, radiates from its lateral surface into a medium at temperature zero, and

$$u(0, t) = u_0, \quad \lim_{x \to \infty} u(x, t) = 0, \ t > 0$$

$$u(x, 0) = f(x), \ x > 0.$$

9. When a vibrating string is subjected to an external vertical force that varies with the horizontal distance from the left end, the wave equation takes on the form

$$a^2 \frac{\partial^2 u}{\partial x^2} + Ax = \frac{\partial^2 u}{\partial t^2},$$

where A is constant. Solve this partial differential equation subject to

$$u(0, t) = 0, \quad u(1, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \ 0 < x < 1.$$

10. A string initially at rest on the x-axis is secured on the x-axis at $x = 0$ and $x = 1$. If the string is allowed to fall under its own weight for $t > 0$, the displacement $u(x, t)$ satisfies

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \ 0 < x < 1, \ t > 0,$$

where g is the acceleration of gravity. Solve for $u(x, t)$.

11. Find the steady-state temperature $u(x, y)$ in the semi-infinite plate shown in **FIGURE 13.6.1** Assume that the temperature is bounded as $x \to \infty$. [*Hint*: Use $u(x, y) = v(x, y) + \psi(y)$.]

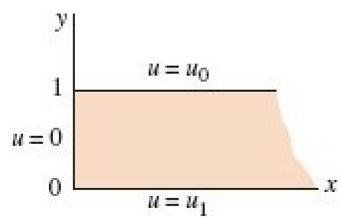


FIGURE 13.6.1 Semi-infinite plate in Problem 11

12. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -h,$$

where $h > 0$ is a constant, occurs in many problems involving electric potential and is known as **Poisson's equation**. Solve the above equation subject to the conditions

$$u(0, y) = 0, \ u(\pi, y) = 1, \ y > 0$$

$$u(x, 0) = 0, \ 0 < x < \pi.$$

Time-Dependent PDE and BCs

In Problems 13–18, solve the given boundary-value problem.

13.

$$\frac{\partial^2 u}{\partial x^2} + xe^{-3t} = \frac{\partial u}{\partial t}, \ 0 < x < \pi, \ t > 0$$

$$u(0, t) = 0, \ u(\pi, t) = 0, \ t > 0$$

$$u(x, 0) = 0, \ 0 < x < \pi.$$

$$\frac{\partial^2 u}{\partial x^2} + xe^{-3t} = \frac{\partial u}{\partial t},\ 0 < x < \pi,\ t > 0$$

14. $$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=\pi} = 0,\ t > 0$$

$$u(x, 0) = 0,\ 0 < x < \pi.$$

15. $$\frac{\partial^2 u}{\partial x^2} - 1 + x - x\cos t = \frac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$$

$$u(0, t) = 0,\ u(1, t) = 0,\ t > 0$$

$$u(x, 0) = x(1 - x),\ 0 < x < 1$$

16. $$\frac{\partial^2 u}{\partial x^2} + \sin x \cos t = \frac{\partial^2 u}{\partial t^2},\ 0 < x < \pi,\ t > 0$$

$$u(0, t) = 0,\ u(\pi, t) = 0,\ t > 0$$

$$u(x, 0) = 0,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,\ 0 < x < \pi$$

17. $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$$

$$u(0, t) = \sin t,\ u(1, t) = 0,\ t > 0$$

$$u(x, 0) = 0,\ 0 < x < 1$$

18. $$\frac{\partial^2 u}{\partial x^2} + 2t + 3tx = \frac{\partial u}{\partial t},\ 0 < x < 1,\ t > 0$$

$$u(0, t) = t^2,\ u(1, t) = 1,\ t > 0$$

$$u(x, 0) = x^2,\ 0 < x < 1$$

Discussion Problems

19. Consider the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$

$$u(0, t) = u_0,\ u(L, t) = u_1$$

$$u(x, 0) = f(x),$$

that is a model for the temperature u in a rod of length L. If u_0 and u_1 are different nonzero constants, what would you intuitively expect the temperature to be at the center of the rod after a very long period of time? Prove your assertion.

20. Read (*i*) of the *Remarks* at the end of this section. Then discuss how to solve

$$k\frac{\partial^2 u}{\partial x^2} + F(x, t) = \frac{\partial u}{\partial t},\ 0 < x < L,\ t > 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0,\ t > 0$$

$$u(x, 0) = f(x),\ 0 < x < L.$$

Carry out your ideas by solving the above BVP with $k = 1$, $L = 1$, $F(x, t) = tx$, and $f(x) = 0$.

13.7 Orthogonal Series Expansions