The Gaussian solution or Poisson solution of the following heat problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad t > 0 \quad (1) \\ u(x,0) = f(x), & -\infty < x < \infty, \quad (2) \end{cases}$$

is given by

$$\boxed{u(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(x-\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta}$$

proof: Applying F.T to both sides of (1) to get

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = k\,\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{dU(\alpha,t)}{dt} = -k\alpha^2 U(\alpha,t)$$

or $\boxed{\dfrac{dU}{dt} + k\alpha^2 U = 0}$ (3), where $U = U(\alpha,t)$.

Next, Applying F.T to the condition (2):

$$\mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$

$$\boxed{U(\alpha,0) = F(\alpha)} \quad (4)$$

Now, Solve (3) with condition (4):

$$U(\alpha,t) = A e^{-k\alpha^2 t}$$

$$U(\alpha,0) = A = F(\alpha)$$

$$\Rightarrow \boxed{U(\alpha,t) = F(\alpha)\, e^{-k\alpha^2 t}} \quad (5)$$

We know that (6) $\boxed{\mathcal{F}\{e^{-\beta x^2}\} = \sqrt{\frac{\pi}{\beta}}\, e^{-\frac{\alpha^2}{4\beta}}, \ \beta > 0}$

See your notes

Here we choose, $\frac{1}{4\beta} = kt$ or $\boxed{\beta = \frac{1}{4kt}}$

$$\xrightarrow{\text{in (6)}} \mathcal{F}\left\{e^{-\frac{x^2}{4kt}}\right\} = \sqrt{4\pi kt}\; e^{-kt\alpha^2}$$

or $\boxed{e^{-k\alpha^2 t} = \mathcal{F}\left\{\frac{1}{2\sqrt{\pi kt}}\, e^{-\frac{x^2}{4kt}}\right\}}$ (7)

(7) in (5) gives

$$U(\alpha,t) = F(\alpha)\, e^{-k\alpha^2 t}$$

$$U(\alpha,t) = \mathcal{F}\{f(x)\}\, \mathcal{F}\left\{\frac{1}{2\sqrt{\pi kt}}\, e^{-\frac{x^2}{4kt}}\right\}$$

$$u(x,t) = \mathcal{F}^{-1}\{U(\alpha,t)\}$$

$$= \mathcal{F}^{-1}\left\{ \mathcal{F}\{f(x)\}\, \mathcal{F}\left\{ \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}} \right\} \right\}$$

$$= f(x) * \frac{1}{\underbrace{2\sqrt{\pi k t}}_{\text{Constant in } x}} e^{-\frac{x^2}{4kt}} \qquad \text{(Convolution theorem)}$$

$$= \frac{1}{2\sqrt{\pi k t}} \left(f(x) * e^{-\frac{x^2}{4kt}} \right)$$

$$= \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} f(x-\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta$$

$\blacksquare$

__Ex.__ Find the Gaussian solution of the problem

$$\begin{cases} u_t = u_{xx}, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = e^{-x}. \end{cases}$$

__Sol.__ Here we have $k = 1$, $f(x) = e^{-x}$.

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-x+\eta}\, e^{-\frac{\eta^2}{4kt}}\, d\eta$$

$$= \frac{e^{-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{\eta^2}{4t}+\eta}\, d\eta$$

Now, $$-\frac{\eta^2}{4t}+\eta = -\frac{1}{4t}\left[\underbrace{\eta^2 - 4t\eta + 4t^2} - 4t^2\right]$$

$$= -\frac{1}{4t}\left[(\eta-2t)^2 - 4t^2\right]$$

$$= t - \left(\frac{\eta-2t}{2\sqrt{t}}\right)^2$$

$$\therefore u(x,t) = \frac{e^{-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{t-\left(\frac{\eta-2t}{2\sqrt{t}}\right)^2}\, d\eta$$

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left(\frac{\eta-2t}{2\sqrt{t}}\right)^2}\, d\eta$$

$$u = \frac{\eta-2t}{2\sqrt{t}} \Rightarrow d\eta = 2\sqrt{t}\, du$$

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-u^2} \cdot 2\sqrt{t}\, du$$

$$= \frac{e^{t-x}}{2\sqrt{\pi t}} \cdot 2\sqrt{t} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{e^{t-x} \cdot \cancel{2\sqrt{t}\sqrt{\pi}}}{\cancel{2\sqrt{\pi t}}}$$

$$= e^{t-x}$$