Chapter 1. The Real number system ($\mathbb{R}$) 1812

1.2 Ordered Field Axioms

Postulate 1: (Field Axioms)
There are functions $+$ and $\cdot$ defined on $\mathbb{R}^2 := \mathbb{R}\times\mathbb{R}$ with the following properties for every $a,b,c \in \mathbb{R}$

A) $a+b$ and $a\cdot b \in \mathbb{R}$ (closure property)

B) $(a+b)+c = a+(b+c)$
$(a\cdot b)\cdot c = a\cdot(b\cdot c)$ Assocative (Laws

C) $a+b = b+a$
$a\cdot b = b\cdot a$ Commutitive

D) $\exists!$ element $0\in\mathbb{R}$ s.t. $a+0=a$, $\forall a\in\mathbb{R}$ Additive identity

E) $a\cdot(b+c) = a\cdot b + a\cdot c$ Distributive law

F) $\exists!$ element $1\neq 0$ in $\mathbb{R}$ s.t. $1\cdot a = a$, $\forall a\in\mathbb{R}$ Multiplicative identity

G) $\forall a\in\mathbb{R}$, $\exists!$ $-a\in\mathbb{R}$ s.t. $a+(-a)=0$

H) $\forall a\in\mathbb{R}\setminus\{0\}$, $\exists!$ $a^{-1}\in\mathbb{R}$ s.t. $a\cdot a^{-1}=1$

from postulate 1, we can derive the following (H.W)

① $(-1)^2 = 1$
$0\cdot a = 0$
$-a = (-1)\cdot a$
$-(-a) = a$, $a\in\mathbb{R}$

② $-(a-b) = b-a$, $\forall a,b\in\mathbb{R}$

③ $a,b\in\mathbb{R}$ and $ab=0 \Rightarrow a=0$ or $b=0$

Postulate 2 (Order Axioms)

There is a relation $<$ on $\mathbb{R}\times\mathbb{R}$ that has the following properties:-

i) $\forall a,b \in \mathbb{R}$, exactly one of the following is True
$a>b$, $a<b$, $a=b$

This is called Trichotomy property:

ii) $\forall a,b,c \in \mathbb{R}$, $a<b$, $b<c$, then $a<c$ (Transitive)

iii) $\forall a,b,c \in \mathbb{R}$ $a<b$ and $c\in\mathbb{R} \Rightarrow a+c<b+c$ (Additive)

iv) For $a,b,c \in \mathbb{R}$, $a<b$ and $c>0 \Rightarrow ac<bc$
$a<b$ and $c<0 \Rightarrow ac>bc$

- $a\leq b$ means $a<b$ or $a=b$
- $a<b<c$ means $a<b$ and $b<c$

ex. $2<x<1$ makes no sense at all

$a\in\mathbb{R}$ is nonnegative if $a\geq 0$ and positive if $a>0$

Remark: The real number system $\mathbb{R}$ contains the following special subsets.

(1) The set of natural number $\mathbb{N}=\{1,2,3,\ldots$

(2) The set of integers $\mathbb{Z}=\{\cdots,-2,-1,0,1,2,\cdots\}$

(3) Rationals (Fractions or quotients) $\mathbb{Q}:=\{\frac{m}{n}: m,n\in\mathbb{Z}, n\neq 0\}$

- Equality in $\mathbb{Q}$ is defined as $\frac{m}{n}=\frac{p}{q} \Leftrightarrow mq=np$

Notice that $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$

(4) Irrationals $\mathbb{Q}^c=\mathbb{R}\setminus\mathbb{Q}$

Remark: The sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy

(i) if $m,n\in\mathbb{Z}$, then $m+n$ and $m\cdot n$, $m,n\in\mathbb{Z}$

(ii) if $n\in\mathbb{Z}$, then $n\in\mathbb{N} \Leftrightarrow n\geq 1$

(iii) There is no $n\in\mathbb{Z}$ s.t. $0<n<1$

Ex. (HW) ① prove that ① satisfies postulate 1:

m in Q is $\frac{a}{b}$ $\qquad$ 0 in Q is $\frac{0}{a}$, $a \neq 0$

ex. if $a \in \mathbb{R}$, prove that $a \neq 0 \Rightarrow a^2 > 0$. In particular, $-1 < 0 < 1$

proof: since $a \neq 0$, then either $a > 0$ or $a < 0$

Case 1: $a > 0 \Rightarrow a.a > 0.a$

$\Rightarrow a^2 > 0$

Case 2: $a < 0 \Rightarrow a.a > 0.a$

$a^2 > 0$

This proves $a^2 > 0$, when $a \neq 0$

$1 \neq 0 \Rightarrow 1 = 1^2 > 0$

$\Rightarrow 1 > 0 \Rightarrow 0 < 1$

$\Rightarrow 0 - 1 < 1 - 1$ $\qquad$ أو نضيف في (-1)

$\Rightarrow -1 < 0$

ex. (H.W) $0 \leq a < b$ and $0 \leq c < d \Rightarrow ac < bd$

② $0 \leq a < b \Rightarrow 0 \leq a^2 \leq b^2$ and $0 \leq \sqrt{a} \leq \sqrt{b}$

③ $0 < a < b \Rightarrow 0 < \frac{1}{b} < \frac{1}{a}$

④ $0 < a < 1 \Rightarrow 0 < a^2 < a$ and $a > 1 \Rightarrow a^2 > a$, $\forall a \in \mathbb{R}$

see textbook page 9.

Df: (The absolute value)

$$|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

ex. (HW) prove that $|ab| = |a||b|$, $\forall a, b \in \mathbb{R}$

see textbook

Thm ① let $a \in \mathbb{R}$ and $M \geq 0$ Then $|a| \leq M \Leftrightarrow -M \leq a \leq M$

Pf: ($\Rightarrow$) suppose that $|a| \leq M$. Then multiply both sides by -1,
we have $-|a| \geq -M$

Case 1: $a \geq 0 \Rightarrow |a| = a$. Thus $-M \leq 0 \leq a = |a| \leq M$

This implies $-M \leq a \leq M$

Case 2: $a < 0$, By definition, $|a| = -a$ $\quad -M \leq |a| = -(-a) = a < 0 \leq M$

This proves $-M \leq a \leq M$ in each case.

Conversely, suppose that $-M \leq a \leq M$
we need to show $|a| \leq M$ ما إلها داعي نبرهنه

Since $-M \leq a \leq M$, then $-M \leq a$ and $a \leq M$. Multiply both sides by -1 (of 1st inequality).

we have $M \geq -a$. Consequently,

$|a| = a \leq M$ if $a \geq 0$,
and $|a| = -a \leq M$ if $a < 0$

This proves $|a| \leq M$, $\forall a \in \mathbb{R}$

ex. (H.W) One can prove $|a| < M \Leftrightarrow -M < a < M$, $\forall a \in \mathbb{R}$, $M \geq 0$

1.2 Continue 2012

Let $a \in \mathbb{R}$ and $M \leq 0$. Then $|a| \leq M \Leftrightarrow -a \leq M \leq a$

Thm 2: The absolute value satisfies

i) $\forall a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0 \Leftrightarrow a = 0$ positive definite

ii) (Symmetric), $\forall a, b \in \mathbb{R}$ $|a-b| = |b-a|$

iii) Triangular inequality

$\forall a, b \in \mathbb{R}$ $|a+b| \leq |a| + |b|$ & $||a| - |b|| \leq |a-b|$

Proof. (i) & (ii) Exercise

iii) To prove the 1st inequality, notice that $|x| \leq |x|$, $\forall x \in \mathbb{R}$

Thus, Thm ①: $-|x| \leq x \leq |x|$

$\Rightarrow \forall a, b \in \mathbb{R}$ $-|a| \leq a \leq |a|$ & $-|b| \leq b \leq |b|$

adding these inequalities ($\leq$'s):

$-(|a| + |b|) \leq a+b \leq |a| + |b|$

Thm① $\Rightarrow$ $|a+b| \leq |a| + |b|$

To prove 2nd inequality apply the 1st inequality to $(a-b)+b$

$|a| - |b| = |(a-b)+b| - |b|$

$\leq |a-b| + \cancel{|b|} - \cancel{|b|}$ $\Rightarrow |a| - |b| \leq |a-b|$ ...(1)

By reversing the order (rules) of a and b, we also obtain $|a| - |b| \leq |b-a| = |a-b|$ ②

$-|a-b| \leq |a| - |b|$ ...②

① & ② lead to

$-|a-b| \overset{②}{\leq} |a| - |b| \overset{①}{\leq} |a-b|$

Thm 1 $\Rightarrow$ $||a| - |b|| \leq |a-b|$

* warning: don't mix absolute values and the additive property to conclude $b < c \Rightarrow |a+b| < |a+c|$

Counter example: $-5 < -1$ $\not\Rightarrow$ $|-5+2| < |-1+2|$

since $3 \not< 1$

Ex. prove that if $-2<x<1$ then $|x^2-x|\leq 6$

Since $-2<x<1 \Longrightarrow |x|<2$, Thm 2 $\Rightarrow |x^2-x| \leq |x^2|+|-x|$
$= |x|^2+|x|$
$< 2^2+2 = 6$ ∎

Thm ③ let $x,y,a \in \mathbb{R}$ Then:
i) $x<y+\epsilon, \forall \epsilon>0$ iff $x\leq y$
ii) $x>y-\epsilon, \forall \epsilon>0 \Leftrightarrow x\geq y$
iii) $|a|<\epsilon \; \forall \epsilon>0 \Leftrightarrow a=0$

Proof: suppose to contrary that $x<y+\epsilon, \forall \epsilon>0$ but $x>y$.
Set $\epsilon_0 = x-y$ and observe that $\epsilon_0+y=x$
Hence by Trichotomy property, x can't be smaller $y+\epsilon_0$
& " " " greater "

This contradicts the hypothesis for $\epsilon=\epsilon_0$. Thus, $x\leq y$
$\Leftarrow$ conversely, suppose that $x\leq y$ & let $\epsilon>0$ be given. (For all given ϵ)
Either $x<y$ or $x=y$
Case 1: if $x<y$, then $x=0+x<0+y<\epsilon+y \Longrightarrow x<\epsilon+y$
Case 2: if $x=y$, $x<y+\epsilon$
Then $x<y+\epsilon \; \forall \epsilon>0$ in either case.

$x<x+\epsilon$, $x>y$
$\Rightarrow x<y+\epsilon$

ii) suppose that $x>y-\epsilon \; , \forall \epsilon>0$
$\Rightarrow -x<-y+\epsilon$ $\xrightarrow{\text{part (i)}} (-x\leq -y) \cdot -1 \Rightarrow x\geq y$.
conversely, بنفس طريقة الأولى

iii) suppose that $|a|<\epsilon, \forall \epsilon>0$ for $|a|<\epsilon=0+\epsilon$
Applying part (i) (with $x=|a|$, $y=0$)
$\Rightarrow |a|\leq 0$. And we know $|a|\geq 0$ is always the case.
— By Trichotomy property, $|a|=0$, Therefore, $a=0$
$\Leftarrow$ conversely, $a=0$ & $\epsilon>0 \Rightarrow |a|<\epsilon$

Now, $|a| = |0| = 0 < \varepsilon$ .. #

* Thus : بنحكي في نهاية البرهان لما نتنهي مش أول البرهان
* Therefor : بنحكي قبل الآخر سطر (مش في البداية)
* $\Rightarrow$ مش محل البرهان

Df. let $a, b \in \mathbb{R}$, A closed interval is a set of the form

$$[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$[a,\infty) := \{x \in \mathbb{R} : x \geq a\}$$
$$(-\infty, b] := \{x \in \mathbb{R} : x \leq b\}$$
or $(-\infty, \infty) = \mathbb{R}$

Open interval
$$(a,b) := \{x \in \mathbb{R} , a < x < b\}$$
$$(a,\infty) := \{x > a : x \in \mathbb{R}\}$$
$$(-\infty, a) := \{x \in \mathbb{R} : x < b\}$$

By an interval, we mean a closed interval, an open interval, or a set of the form $[a,b)$, $(a,b]$

- An interval I is bounded iff it has the form $[a,b]$, (a,b), $[a,b)$ or $(a,b]$ for $-\infty < a \leq b < \infty$; a,b : end points of I. All other intervals will be unbounded

if $a = b \Rightarrow I$ is degenerate (تافهة)
non degenerate iff $a < b$

1.3 Completeness Axiom

Df ① let $E \subseteq \mathbb{R}$ be a nonempty set $\Rightarrow$

(i) E is said to be bounded above $\Leftrightarrow$ $\exists$ an $M \in \mathbb{R}$ s.t. $x \leq M$, $\forall x \in E$

in this case, M is called an upperbound of E.

(ii) A number β is called a supremum of E iff β is an upper bound of E & $\beta \leq M$ for all upperbounds M of E

Here, we say E has a finite supremum.

& write $\beta = \sup E$

Remark: (1) by last Def ① $\sup E$ (when it exists) is the smallest (least) upper bound of E.

(2) In order to prove $\beta = \sup E$ for some $E \subset \mathbb{R}$

We must show 2 things:

① β is an upper bound of E

② β is the smallest upper bound (i.e., if M is any upperbound, then $\beta \leq M$)

Ex. let $E = [0,1]$, prove that $\sup E = 1$.

PF: ① By the df of interval 1 is an upper bound of E.

② let M be any upper bound of E. We have to show $M \geq 1$

* sup may not equal max (if $\sup \in E$ then $\sup E = \max$)

Since M is an upper bound of E, then $M \geq x$, $\forall x \in E$. In particular, take $x = 1 \in E$. Therefore, $M \geq 1$. Hence, $\sup E = 1$.

ex. Let $E_1 = \mathbb{R}^-$, $E_2 = \mathbb{Z}^-$ then $\sup E_1 = 0$, $\sup E_2 = -1$

Question: How many upperbounds & suprema can a given set have?

Ans. See the remark below

Rmk ③. If a set has one upper bound, it has infinitly many upperbounds.

PF. If M_0 is an upperbound for a set E, then so is M for all $M > M_0$. ∎

Rmk ④. if a set E has a sup β (i.e. $\sup E = \beta$), then it is unique.

pf. let β_1 & β_2 suprema of E. we have to show $\beta_1 = \beta_2$.

Then, both β_1 & β_2 are upper bounds

if E, W hence by Def ①

$\beta_1 \leq \beta_2$ and $\beta_2 \leq \beta_1$, we conclude that $\beta_1 = \beta_2$

Thm 1 [Approximation property for suprema] $\rightarrow$ نصاً وبرهاناً

if E has a finite sup β & $\epsilon > 0$ an any positive number, then there is a point $x \in E$ s.t. $\beta - \epsilon < x \leq \beta$

pf. suppose that the theorem is False. Then there exists an $\epsilon_0 > 0$ s.t. no element of E lies between $\beta - \epsilon$ & β.

Since $\beta = \sup E$ is an upper bound of E. *every sup is an upper bound

It follows $x \leq \beta - \epsilon_0$, $\forall x \in E$, i.e., $\beta - \epsilon_0$ is an upper bound of E.

no point here

$\beta - \epsilon$ β

Thus by def① , $\beta \leq \beta - \epsilon_0$. It follows, $\epsilon_0 \leq 0$ and a contradiction. ∎

Thm ②. * مهم جداً

If $E \subset \mathbb{Z}$ has a sup β, $\beta \in E$

(if the sup of a set, which contains only integers, exits, that sup must be an integer).

Pf. suppose that $\beta = \sup E$. Apply the approximation property for $\epsilon = 1$

$\exists x_0 \in E$ s.t. $\beta - 1 < x_0 \leq \beta$ * بنطبق، بناخد أي قيمة لل ϵ، أخذنا $\epsilon = 1$

If $\beta = x_0$ then $\beta = x_0 \in E$

Otherwise, $\beta - 1 < x_0 < \beta$. Apply theorem 1 (Thm) again (Approximation property), $\exists x_1 \in E$ s.t. $x_0 < x_1 \leq \beta$

x_0 x_1

E $\beta - 1$ β

The last inequality gives $0 < x_1 - x_0 < \beta - x_0$.

Since $-x_0 < 1 - \beta$, it follows that $0 < x_1 - x_0 < \beta + 1 - \beta = 1$

$\Rightarrow x_1 - x_0 \in \mathbb{Z} \cap (0, 1)$, a contradiction, we conclude that $\beta \in E$.

Postulate ③ [Completness Axiom] مهم

if E is a nonempty subset of $\mathbb{R}$ that is bounded above then E has a finite sup.

Rmk: From postulate 1 & 2 (sec. 1.2), and 3 (sec 1.3) we say that $\mathbb{R}$ is a complete ordered field.

1.3 continue.

Th.3 (Archimedean property) مهما b كانت أكبر بنلاقي

Given real numbers a & b, then $\exists$ an integer $n \in \mathbb{N}$ s.t. $na > b$

PF. if $b < a$ set $n = 1$, (case 1)

if $b \geq a$ and $a > 0$ (case 2)

consider the set $E = \{k \in \mathbb{N} : ka \leq b\}$

$E \neq \emptyset$ since $1 \in E$ → $b \geq a$ assumption ال

Let $k \in E \Rightarrow ka \leq b$. Since $a > 0$, it Follows $k \leq \frac{b}{a}$.

This prove that E is bounded above. Thus, completeness Axiom and Thm 2, E has a finite sup β and $\beta \in E$.

In particular $\beta \in \mathbb{N}$. Set $n = \beta + 1$. Then $n \in \mathbb{N}$ and $n > \beta \Rightarrow n \notin E$

$\Rightarrow na > b$ ■

* إذا ال sup عنصر في ال set بكون هو ال max
* " " مش " " " ما بكون max.
* ال sup ممكن يكون في ال set وممكن لا بس في حالة ال set ← int
أكيد ال sup موجود في الفترة

Rmk: Sup E is not always belong to E.

Ex. let $A = \{1, \frac{1}{2}, \frac{1}{4}, \dots\}$ $\quad B = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

prove that $\sup A = 1 \in A$ & $\sup B = 1$

PF. For A, it is clear that 1 is an upper bound of A. let M be any upper bound of A. This means $M \geq a, \forall a \in A$. In particular, $a = 1$
$\Rightarrow M \geq 1$. Hence 1 is least upper bound (sup)

واضح

for B, it is clear that 1 is an upper bound.
Let N be any upperbound of B. Suppose that $N < 1$
This gives $1 - N > 0 \Longrightarrow \frac{1}{1-N} \in \mathbb{R}$. By Archmedian principle
$\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{1-N}$

It follows $x_0 := 1 - \frac{1}{n} > N$
$\rightarrow x_0 \in B$ by definition of B
$x_0 > N$ for $x_0 \in B$

This Contradict, the assumption that N is an upperbound of B
$(b \leq N, \forall b \in B)$
Hence, $N \geq 1$. This gives, $1 = \sup B$. ■

Thm 3 (Density of Rationals) irrational → H.W

The rational numbers $\mathbb{Q}$ are dense in the reals $\mathbb{R}$ i.e, if $a, b \in \mathbb{R}$ with $a < b$, then there is a rational $q \in \mathbb{Q}$ s.t. $a < q < b$
irrational → H.W
proof:

1.3. continue

Theorem 4 (Density Theorem)

if $a, b \in \mathbb{R}$ s.t $a<b$ & $\exists q \in \mathbb{Q}$ s.t. $a<q<b$

befor proof:-

أول اشي منثبت انه في sup ⟺ nonempty, boundedabove

$\sup \leq M$

، إذا معطى ني ال set (أو منعطيها)

proof:-

suppose that $a>0$

Case ($0<a<b$)

Use Archimedean principle to choose an $n \in \mathbb{N}$ s.t.

$$\max\left\{\frac{1}{a}, \frac{1}{b-a}\right\} < n.$$

This implies $\frac{1}{n}<a$ & $\frac{1}{n}<b-a$

consider the set

$E = \{k \in \mathbb{N} : \frac{k}{n} \leq a\}$, $E \neq \phi$ since $1 \in E$ ($\frac{1}{n}<a$)

since $n>0$, E is bounded above by an. Hence by Theorem 2,

$k_0 := \sup E$ exists, and $k_0 \in E$ (set of Integers).

In particular $k_0 \in \mathbb{N}$

set $m = k_0+1$ and $q = \frac{m}{n}$ since $k_0 := \sup E$, then $m \notin E$

Thus by def. of q we have $q = \frac{m}{n} = \frac{k_0+1}{n} > a$

On the other hand, since $k_0 \in E$, it follows that $b = a+b-a$

$$> \frac{k_0}{n} + \frac{1}{n} = \frac{k_0+1}{n} = \frac{m}{n} = q$$

$\Rightarrow b>q$

Thus, $\exists q \in \mathbb{Q}$ s.t. $a<q<b$ $\forall a>0$

suppose that $a \leq 0$.

Choose by Archimedean principle, $k \in \mathbb{N}$ s.t.

$$0 < k+a < k+b$$

and by the case already proved, $\exists q \in \mathbb{Q}$ s.t.

$$k+a < r < k+b$$
$$a < r-k < b$$

Therefore, $q := r-k \in \mathbb{Q}$ satisfy $a<q<b$. ■

Exercise: If $a,b \in \mathbb{R}$ with $a<b$, $\exists$ an irrational $y \in \mathbb{Q}$ s.t. $a<y<b$

proof:- (use Thm 4).

Infimum of a set:-

Df ② . let $\varphi \neq E \subseteq \mathbb{R}$

(i) E is said to be bounded below iff $\exists\, m \in \mathbb{R}$ s.t. $m \leq x, \forall x \in E$
in this case, m is said to be a lowe bound of E.

(ii) A number α is called an infimum of E iff α is a lowerbound of E and $\alpha \geq \gamma$, $\forall$ lower bound γ of E ($\alpha = \inf E$)

* When inf E exists, it is the greatest lower bound.

iii) E is said to be bounded iff bounded above & bounded below.
(i.e $\exists$ #s m, M s.t. $m \leq x \leq M$ $\forall x \in E$).
or ($\exists M > 0$ s.t. $|x| \leq M$, $\forall x \in E$)

Remark:-

A bounded nonempty set E has a unique sub & unique inf
Moreover, $\inf E \leq \sup E$

① prove this
② give necessary and sufficient conditions for equality.
③ When a set E contains its sup, we write $\max E = \sup E$
Similarly if $\inf E \in E$, we write $\inf E = \min E$

Theorem 5: (Reflection principle)
let $E \subseteq \mathbb{R}$ be nonempty

(i) E has a sup iff $-E$ has an inf
In which case, $\inf(-E) = -\sup E$
$\sup(-E) = -\inf E$

(ii) E has an inf iff $-E$ has a sup
In which case, $\sup(-E) = -\inf E$

(i) $\boxed{\Rightarrow}$ suppose that $\beta = \sup E$ exists
since β is an upper bound for E, $x \leq \beta$ $\forall x \in E$
This gives, $-\beta \leq -x$, $\forall x \in E$ ($-\beta$ is a lower bound of E)

suppose that m is any lower bound of $-E$, Then
$m \leq -x$, $\forall x \in E$
$-m \geq x$, $\forall x \in E$
$-m$ is an upper bound of E. Since $\sup E = \beta$, $\beta \leq -m \Rightarrow m \leq -\beta$

Thus, $-\beta = \inf(-E)$ and $\sup E = \beta = --(\beta)$
$\Rightarrow \sup E = -\inf(-E)$
$-\sup E = \inf(-E)$

$\boxed{\Leftarrow}$ Conversely,

$\boxed{\Leftarrow}$ Conversly, suppose that $-E$ has an inf α. We need to show that E has sup

Since $\inf -E = \alpha$, by definition, $\alpha \leq -x \quad \forall x \in E$

Thus, $-\alpha > x \quad \forall x \in E$

Thus, $-\alpha$ is an upper bound of E (i.e, E is bounded above).

Since $E \neq \phi$ (given) and bounded above,

Then by Completness Axiom sup E exists.

Note :-

* $\sup(kE) = k \sup E$

* $\sup(x + A) = \sup A + x$

Theorem 6 (Monotone Property)

suppose that $A \subseteq B$ are nonempty sets of $\mathbb{R}$

(i) If B has a sup then $\sup A \leq \sup B$

(ii) If B has an inf then $\inf A \geq \inf B$

Proof :-

(i) Since $A \subseteq B$, any upper bound of B is an upper bound of A.

Therefore, sup B is an upper bound of A.

It follows that by completness Axiom that sup A exist's.

Thus, by definition of sup A, $\sup A \leq \sup B$

(ii) Since $A \subseteq B$ then $-A \subseteq -B$; using part (i)

by Thm 5 :- $\sup(-A) \leq \sup(-B)$

$\Rightarrow -\inf A \leq -\inf B$

$\Rightarrow \inf A \geq \inf B$

Theorem 7 (Approximation property for infimum).

If a set $E \subseteq \mathbb{R}$, has a finite inf α & $\epsilon > 0$ is any positive number, then $\exists$ a point $x \in E$ s.t.

$$\alpha \leq x < \alpha + \epsilon$$

Proof: (Exercise)

Completeness Axiom for infima.

If $E \subseteq \mathbb{R}$ s.t. $E \neq \phi$ that is bounded below, then $\inf E$ exists

The Extended real number

Definition :-

① $\phi \neq E \subseteq \mathbb{R}$ is unbounded above if it has no upper bound or unbounded below if it has no lower bound.

② The set of extended real numbers is $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$

i.e $x \in \overline{\mathbb{R}}$ iff $x \in \mathbb{R}$, or $x = \pm\infty$.

③ let $\phi \neq E \subseteq \mathbb{R}$. We define $\sup E = \infty$ if E is unbounded above and $\inf E = -\infty$ if E is unbounded below.

Ex.

$E = (-\infty, 2) \Rightarrow \sup E = 2$, $\inf E = -\infty$

Ex.

$E = (2, \infty) \Rightarrow \sup E = \infty$, $\inf E = 2$

[4] $\sup \phi = -\infty$, $\inf \phi = \infty$ provided we use $-\infty < \infty$

Remember :-

$\phi \subseteq A \Rightarrow \sup \phi \leq \sup A$

$\inf \phi \geq \inf A$

ex.

$\sup \mathbb{Z} = \infty$, $\inf \mathbb{Z} = -\infty$

$\sup \mathbb{N} = \infty$, $\inf \mathbb{N} = 1$

$\sup \mathbb{R} = \infty$, $\inf \mathbb{R} = -\infty$

CH2. Sequences in $\mathbb{R}$

2.1 limits of sequences:

- An infinite sequence (briefly, a sequence) is a function whose domain is $\mathbb{N}$.
- A sequence $x_n := f(n)$ will be denoted by

$x_1, x_2, \ldots$ OR $\{x_n\}_{n\in\mathbb{N}}$, OR $\{x_n\}$.

Ex. ① $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ represents the sequence $\left\{\frac{1}{2^{n-1}}\right\}_{n\in\mathbb{N}}$

② $\{-1, 1, -1, 1, \ldots\}$ is the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$.

③ $\{1, 2, 3, 4, \ldots\}$ " " " $\{n\}_{n\in\mathbb{N}}$.

Important $\{x_n\}_{n\in\mathbb{N}}$ is not the set $\{x_n : n\in\mathbb{N}\}$.

ex. $\{1, 2, 3, \ldots\}$ is <u>different</u> from $\{2, 1, 3, \ldots\}$ as sequences. <u>But</u> as sets $\{1, 2, 3, \ldots\}$ is identical with $\{2, 1, 3, \ldots\}$.

ex. $\{1, -1, 1, -1, \cdots\}$ is infinite but the set $\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ only.

Df: ① A sequence of real numbers $\{x_n\}$ is said to converge to $a \in \mathbb{R}$ iff

$\forall \varepsilon > 0$, $\exists$ an $K \in \mathbb{N}$ (in general $K(\varepsilon)$)

such that $\boxed{n \geq K \Rightarrow |x_n - a| < \varepsilon}$

Notations (a) $\{x_n\}$ converges to a.

(b) x_n converges to a.

(c) $\lim\limits_{n \to \infty} x_n = a$

(d) $x_n \longrightarrow a$ as $n \to \infty$

(e) the limit of $\{x_n\}$ exists and equals a.

Rmks. (1) when $x_n \longrightarrow a$ as $n \to \infty$, you can think of x_n as a sequence of approximations to a and ε as an upper bound for the error

(2) the number K in Df (1) is choosen so that the error is less than ε when $n \geq K$

In general, the smaller ε gets, the Larger K must be.

(3) $x_n \longrightarrow a$ iff $|x_n - a| \longrightarrow 0$ as $n \to \infty$

In particular, $x_n \longrightarrow 0$ iff $|x_n| \longrightarrow 0$ as $n \to \infty$.

(4) K depends on $\underline{\underline{\varepsilon}}$ CANNOT depend on $\underline{\underline{n}}$

(5) (Summary of Df (1)). $x_n \longrightarrow a \iff |x_n - a|$ is small for large n.

Ex 1. Prove that $\frac{1}{n} \longrightarrow 0$ as $n \to \infty$.

Pf. Let $\varepsilon > 0$ be given. We need to find $K \in \mathbb{N}$ s.t. $n \geq K \Longrightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$. Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t $K > \frac{1}{\varepsilon}$.

Now, $n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K} < \varepsilon$. It follows that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$, $\forall n \geq K$. ▩

Ex 2. If $\lim_{n\to\infty} x_n = 2$, prove that

$$\lim_{n\to\infty}\left(\frac{2x_n+1}{x_n}\right) = \frac{5}{2}.$$

Proof. Let $\varepsilon > 0$ be given. Since $x_n \longrightarrow 2$, Apply Df ① to this $\varepsilon > 0$, $\exists\, K_1 \in \mathbb{N}$ such that

$n \geq K_1 \implies |x_n - 2| < \varepsilon$. Next, apply Df ① with $\varepsilon = 1$, $\exists\, K_2 \in \mathbb{N}$ such that

$n \geq K_2 \implies |x_n - 2| < 1$. That is,

$n \geq K_2 \implies x_n > 1$ (i.e., $2x_n > 2$).

Set $K = \max\{K_1, K_2\}$ and suppose that $n \geq K$. Since $n \geq K_1$, we have

$|2 - x_n| = |x_n - 2| < \varepsilon$. Since $n \geq K_2$,

we have $0 < \frac{1}{2x_n} < \frac{1}{2} < 1$. It follows that

$$\left|\frac{2x_n+1}{x_n} - \frac{5}{2}\right| = \left|\frac{2-x_n}{2x_n}\right| = \frac{|x_n-2|}{2x_n} < \frac{\varepsilon}{2x_n} < \varepsilon,$$

for all $n \geq K$. ∎

Ex: Show that the sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ has no limit.

Proof: Spse that $(-1)^n \longrightarrow \alpha$ as $n\to\infty$ for some $\alpha\in\mathbb{R}$. Given $\varepsilon=1$, $\exists$ a $K\in\mathbb{N}$ s.t.

$$n\geq K \Longrightarrow |(-1)^n-\alpha|<\varepsilon.$$

For n odd this implies $|1+a|=|-1-a|<1$, and for n even this implies $|1-a|<1$.

Hence, $2=|1+1|=|1-\alpha+\alpha+1|$
$$\leq |1-\alpha|+|1+\alpha|$$
$$< 1+1=2, \quad \text{i.e } 2<2,$$
a contradiction ∎

Rmk: A sequence can have at most one limit.

Proof: Spse that $x_n\longrightarrow\alpha$ and $x_n\longrightarrow\beta$ as $n\to\infty$

By def'n; $\forall\varepsilon>0$, $\exists$ a $K\in\mathbb{N}$ s.t.

$$n\geq K \Longrightarrow |x_n-\alpha|<\tfrac{\varepsilon}{2} \text{ and } |x_n-\beta|<\varepsilon/2$$

Thus, it follows from triangle inequality that

(6)

$$|\alpha-\beta| = |(\alpha - x_n)+(x_n-\beta)|$$
$$\leq |\alpha - x_n| + |x_n-\beta|$$
$$= |x_n-\alpha| + |x_n-\beta|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

i.e., $|\alpha-\beta| < \varepsilon$, $\forall \varepsilon > 0$. We conclude that $\alpha = \beta$ (see Thm in section 1.2). ∎

Df ②. A subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots\cdots$

Thus, a subsequence $x_{n_1}, x_{n_2}, \cdots$ of $x_1, x_2, \cdots$ is obtained by deleting from $x_1, x_2, \cdots$ all x_n's except those such that $n = n_k$ for some k.

Ex. $\{1, 1, 1, \cdots\}$ is a subsequence of $\{-1, 1, -1, 1, \cdots\}$ by deleting every other terms (set $n_k = 2k$).

توضیح

$$\{\underset{a_1}{-1}, \underset{\substack{a_2 \\ a_{n_1}}}{1}, \underset{a_3}{-1}, 1, \cdots\cdots\} \qquad 1 \to a_4 = a_{n_2}$$

Rmk. If $\{x_n\}_{n\in\mathbb{N}}$ converges to α and $\{x_{n_k}\}_{k\in\mathbb{N}}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, then $x_{n_k} \longrightarrow \alpha$ as $k \longrightarrow \infty$.

Pf. Let $\varepsilon > 0$ be given. Since $x_n \longrightarrow \alpha$, then

$$\exists K \in \mathbb{N} \text{ s.t } n \geq K \Longrightarrow |x_n - \alpha| < \varepsilon.$$

Since $n_k \in \mathbb{N}$ and $n_1 < n_2 < \cdots\cdots$, then by induction, $n_k \geq k$, $\forall k \in \mathbb{N}$.

Hence, $k \geq K \Longrightarrow |x_{n_k} - \alpha| < \varepsilon$

i.e., $x_{n_k} \longrightarrow \alpha$ as $k \longrightarrow \infty$. ∎

Df ③. Let $\{x_n\}$ be a sequence of real numbers. Then

(i) $\{x_n\}$ is said to be bounded above iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded above

i.e., iff $\exists$ an $M \in \mathbb{R}$ s.t $x_n \leq M$, $\forall n \in \mathbb{N}$.

(ii) $\{x_n\}$ is said to be bounded below iff the set $\{x_n : n \in \mathbb{N}\}$ is bounded below i.e., iff $\exists$ an $m \in \mathbb{R}$ s.t. $x_n \geq m$, $\forall n \in \mathbb{N}$.

(iii) $\{x_n\}$ is said to be bounded iff it is bounded both above and below.

i.e., $\exists$ a $C > 0$ such that $|x_n| \leq C$, $\forall n \geq N$.

Question. Is there a relationship between convergent sequences and bounded sequences

Ans. See thm below.

Thm. Every convergent sequence is bounded

proof. Let $\{x_n\}$ be a sequence s.t. $x_n \to \alpha \in \mathbb{R}$.
Let $\varepsilon = 1$ be given, then $\exists$ a $K \in \mathbb{N}$ s.t.

$$n \geq K \implies |x_n - \alpha| < 1.$$

Hence, by triangle inequality,

$$\begin{aligned} |x_n| &= |(x_n - \alpha) + \alpha| \\ &\leq |x_n - \alpha| + |\alpha| \\ &< 1 + |\alpha|, \quad \forall n \geq K. \end{aligned}$$

On the otherhand, if $1 \le n < K$, then

$$|x_n| \le M := \max\{|x_1|, |x_2|, \dots, |x_K|\}$$

Therefore, $|x_n| \le \max\{M, 1+|\alpha|\}, \forall n \ge K$

That is $\{x_n\}$ is <u>bounded</u> and dominated by $\max\{M, 1+|\alpha|\}$. ■

<u>Rmk.</u> the Converse of last thm is <u>False</u>

<u>Counterexample</u>. Let $x_n = (-1)^n$

Notice that $|x_n| = |(-1)^n| = 1$, so it is <u>bounded</u> but it is diverges (see p.5).

<u>HWs.</u> 0, 1, 2, 3, 4, 5, 6, 7, 8

وفقكم الله

Exercises

1. prove that If x_n converges to $a \in \mathbb{R}$, then $\frac{x_n}{n} \to 0$.

Pf. spse that $x_n \to a$ as $n \to \infty$. Since x_n converges, then it is bdd, i.e., $\exists$ an $M > 0$ s.t. $|x_n| \leq M$, $\forall n \in \mathbb{N}$. Let $\varepsilon > 0$ be given, we need to find $K \in \mathbb{N}$ s.t

$$n \geq K \implies \left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n} < \varepsilon.$$

Use the Archimedean principle $\exists$ $K \in \mathbb{N}$ s.t $K > \frac{M}{\varepsilon}$. Then $n \geq K$ implies

$$\left|\frac{x_n}{n} - 0\right| = \frac{|x_n|}{n} \leq \frac{|x_n|}{K} \leq \frac{M}{K} < \varepsilon.$$

2. (True) or (False)

a) If x_n conv. & y_n bdd, then $x_n y_n$ conv.

Ans. False Take $x_n = 1$ conv., $y_n = (-1)^n$ bdd but $x_n y_n = (-1)^n$ doesnot conv.

b) If $x_n \to 0$ and $y_n > 0$, $\forall n \in \mathbb{N}$, then $x_n y_n$ converges.

Ans. False. Take $x_n = \frac{1}{n}$ conv., $y_n = n^2 > 0$ but $x_n y_n = n$ div.

(3) Prove that $\lim_{n\to\infty} \frac{2n^2+1}{3n^2} = \frac{2}{3}$

Pf: Let $\varepsilon > 0$ be given. We need to find a $K \in \mathbb{N}$ s.t $n \geq K \Longrightarrow \left| \frac{2n^2+1}{3n^2} - \frac{2}{3} \right| < \varepsilon$. Use the Archimedean principle $\exists K \in \mathbb{N}$ s.t. $K > \frac{1}{\sqrt{3\varepsilon}}$. Thus $n \geq K$ implies

$$\left| \frac{2n^2+1}{3n^2} - \frac{2}{3} \right| = \left| \frac{6n^2+3-6n^2}{9n^2} \right|$$

$$= \frac{1}{3n^2} \leq \frac{1}{3K^2} < \frac{(\sqrt{3\varepsilon})^2}{3} = \varepsilon$$

■

(4) Let c be fixed, positive constant. If $\{b_n\}$ is a seq. s.t $b_n \geq 0$ and $b_n \to 0$, and $\{x_n\}$ is a real seq. that satisfies

$$|x_n - a| \leq c\, b_n \text{ for large } n.$$

Prove that $x_n \to a$ as $n \to \infty$.

Proof: Let $\varepsilon > 0$ be given. Since $b_n \to 0$, then $\exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |b_n - 0| = b_n < \frac{\varepsilon}{c}$.

Hence by hypothesis, $n \geq K$ implies

$|x_n - a| \leq C b_n < C \cdot \frac{\varepsilon}{C} = \varepsilon$. therefore, by def'n, $x_n \to a$ as $n \to \infty$ ∎

(5) prove that if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \beta$, then

$$\lim_{n\to\infty} (x_n - y_n) = 0$$

Pf: Let $\varepsilon > 0$ be given. Since $x_n \to \beta$ and $y_n \to \beta$, by def'n, $\exists$ a $K \in \mathbb{N}$ s.t.

$$n \geq K \implies |x_n - \beta| < \varepsilon/2 \text{ and } |y_n - \beta| < \varepsilon/2.$$

By the triangle inequality, $n \geq K$ implies

$$\begin{aligned} |(x_n - y_n) - 0| = |x_n - y_n| &= |x_n - \beta + \beta - y_n| \\ &\leq |x_n - \beta| + |y_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By def'n, $x_n - y_n \to 0$ as $n \to \infty$ ∎

2.2 limit Thms

Thm 1 (Squeeze thm).
Suppose that $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are real sequences. then

(i) If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \alpha$, and if $\exists$ an $N_0 \in \mathbb{N}$ s.t.

$$x_n \leq w_n \leq y_n \quad , \forall n \geq N_0,$$

then $\lim_{n\to\infty} w_n = \alpha$.

(ii) If $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bdd, then $\lim_{n\to\infty} (x_n y_n) = 0$.

Proof. (i) Let $\varepsilon > 0$ be given. Since x_n and y_n converge to α, by def'n, $\exists$ $N_1, N_2 \in \mathbb{N}$

s.t. $n \geq N_1 \Longrightarrow -\varepsilon < x_n - \alpha < \varepsilon$

$n \geq N_2 \Longrightarrow -\varepsilon < y_n - \alpha < \varepsilon$.

Set $N = \max\{N_0, N_1, N_2\}$. If $n \geq N$, we have by the hypothesis & the choise of

N_1 and N_2 that

$$\alpha - \varepsilon < x_n \leq w_n \leq y_n < \alpha + \varepsilon$$

i.e., $\alpha - \varepsilon < w_n < \alpha + \varepsilon$, for $n \geq N$.

or $|w_n - \alpha| < \varepsilon$, for $n \geq N$.

We conclude that $w_n \to \alpha$ as $n \to \infty$.

(ii) Spse that $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}$ is bdd.

Since $\{y_n\}$ is bdd, this means that $\exists$ an $M > 0$ s.t. $|y_n| \leq M$, $\forall n \in \mathbb{N}$. Let $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |x_n| < \frac{\varepsilon}{M}$ (since $x_n \to 0$).

then $n \geq N$ implies

$$|x_n y_n - 0| = |x_n y_n| = |x_n||y_n| < \frac{\varepsilon}{M} M = \varepsilon.$$

We conclude that $x_n y_n \to 0$ as $n \to \infty$. ∎

Ex. Find $\lim_{n\to\infty} \frac{\cos(n^3 - n^2 + n - 13)}{2^n}$

Sol: Since $|\cos x| \leq 1$, $\forall x \in \mathbb{R}$, then

$$\left| \frac{\cos(n^3 - n^2 + n - 13)}{2^n} \right| \leq 2^{-n}$$

Since $2^n > n$ (why?), it is clear $2^{-n} < \frac{1}{n}$.

$$\Rightarrow \quad \underset{\substack{\to 0 \\ \text{as } n\to\infty}}{-\frac{1}{n}} < \frac{\cos(n^3 - n^2 + n - 13)}{2^n} < \underset{\substack{\to 0 \\ \text{as } n\to\infty}}{\frac{1}{n}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\cos(n^3 - n^2 + n - 13)}{2^n} = 0 \text{ by Squeeze thm}$$

Rmk. The squeeze thm can be used to construct convergent sequences with certain properties.

We now establish a result that connects suprema & infima with convergent sequences.

Thm 2. Let $E \subset \mathbb{R}$. If E has a finite sup (resp., a finite inf), then $\exists$ a seq. $x_n \in E$ s.t. $\lim_{n\to\infty} x_n = \sup E$ (resp., a seq. $y_n \in E$ s.t. $\lim_{n\to\infty} y_n = \inf E$)

proof: Spse that E has a finite sup β.

For each $n \in \mathbb{N}$, $\exists$ (by the Approximation Property for suprema), an $x_n \in E$ s.t.

$$\beta - \frac{1}{n} < x_n \leq \beta.$$

Then by the squeeze Thm (Thm 1),

$$\lim_{n\to\infty} x_n = \beta = \sup E.$$

Similarly, $\exists$ a seq. $y_n \in E$ s.t

$$\inf E \leq y_n < \inf E + \frac{1}{n}.$$

Then by the squeeze thm $\lim_{n\to\infty} y_n = \inf E$ ∎

<u>Thm 3</u>. Spse that $\{x_n\}$ & $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are <u>convergent</u>, then

(i) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

(ii) $\lim_{n\to\infty}(\alpha x_n) = \alpha \lim_{n\to\infty} x_n$

and

(iii) $\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right)$.

If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then

(iv) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$.

Proof: Spse that $x_n \to x$ and $y_n \to y$ as $n \to \infty$

(i) Let $\varepsilon > 0$ be given. then $\exists K \in \mathbb{N}$ s.t

$n \geq K \implies |x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$.

Thus $n \geq K$ implies

$$\begin{aligned} |(x_n + y_n) - (x+y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \quad \text{(triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We conclude that $x_n + y_n \to x + y$ as $n \to \infty$

(ii) & (iv) (Exercises)

(iii) Since $\{x_n\}$ conv., then it is bdd.

Hence by the sequence Thm (ii), then sequences

$$\underbrace{x_n}_{\text{bdd}}\underbrace{(y_n - y)}_{\to 0} \longrightarrow 0 \quad \text{as } n \to \infty$$

and $\underbrace{(x_n - x)}_{\to 0}\underbrace{y}_{\text{bdd}} \longrightarrow 0$ as $n \to \infty$. thus,

$$\begin{aligned} \Rightarrow \lim_{n\to\infty} (x_n y_n - xy) &= \lim_{n\to\infty} \left[x_n (y_n - y) + (x_n - x) y\right] \\ &= \lim_{n\to\infty} x_n (y_n - y) + \lim_{n\to\infty} (x_n - x) y \quad \text{(part (i))} \\ &= 0 \end{aligned}$$

. We conclude that $x_n y_n \to xy$ as $n \to \infty$ ∎

<u>Df</u> 1 Let $\{x_n\}$ be a sequence of real numbers.

(i) $\{x_n\}$ is said to be <u>diverge</u> to $+\infty$

(<u>notation</u>: $x_n \longrightarrow +\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = +\infty$)

iff $\forall M \in \mathbb{R}$, $\exists$ an $N \in \mathbb{N}$ s.t.

$$n \geq N \Longrightarrow x_n > M.$$

(ii) $\{x_n\}$ is said to be <u>diverge</u> to $-\infty$

(<u>notation</u>: $x_n \longrightarrow -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$)

iff $\forall M \in \mathbb{R}$, $\exists$ an $N \in \mathbb{N}$ s.t.

$$n \geq N \Longrightarrow x_n < M.$$

<u>Rmk. (Def 1)</u> $x_n \longrightarrow +\infty$ iff given $M \in \mathbb{R}$, x_n is greater than M for sufficiently large n, i.e., eventually x_n exceeds every number M (no matter how large and positive M is).

2 $x_n \to -\infty$ iff x_n eventually is <u>less</u> than every number M (no matter how large and negative M).

<u>Examples [page 12]</u>

<u>Thm 4</u> (An extension of th 3)

Spse that $\{x_n\}$ and $\{y_n\}$ are real sequences s.t. $x_n \to +\infty$ (resp. $x_n \to -\infty$) as $n \to \infty$.

(1) If y_n is bdd below (resp., y_n is bdd above), then $\lim_{n\to\infty} (x_n + y_n) = +\infty$ (resp. $\lim_{n\to\infty} (x_n+y_n) = -\infty$).

(2) If $\alpha > 0$, then

$$\lim_{n\to\infty} (\alpha x_n) = +\infty \; (\text{resp. } \lim_{n\to\infty} (\alpha x_n) = -\infty).$$

(3) If $y_n > M_0$, for some $M_0 > 0$ and all $n \in \mathbb{N}$, then

$\lim_{n\to\infty} (x_n y_n) = +\infty$ (resp. $\lim_{n\to\infty} (x_n y_n) = -\infty$).

(4) If $\{y_n\}$ is bdd and $x_n \neq 0$, then $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$.

Proof: Spse for simplicity that $x_n \to +\infty$ as $n \to \infty$.

(1) By hypothesis, $y_n \geq M_0$ for some $M_0 \in \mathbb{R}$.

Let $M \in \mathbb{R}$ and set $M_1 = M - M_0$.

Since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow x_n > M_1$.

Then $n \geq N \Rightarrow x_n + y_n > M_1 + M_0 = M$.

(2) Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{\alpha}$.

Since $x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t

$n \geq N \Rightarrow x_n > M_1$. Since $\alpha > 0$,

we conclude that $\alpha x_n > \alpha M_1 = M$, $\forall n \geq N$.

(3) Let $M \in \mathbb{R}$ and set $M_1 = \frac{M}{M_0}$. Since

$x_n \to +\infty$, $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$

then $n \geq N \Rightarrow x_n y_n > M_1 y_n > M_1 M_0 = M$.

(4) Let $\varepsilon > 0$. Since $\{y_n\}$ is bdd, then $\exists M_0 > 0$

s.t. $|y_n| \leq M_0$. Since $x_n \to \infty$ as $n \to \infty$,

then $\exists N \in \mathbb{N}$ s.t $n \geq N \Rightarrow x_n > M_1$.

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Choose M_1 so large that $\frac{M_0}{M_1} < \varepsilon$.

Then $n \geq N \implies \left|\frac{y_n}{x_n}\right| = \frac{|y_n|}{x_n} < \frac{M_0}{M} < \varepsilon$.

$\blacksquare$

Rmk. We use the conventions

1) $x + \infty = \infty$, $x - \infty = -\infty$, $x \in \mathbb{R}$.

2) $x \cdot \infty = \infty$, $x \cdot (-\infty) = -\infty$, $x > 0$.

3) $x \cdot \infty = -\infty$, $x \cdot (-\infty) = \infty$, $x < 0$.

4) $\infty + \infty = \infty$, $-\infty - \infty = -\infty$.

5) $\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$, and

$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$.

If we use the above conventions, then Thm 4 contains the following corollary.

Corollary Let $\{x_n\}, \{y_n\}$ be real sequences and α, x, y be extended real numbers.
If $x_n \longrightarrow x$ and $y_n \longrightarrow y$, as $n \to \infty$, then

$\lim_{n \to \infty} (x_n + y_n) = x + y$, provided $x + y$ is not of the form $\infty - \infty$ and

$$\lim_{n\to\infty}(\alpha x_n) = \alpha x, \quad \lim_{n\to\infty}(x_n y_n) = xy$$

provided that none of these products is of the form $0 \cdot \pm\infty$.

Thm [Comparison Thm]

Spse that $\{x_n\}$ & $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ s.t.

(*) $\boxed{x_n \leq y_n \text{ for } n \geq N_0}$, then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a,b]$, converges to some point c, then c must belong to $[a,b]$.

Proof: Spse that the first statement is false, i.e., that (*) holds but $x := \lim_{n\to\infty} x_n > y := \lim_{n\to\infty} y_n$.

Set $\varepsilon = \frac{x-y}{2}$. Choose $N > N_0$ s.t. $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ for all $n \geq N_1$.

Then for such an $n \geq N$,

$$x_n > x - \varepsilon = x - \left(\frac{x-y}{2}\right) = y + \left(\frac{x-y}{2}\right) = y + \varepsilon > y_n,$$

$\Rightarrow$ $x_n > y_n$, which contradicts (*). this proves the first statement.

To prove the second statement, we conclude, $a \leq x_n \leq b$, then by the fist statement $\lim\limits_{n\to\infty} a \leq \lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} b$. this implies $a \leq c \leq b$. ∎

انتبه!!

<u>Rmk.</u> $x_n < y_n$, $n \geq N_0$ <u>Does Not imply</u> that $\lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} y_n$.

<u>counter example</u>, $\frac{1}{n^2} < \frac{1}{n}$, but $\lim\limits_{n\to\infty} \frac{1}{n^2} \not< \lim\limits_{n\to\infty} \frac{1}{n}$

2.2.2 (a) prove that $\lim_{n\to\infty}(n^2-n)=\infty$.

Pf. Let $M\in\mathbb{R}$. Use Archimedean principle $\exists$ an $N\in\mathbb{N}$ s.t. $N>\max\{M,2\}$. Then

$n\geq N \Rightarrow x_n=n^2-n=n(n-1)>N(N-1)>M(2-1)=M$. $\blacksquare$

(b) $\lim_{n\to\infty}(n-3n^2)=-\infty$

Pf. Let $M\in\mathbb{R}$, by Archimedean principle, $\exists$ an $N\in\mathbb{N}$ s.t $N>-\frac{M}{2}$. Notice that

$n\geq 1 \Rightarrow -3n\leq -3$ so $1-3n<-2$. Thus,

$n\geq N \Rightarrow x_n=n-3n^2=n(1-3n)\leq -2n\leq -2N<M$. $\blacksquare$

Ex. show that $\lim_{n\to\infty}\left(\frac{n}{2}+\frac{1}{n}\right)=\infty$.

Pf. Let $M\in\mathbb{R}$, by Archimedean principle, $\exists$ an $N\in\mathbb{N}$ s.t $N>2M$. Then

$n\geq N \Rightarrow x_n=\frac{n}{2}+\frac{1}{n}\geq\frac{N}{2}+\frac{1}{n}>\frac{N}{2}>\frac{2M}{2}=M$. $\blacksquare$

$\boxed{\text{H.w's}\quad 0,1,2,3,4,5,6,7,8}$

2.3 Bolzano – Weirstrass theorem

Notice that the seq $\{(-1)^n\}$ does not converge but it has convergent subsequence. In this section, we will prove that this is a general principle. That is every bounded seq. has a convergent subsequence.

Df(1). Let $\{x_n\}_{n\in\mathbb{N}}$ be a seq. of real numbers

(i) $\{x_n\}$ is said to be increasing (resp., strictly increasing) iff $x_1 \leq x_2 \leq \cdots$ (resp. $x_1 < x_2 < \cdots$)

(ii) $\{x_n\}$ is said to be decreasing (resp., strictly decreasing) iff $x_1 \geq x_2 \geq \cdots$ (resp. $x_1 > x_2 > \cdots$)

(iii) $\{x_n\}$ is said to be monotone iff it is either increasing or decreasing.

Rmk. (1) Some times, we call decreasing seq. nonincreasing and increasing seq. nondecreasing.

(2) If $\{x_n\}$ is increasing (resp., decreasing) and $x_n \longrightarrow a$ as $n \longrightarrow \infty$, we shall write $x_n \uparrow a$ (resp. $x_n \downarrow a$), as $n \longrightarrow \infty$.

(3) Every strictly increasing seq. is increasing and every strictly decreasing seq. is decreasing.

(4) $\{x_n\}$ is increasing iff the sequence $\{-x_n\}$ is decreasing.

We know that any convergent seq. is bounded. We now establish the converse for ~~the~~ monotone sequences.

Thm (1). [Monotone convergence Thm]
If $\{x_n\}$ is increasing and bounded above, or $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

proof. Spse first that $\{x_n\}$ is increasing and bounded above. By the completeness

Axiom, the supremum $\beta := \sup\{x_n : n \in \mathbb{N}\}$ exists and is finite. Let $\varepsilon > 0$. By the Approximation property for Suprema, choose $N \in \mathbb{N}$ s.t

$$\beta - \varepsilon < x_N \leq \beta.$$

Since $x_N \leq x_n$ for $n \geq N$ and $x_n \leq \beta$ for all $n \in \mathbb{N}$, it follows that

$$\beta - \varepsilon < x_n \leq \beta, \text{ for all } n \geq N.$$

In particular, $x_n \uparrow a$ as $n \to \infty$.

If $\{x_n\}$ is decreasing with $\alpha := \inf\{x_n : n \in \mathbb{N}\}$, then $\{-x_n\}$ is increasing with supremum $-\alpha$. Hence, by the first case,

$$\alpha = -(-\alpha) = -\left(\lim_{n\to\infty} -x_n\right) = \lim_{n\to\infty} x_n$$

$\blacksquare$

Ex. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Pf. It suffices to prove that

$$|x|^n \to 0 \text{ as } n \to \infty.$$

④

First, we notice that $|x|^n$ is monotone decreasing since, $|x|<1$ implies $|x|^{n+1} < |x|^n$, $\forall n \in \mathbb{N}$.

Next, notice that $|x|^n$ is bounded below (by 0). Hence by the Monotone Convergence Thm, $\lim |x|^n := L$ exists. To find this limit, $\lim_{n\to\infty} |x|^{n+1} = \lim_{n\to\infty} |x|^n \cdot |x|$.

This implies $L = |x| \cdot L$, thus either $L = 0$ or $|x| = 1$. Since $|x| < 1$, we conclude that $L = 0$. ∎

ex. If $x > 0$, then $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$.

proof. We consider three cases:

Case 1. $x = 1$. Then $x^{\frac{1}{n}} = 1$, $\forall n \in \mathbb{N}$ and it follows that $\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$.

Case 2. $x > 1$. We shall apply the MCT. We shall show that $\{x^{\frac{1}{n}}\}$ is decreasing and bdd below. Indeed, Since $x > 1$, then $x^{n+1} > x^n$. Taking the $n(n+1)$st root of this inequality, we obtain $x^{\frac{1}{n}} > x^{\frac{1}{n+1}}$, i.e., $\{x^{\frac{1}{n}}\}$ is <u>decreasing</u>. Since $x > 1$ implies $x^{\frac{1}{n}} > 1$, it follows that $\{x^{\frac{1}{n}}\}$ is <u>bounded below</u>.

Hence, by the MCT, $L := \lim_{n\to\infty} x^{\frac{1}{n}}$ exists.

To find its value L, we have

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} \left(x^{\frac{1}{2n}}\right)^2 = \left(\lim_{n\to\infty} \frac{1}{x^{2n}}\right)^2$$

$$\Rightarrow L = L^2, \text{ i.e., } L = 0 \text{ or } L = 1.$$

Since $x^{\frac{1}{n}} > 1$, the comparison thm shows that $\lim_{n\to\infty} x^{\frac{1}{n}} \geq \lim_{n\to\infty} 1$; i.e., $L \geq 1$.

Hence $L = 1$.

<u>Case 3</u>. $0 < x < 1$. Then $\frac{1}{x} > 1$. It follows from Case 2 that

$$\lim_{n\to\infty} x^{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} \left(\frac{1}{x}\right)^{\frac{1}{n}}} = 1$$

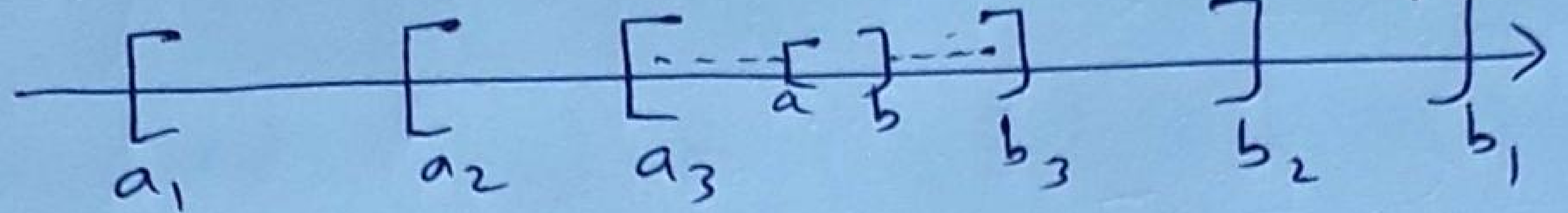
Df. 2 A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be <u>nested</u> iff $I_1 \supseteq I_2 \supseteq \cdots$

Rmk. this is a monotone property for sequence of sets.

Thm 2. [<u>Nested Interval property</u>]

If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bdd intervals, then $E := \bigcap_{n=1}^{\infty} I_n \neq \phi$. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E is a single point.

<u>proof</u>: Let $I_n = [a_n, b_n]$. Since $\{I_n\}$ is nested, then the real seq. $\{a_n\}$ is <u>increasing</u> and <u>bdd above by b_1</u>, and $\{b_n\}$ is <u>decreasing</u> and <u>bdd below</u> by <u>a_1</u>



Thus, by MCT, $\exists$ $a, b \in \mathbb{R}$ such that $a_n \uparrow a$ and $b_n \downarrow b$ as $n \to \infty$. Since $a_n \leq b_n$, $\forall n \in \mathbb{N}$, it follows from the Comparison thm that $a_n \leq a \leq b \leq b_n$.

Hence, $x \in I_n$, $\forall n \in \mathbb{N}$ iff $x \in [a, b]$.

In partical, any $x \in [a,b]$ belongs to all the I_n's i.e., any $x \in [a,b]$, $x \in \bigcap_{n=1}^{\infty} I_n$ (i.e., $\bigcap_{n=1}^{\infty} I_n \neq \phi$).

Next, If $|I_n| \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} (b_n - a_n) = 0 \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$$

$$b = a.$$

But we have proved that $x \in \bigcap_{n=1}^{\infty} I_n$ iff $x \in [a,b]$.

Hence, $\pounds$ is a single point if $\lim_{n\to\infty} |I_n| = 0$ ∎

<u>Rmk ①</u>. the Nested Interval property (thm 2) might not hold if "closed" is <u>omitted</u>.

<u>proof</u>: $I_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$ are bdd and nested ($I_1 = (0,1) \supset I_2 = (0, \frac{1}{2}) \supset \cdots$). but not closed. If there were an $x \in I_n$, $\forall n \in \mathbb{N}$, then $0 < x < \frac{1}{n}$, i.e., $n < \frac{1}{x}$, for all $n \in \mathbb{N}$. Since this contradicts the Archimedean principle, it follows that

$$\bigcap_{n=1}^{\infty} I_n = \phi . \quad \blacksquare$$

(2) the <u>Nested Interval property</u> (thm2) might not hold if "bounded" is omitted.

<u>proof</u>: The intervals $I_n = [n, \infty)$, $n \in \mathbb{N}$ are <u>closed</u> and <u>nested</u> but <u>not bdd</u>.

Again, $\bigcap_{n=1}^{\infty} I_n = \phi$. $\blacksquare$

We are now prepared to establish the main result of <u>this section</u>.

Thm 3. [Bolzano – Weirstrass Thm]

Every bounded sequence of real numbers has a convergent subsequence

proof. Let $\{x_n\}$ be a bdd sequence. Choose $a, b \in \mathbb{R}$ such that $x_n \in [a, b]$, $\forall n \in \mathbb{N}$, and set $I_0 = [a, b]$. Divide I_0 into two halves, say $I' = [a, \frac{a+b}{2}]$ and $I'' = [\frac{a+b}{2}, b]$. Since $I_0 = I' \cup I''$, then $x_n \in I'$ or $x_n \in I''$, for infinitely many n. Say $x_n \in I_1$ and choose $n_1 > 1$ such that $x_{n_1} \in I_1$. Notice that $|I_1| = \frac{|I_0|}{2} = \frac{b-a}{2}$.

Spse that closed intervals $I_0 \supset I_1 \supset \dots \supset I_m$ and natural numbers $n_1 < n_2 < \dots < n_m$ have been choosen s.t, for each $0 \le k \le m$,

$|I_k| = \frac{b-a}{2^k}$, $n_k \in I_k$ and $x_n \in I_k$ for infinitely many n. (2)

To choose I_{m+1}, divide $I_m = [a_m, b_m]$ into two halves, say $I' = [a_m, \frac{a_m+b_m}{2}]$ and $I'' = [\frac{a_m+b_m}{2}, b_m]$. Since $I_m = I' \cup I''$, at least one of these halves contains x_n for infinitely many n. Call it I_{m+1}, and choose $n_{m+1} > n_m$ s.t; $x_{n_{m+1}} \in I_{m+1}$;

Since $|I_{m+1}| = \frac{|I_m|}{2} = \frac{b-a}{2^{m+1}}$,

it follows by induction that there is a nested seq. $\{I_k\}_{k\in\mathbb{N}}$ of nonempty closed bdd intervals that satify (2) for all $k\in\mathbb{N}$. By the Nested Interval property, there is an $x\in\mathbb{R}$ that belongs to I_k, $\forall k\in\mathbb{N}$. Since $x \in I_k$, we have by (2) that

$$0 \le |x_{n_k} - x| \le |I_k| \le \frac{b-a}{2^k}, \forall k\in\mathbb{N}$$

Hence by the squeeze, $x_{n_k} \longrightarrow x$ as $k\to\infty$ ∎

H.W's 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (i.e; All).

2.4 Cauchy sequences

Df 1 A sequence of points $x_n \in \mathbb{R}$ is said to be Cauchy (in $\mathbb{R}$) iff $\forall \varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $n, m \geq N \Longrightarrow |x_n - x_m| < \varepsilon$.

Rmk. 1 If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Proof. Spse that $x_n \longrightarrow a$ as $n \to \infty$. then, given $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. $|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence, if $n, m \geq N$, then

$$|x_n - x_m| = |x_n - a + a - x_m|$$
$$\leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \blacksquare$$

the following result shows that the converse of the above remark is also true (for real sequences).

Thm 1 [Cauchy]. Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy iff $\{x_n\}$ converges (to some point a in $\mathbb{R}$).

Proof: By Rmk 1, we need only show that every Cauchy sequence converges.

Suppose that $\{x_n\}$ is Cauchy. Given $\varepsilon = 1$, $\exists N \in \mathbb{N}$ s.t. $|x_N - x_m| < 1$, for all $m \geq N$.

By the triangle inequality,

$$\begin{aligned} |x_m| &= |x_m - x_N + x_N| \\ &\leq |x_N - x_m| + |x_N| \\ &< 1 + |x_N| \text{ , for } m \geq N. \end{aligned}$$

Also, $|x_m| \leq \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|\} := M$,

for $m = 1, 2, \ldots, N-1$

Therefore, $|x_n| \leq \max\{M, 1 + |x_N|\}$, $\forall n \in \mathbb{N}$

this means $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \longrightarrow a$ as $k \to \infty$. Let $\varepsilon > 0$. Since $\{x_n\}$ is Cauchy, $\exists N_1 \in \mathbb{N}$ s.t. $n, m \geq N_1 \implies |x_n - x_m| < \frac{\varepsilon}{2}$

Since $x_{n_k} \to a$ as $k \to \infty$, $\exists\, N_2 \in \mathbb{N}$ s.t.

$$k \geq N_2 \implies |x_{n_k} - a| < \frac{\varepsilon}{2}.$$

Fix $k \geq N_2$ s.t. $n_k \geq N_1$. then

$$\begin{aligned} |x_n - a| &= |x_n - x_{n_k} + x_{n_k} - a| \\ &\leq |x_n - x_{n_k}| + |x_{n_k} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for all } n \geq N_1. \end{aligned}$$

Thus, $x_n \to a$ as $n \to \infty$ $\blacksquare$

<u>Rmk (2)</u> this result is extremely useful because it is often ~~easier~~ easier to show that a sequence is Cauchy than to show that it converges. Let us see the following example.

<u>Example</u> Prove that any real sequence $\{x_n\}$ satisfies $|x_n - x_{n+1}| \leq \frac{1}{2^n}$, $n \in \mathbb{N}$, is <u>convergent</u>.

4

Proof: If $m > n$, then

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m|$$

$$\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m|$$

$$\leq \frac{1}{2^n} + \cdots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^{n-1}}\Big[\underbrace{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n}}}_{\text{Geometric series}}\Big]$$

$a_1 \to$ First term $= \frac{1}{2}$

Ratio $= r = \frac{1}{2}$

$$= \frac{1}{2^{n-1}}\left(\left[\frac{a_1(1-r^{m-n})}{1-r}\right]\right.$$

$$= \frac{1}{2^{n-1}}\left[\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{m-n}\right)}{1-\frac{1}{2}}\right]$$

$$\Rightarrow |x_n - x_m| \leq \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^{m-n}}\right), \text{ if } m > n.$$

It follows that $|x_n - x_m| < \frac{1}{2^{n-1}}$, for all integers $m > n \geq 1$. But given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ so large that

$$n \geq N \implies \frac{1}{2^{n-1}} < \varepsilon$$

(5)

We have proved that $\{x_n\}$ is Cauchy.

By Thm (1), therefore, it converges to some real number. ∎

Rmk (3). A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Proof: Consider the sequence $x_n := \log n$.

$$x_{n+1} - x_n = \log(n+1) - \log n = \log \frac{n+1}{n} \to \log 1 = 0$$

as $n \to \infty$. $\{x_n\}$ cannot be Cauchy because it does not conv. ($\lim_{n\to\infty} x_n = \lim_{n\to\infty} \log n = \infty$). ∎

H.W's Exercises page 60 0, 1, 2, 3, 4, 5.

CH3 Functions on $\mathbb{R}$. ①

3.1 Two-sided limits

Df ① Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f be a real function defined on I except possibly at a. Then we say that $f(x)$ converges (approaches) to L as x approaches a, and write

$\lim_{x \to a} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ (which in general depends on ε, f, I, and a) such that

$$\boxed{0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon.} \quad (*)$$

Rmk. 1) ε represents the maximal error allowed in the approximation $f(x)$ to L.

2) According to Df ①, to show that a function has a limit, we must begin with a general $\varepsilon > 0$ and describe how to choose a δ which satisfies $(*)$

Ex.① Let $f(x) = mx + b$, where $m, b \in \mathbb{R}$.

prove that $\lim_{x \to a} f(x) = f(a)$, $\forall a \in \mathbb{R}$.

Proof. If $m=0$, then $|f(x)-f(a)| = |b-b| = 0 < \varepsilon$ for all x. If $m \neq 0$, given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{|m|}$.

If $|x-a| < \delta$, then

$$|f(x)-f(a)| = |mx+b-(ma+b)| = |m||x-a| < |m|\delta = |m|\frac{\varepsilon}{|m|} = \varepsilon.$$

Thus, by Df ①, $\lim\limits_{x \to a} f(x) = f(a)$. ∎

Ex ②. If $f(x) = x \sin\frac{1}{x}$, $x \neq 0$, then

$$\lim_{x \to 0} f(x) = 0$$

Pf. Let $\varepsilon > 0$, set $\delta = \varepsilon$. If $|x| < \delta = \varepsilon$,

then $$|f(x) - 0| = |x \sin\frac{1}{x} - 0| \leq |x| < \varepsilon.$$

Ex ③. If $f(x) = x^2+x-3$, prove that $\lim\limits_{x \to 1} f(x) = -1$

Let $\varepsilon > 0$. Notice that

$$f(x) - L = x^2+x-3+1 = x^2+x-2 = (x+2)(x-1)$$

If $0 < \delta \leq 1$, then $|x-1| < \delta \Rightarrow 0 < x < 2$, so

$|x+2| \leq |x|+2 < 4$. Set $\delta = \min\{1, \frac{\varepsilon}{4}\}$.

It follows that if $|x-1| < \delta$, then

$$|f(x)-L| = |x-1|\,|x+2| < 4|x-1| < 4\delta \leq 4\cdot\frac{\varepsilon}{4} = \varepsilon.$$

Thus, by Df(1), $\lim\limits_{x\to 1} f(x) = -1$.

<u>Thm (1)</u>. If $\lim\limits_{x\to a} f(x)$ exists, then it is unique, i.e.,

if $\lim\limits_{x\to a} f(x) = L_1$ and $\lim\limits_{x\to a} f(x) = L_2$, then

$L_1 = L_2$.

<u>Proof</u>. Spse that $\lim\limits_{x\to a} f(x) = L_1$ & $\lim\limits_{x\to a} f(x) = L_2$

and let $\varepsilon > 0$. From Df(1), $\exists$ $\delta_1, \delta_2 > 0$

such that $|f(x)-L_1| < \varepsilon$ if $0<|x-a|<\delta_1$

and $|f(x)-L_2| < \varepsilon$ if $0<|x-a|<\delta_2$.

If $\delta = \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned}|L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x)-L_1| + |f(x)-L_2| \\ &< \varepsilon + \varepsilon = 2\varepsilon \quad \text{if } |x-a| < \delta,\end{aligned}$$

i.e., $|L_1 - L_2| < 2\varepsilon$, $\forall \varepsilon > 0$

$$\Rightarrow L_1 = L_2 \quad \blacksquare$$

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The next result shows that even when a function f is defined at a, $\lim_{x \to a} f(x)$, in general, independent of the value of $f(a)$.

__Lemma.__ Let $a \in \mathbb{R}$, let I be an open interval which contains a, and let f, g be real functions defined $\forall x \in I$ except possibly at a.
If $f(x) = g(x)$, $\forall x \in I \setminus \{a\}$ and $\lim_{x \to a} f(x) = L$,
then $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$.

__PF.__ Let $\varepsilon > 0$ and choose $\delta > 0$ small enough so that (*) holds and $|x-a| < \delta \Rightarrow x \in I$.
Suppose that $0 < |x-a| < \delta$. We have $f(x) = g(x)$ by hypothesis and $|f(x) - L| < \varepsilon$ by (*). It follows that $|g(x) - L| < \varepsilon$. That is, $\lim_{x \to a} g(x) = L$. ■

__Ex(4)__ Prove that $\lim_{x \to 1} g(x)$ exists, if

$$g(x) = \frac{x^3 + x^2 - x - 1}{x^2 - 1}.$$

Pf. Set $f(x) = x+1$ and observe that

$$g(x) = \frac{x^3+x^2-x-1}{x^2-1} = \frac{x^2(x+1)-(x+1)}{x^2-1}$$

$$= \frac{(x^2-1)(x+1)}{x^2-1} = x+1, \quad x \neq \pm 1$$

$$= f(x).$$

and observe that, by Ex1, $\lim\limits_{x \to 1} f(x) = 2$.

It follows from previous lemma that $g(x)$ has a limit at $x=1$ and $\lim\limits_{x \to 1} g(x) = 2$ ∎

Theorem (2). [Sequential characterization of limits].

Let $a \in \mathbb{R}$, let I be an open interval contains a, and let f be a real function defined $\forall x \in I$ except possibly at a. then

$\lim\limits_{x \to a} f(x) = L$ iff $f(x_n) \longrightarrow L$ as $n \to \infty$

for every sequence $x_n \in I \setminus \{a\}$ which converges to a as $n \to \infty$.

Proof. $(\Rightarrow)$ Suppose that $\lim_{x\to a} f(x) = L$. Then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon. \quad \cdots (I)$$

Let $x_n \in I \setminus \{a\}$ s.t. $\lim_{n\to\infty} x_n = a$, then $\exists$ an $N \in \mathbb{N}$ s.t $n \geq N \Longrightarrow |x_n - a| < \delta$.

Since $x_n \neq a$, it follows from (I) that $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. Therefore, $f(x_n) \longrightarrow L$ as $n \longrightarrow \infty$.

$(\Leftarrow)$ Conversely, suppose that $f(x_n) \longrightarrow L$ as $n \to \infty$ for every sequence $x_n \in I \setminus \{a\}$ which converges to a (i.e., $x_n \to a$). Suppose that $\lim_{x\to a} f(x) \neq L$, then there is an $\varepsilon > 0$ (say ε_0) such that the implication

" $0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$ " does not hold for any $\delta > 0$. Thus, for each $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, $\exists$ a point $x_n \in I$ which satisfies two conditions:

$0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$.

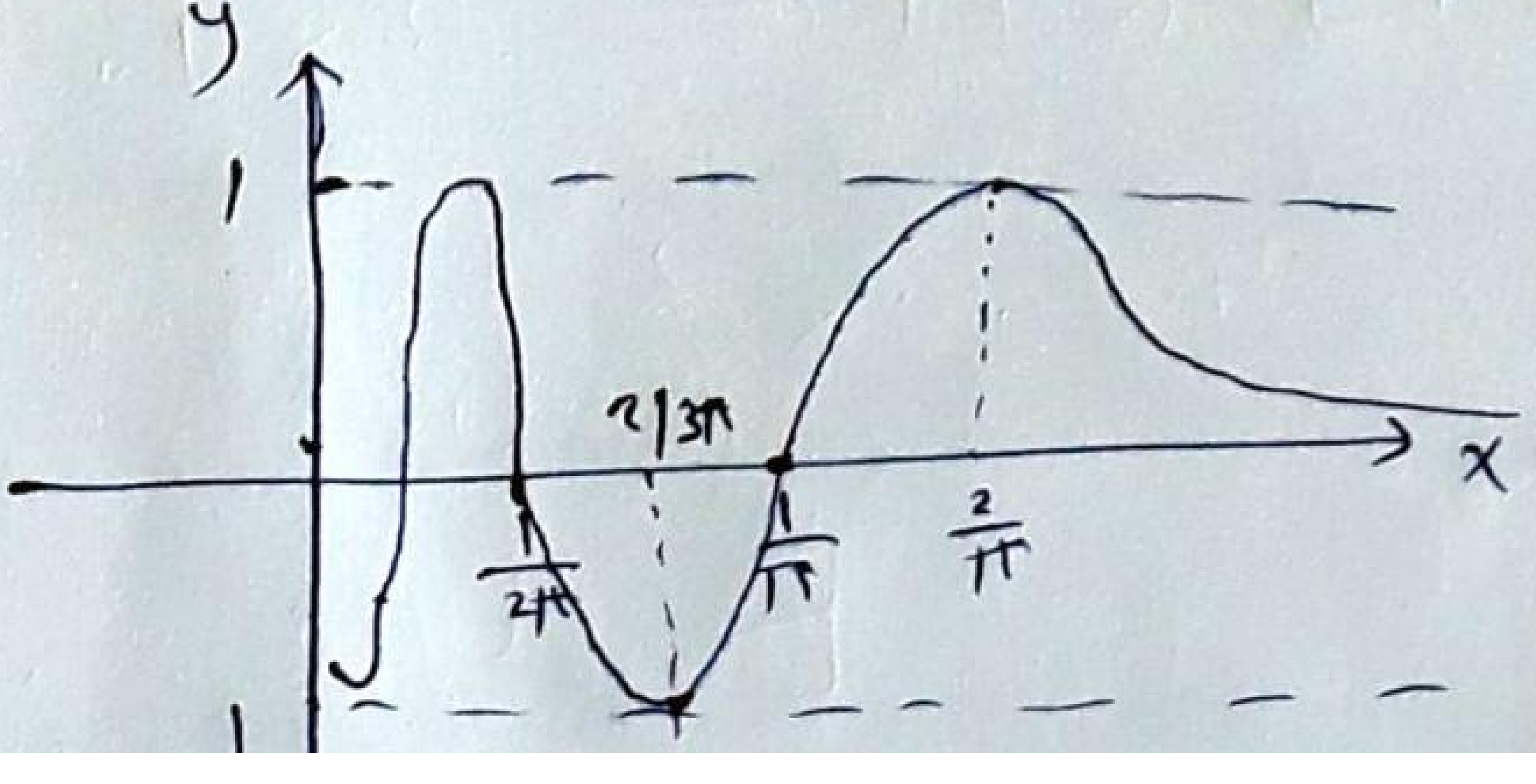
Now the first condition and the squeeze thm imply that $x_n \neq a$, $x_n \longrightarrow a$ as $n \to \infty$, so by hypothesis, $f(x_n) \longrightarrow L$, as $n \to \infty$.
In particular, $|f(x_n) - L| < \varepsilon_0$ for n large, which contradicts the second condition ∎

__Rmk.__ To show that the limit of a function f does not exist as $x \longrightarrow a$, using this thm, we need to find two sequences converges to a (say $x_n \longrightarrow a$ and $y_n \longrightarrow a$) whose images under f have __different__ limits.
(i.e., $f(x_n) \longrightarrow L_1$ and $f(y_n) \longrightarrow L_2$, where $L_1 \neq L_2$). $\neq \lim_{x \to a} f(x)$

__Ex. 5__. prove that $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

has no limit as $x \longrightarrow 0$.

__Pf.__



By examining the graph of $y = f(x)$ (see the above Fig.), we consider

$$x_n := \frac{2}{(4n+1)\pi} \text{ and } y_n := \frac{2}{(4n+3)\pi}, \; n \in \mathbb{N}.$$

Clearly x_n and $y_n \longrightarrow 0$ as $n \longrightarrow \infty$.

But, since $f(x_n) = 1$ and $f(y_n) = -1$ for all $n \in \mathbb{N}$, $f(x_n) \longrightarrow 1$

and $f(y_n) \longrightarrow -1$

as $n \longrightarrow \infty$. Thus by Thm(2), $\lim_{x \to 0} f(x)$ DNE $\blacksquare$

Rmk. Thm(2) allows us to translate results about limits of sequences to results about limits of functions. Let us see the following theorems.

Thm③. Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a and that f, g are real functions defined $\forall x \in I$ except possibly at a. If $f(x)$ and $g(x)$ converges as $x \longrightarrow a$ (i.e.,

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist)., then so do $(f+g)(x) = f(x)+g(x)$, $(fg)(x) = f(x)g(x)$, $(\alpha f)(x) = \alpha f(x)$, and $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (when $\lim_{x\to a} g(x) \neq 0$). In fact,

(i) $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a} (\alpha f)(x) = \alpha \lim_{x\to a} f(x)$

(iii) $\lim_{x\to a} (fg)(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$.

and when $\lim_{x\to a} g(x) \neq 0$,

(iv) $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Prof. Let $\lim_{x\to a} f(x) := L$, and $\lim_{x\to a} g(x) := M$.

(i) If $x_n \in I\setminus\{a\}$ s.t. $x_n \longrightarrow a$, then by thm 2, $f(x_n) \longrightarrow L$ and $g(x_n) \longrightarrow M$ as $n\to\infty$. By Thm (ch2),

$(f+g)(x_n) = f(x_n) + g(x_n) \longrightarrow M+L$ as $n \to \infty$

Since this holds for any sequence $x_n \in I \setminus \{a\}$ which converges to a, we conclude by thm 3,

$$\lim_{x \to a} (f+g)(x) = L + M$$
$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

(ii), (iii), & (iv) Exercises. ∎

<u>Thm ④</u>: [<u>Squeeze thm for functions</u>]

Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g, h are real functions defined $\forall x \in I$ except possibly at a.

(i) If $g(x) \leq h(x) \leq f(x)$ for all $x \in I \setminus \{a\}$,

and $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$,

then $\lim_{x \to a} h(x)$ exists, and $\lim_{x \to a} h(x) = L$.

(ii) If $|g(x)| \leq M$ for all $x \in I \setminus \{a\}$

(i.e., g is bdd) and $\lim_{x \to a} f(x) = 0$,

then $\lim_{x \to a} f(x) g(x) = 0$.

Thm(5): [Comparison thm for functions]
Suppose that $a \in \mathbb{R}$, that I is an open interval which contains a, and that f, g are real functions defined $\forall x \in I$ except possibly at a. If f and g have limit as $x \to a$ and $f(x) \leq g(x)$ $\forall x \in I \setminus \{a\}$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

Rmk. We shall refer to thm 5 as "taking the limit of an inequality".

Rmk. The limit thms (Thm 3, 4, and 5) allow us to prove that limits exist without using (ε-δ) definition (Df 1).

Ex. 6 Prove that $\lim_{x \to 1} \frac{x-1}{3x+1} = 0$

Pf. By Example (1), $\lim_{x \to 1} (x-1) = 0$ and $\lim_{x \to 1} (3x+1) = 4$. Hence, by Thm (3)(iv),

$$\lim_{x \to 1} \frac{x-1}{3x+1} = \frac{\lim_{x \to 1} (x-1)}{\lim_{x \to 1} (3x+1)} = \frac{0}{4} = 0.$$

H.W's 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (All).

3.2 One-sided limits and limits at infinity

Df 1. Let $a \in \mathbb{R}$ and f be a real function.

(i) $f(x)$ is said to converge to L as x approaches a from the right iff f is defined on some open interval I with left endpoint a and for every $\varepsilon > 0$ $\exists$ a $\delta > 0$ (which in general depends on ε, I, f and a) such that

$$a+\delta \in I \text{ and } a < x < a+\delta \implies |f(x) - L| < \varepsilon.$$

Here, L is called the right-hand limit of f at a, and denote it by

$$f(a^+) := L =: \lim_{x \to a^+} f(x).$$

(ii) $f(x)$ is said to converge to L as x approaches a from the left iff f is defined on some open interval I with right endpoint a and for every $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$a-\delta \in I \text{ and } a - \delta < x < a \implies |f(x) - L| < \varepsilon.$$

Here, L is called the left-hand limit of f at a

and denote by

$$f(a^-) := L =: \lim_{x \to a^-} f(x)$$

Example [1] (i) prove that $f(x) = \begin{cases} x+1, & x \geq 0 \\ x-1, & x < 0 \end{cases}$

has one-sided limits at $x = 0$ but

$\lim_{x \to 0} f(x)$ DNE.

(ii) prove that $\lim_{x \to 0^+} \sqrt{x} = 0$

proof. (i) Let $\varepsilon > 0$ and set $\delta = \varepsilon$.

If $0 < x < \delta$, then $|f(x) - L| = |x+1-1| = |x| < \delta = \varepsilon$.

Hence $\lim_{x \to 0^+} f(x)$ exists and equals 1.

Similarly, $\lim_{x \to 0^-} f(x)$ exists and equals -1

Indeed, set $\delta = \varepsilon$. If $-\delta < x < 0$, then

$|f(x) - L| = |x-1+1| = |x| < \delta = \varepsilon$.

However, $\lim_{x \to 0} f(x)$ DNE since,

$$x_n = \frac{(-1)^n}{n} \longrightarrow 0 \quad \text{(by squeeze thm)}$$

but $f(x_n) = f\left(\frac{(-1)^n}{n}\right) = (-1)^n\left(1 + \frac{1}{n}\right)$ does not converge as $n \to \infty$

Hence, by the sequential characterization of Limits, $\lim_{x \to 0} f(x)$ DNE.

(ii) Let $\varepsilon > 0$ and set $\delta = \varepsilon^2$. If $0 < x < \delta$, then $|f(x) - L| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon$. ∎

Rmk. Not every function has one-sided limits $\left(\text{ex. } f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \right)$. The last example show that even a function has one-sided limits, it may not have a two-sided limit. The following Thm show that if both one sided limits at a exist and are EQUAL, then the two-sided limit at a exists.

Thm (1) Let f be a real function. Then the limit $\lim_{x \to a} f(x)$ exists and equals L iff

$$L = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x).$$

Proof. ($\Rightarrow$) Suppose that $\lim_{x \to a} f(x) = L$ exists. Then given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

If $a < x < a+\delta$, then $|x-a| < \delta$ and this implies $|f(x)-L| < \varepsilon$. Hence $\lim_{x\to a^+} f(x) = L$ exists.

Similarly, if $a-\delta < x < a$, then $|x-a| < \delta$ and this implies $|f(x)-L| < \varepsilon$, which means

$\lim_{x\to a^-} f(x) = L$ exists.

($\Leftarrow$) Conversely, suppose that $L = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

Then given $\varepsilon > 0$, $\exists$ a $\delta_1 > 0$ s.t.

$$a < x < a+\delta_1 \implies |f(x)-L| < \varepsilon. \quad (1)$$

and $\exists$ a $\delta_2 > 0$ s.t.

$$a-\delta_2 < x < a \implies |f(x)-L| < \varepsilon. \quad (2)$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then

$|x-a| < \delta \Rightarrow a-\delta < x < a+\delta$ which implies $a < x < a+\delta_1$ or $a-\delta_2 < x < a$ (depending on whether x is to the right or to the left of a). Hence, (1) & (2) give $|f(x)-L| < \varepsilon$

that is $\lim_{x\to a} f(x) = L$ ∎

Df ②. (limits at infinity).

Let $a, L \in \mathbb{R}$ & let f be a real function.

(i) $f(x)$ is said to converge to L as $x \to \infty$ iff $\exists$ a $c > 0$ such that $(c, \infty) \subset Dom(f)$ and given $\varepsilon > 0$ there is an $M \in \mathbb{R}$ s.t.,

$$x > M \implies |f(x) - L| < \varepsilon,$$ in which case we shall write $\lim_{x \to \infty} f(x) = L$ or $f(x) \to L$ as $x \to \infty$.

Similarly, $f(x) \to L$ as $x \to -\infty$ iff $\exists$ a $c > 0$ s.t., $(-\infty, -c) \subset Dom(f)$ and given $\varepsilon > 0$ there is an $M \in \mathbb{R}$ such that

$$x < M \implies |f(x) - L| < \varepsilon,$$ in which case we shall write $\lim_{x \to -\infty} f(x) = L$ or $f(x) \to L$ as $x \to -\infty$.

(ii) $f(x)$ is said to converge to ∞ as $x \to a$ (i.e., $\lim_{x \to a} f(x) = \infty$) iff there is an open interval I containing a such that $I \setminus \{a\} \subset Dom(f)$ and given $M \in \mathbb{R}$

there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies f(x) > M,$$ in which

case we write $\lim_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$.

Similarly, $f(x)$ is said to converge to $-\infty$ as $x \to a$ (i.e., $\lim_{x \to a} f(x) = -\infty$) iff $\exists$ an open interval I containing a such that

$I \setminus \{a\} \subset \mathrm{Dom}(f)$ and given $M \in \mathbb{R}$

there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \implies f(x) < M.$$

$\boxed{\text{Rmk. Obvious modifications of this Df, we define } f(x) \to \pm\infty \text{ as } x \to a^+ \text{ \& } x \to a^-, \text{ and } f(x) \to \pm\infty \text{ as } x \to \pm\infty.}$

Ex. (2) (i) Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

(ii) prove that $\lim_{x \to 1^-} \frac{x+2}{2x^2-3x+1} = -\infty$

Proof. (i) Given $\varepsilon > 0$, set $M = \frac{1}{\varepsilon}$. If

$x > M$, then $|f(x) - L| = |\frac{1}{x} - 0| = \frac{1}{x} < \frac{1}{M} = \varepsilon$.

Thus, $\frac{1}{x} \to 0$ as $x \to \infty$.

(ii) Let $M \in \mathbb{R}$. We need to find $\delta > 0$

s.t. $1-\delta < x < 1 \Rightarrow f(x) < M$,

where $f(x) = \dfrac{x+2}{2x^2-3x+1}$. Without loss

of generality, assume that $M < 0$.

As $x \to 1^-$, $2x^2-3x+1$ is negative and

$2x^2-3x+1 \to 0$ (observe that $2x^2-3x+1$

is aparabola opening upward with roots $\frac{1}{2}$ and 1).

Therefore, choose $\delta \in (0,1)$ such that

$1-\delta < x < 1 \Rightarrow \frac{2}{M} < 2x^2-3x+1 < 0$, i.e.,

$\dfrac{-1}{2x^2-3x+1} > -\frac{M}{2} > 0$. Since $0 < x < 1$ also

implies $2 < x+2 < 3$, it follows that

$-\dfrac{x+2}{2x^2-3x+1} > -M$, i.e.,

$f(x) = \dfrac{x+2}{2x^2-3x+1} < M$, for all $1-\delta < x < 1$. $\blacksquare$

<u>Notation</u> $\lim_{\substack{x \to a \\ x \in I}} f(x)$ $(*)$ [a is an extended real number].

- $(*)$ will denote $\lim_{x \to a} f(x)$ (when it exists)
- If a is a finite left endpoint of I, then $(*)$ will denote $\lim_{x \to a+} f(x)$ (when it exists).
- If a is a finite right endpoint of I, then $(*)$ will denote $\lim_{x \to a-} f(x)$ (when it exists)
- If $a = \pm\infty$ is an endpoint of I, then $(*)$ will denote $\lim_{x \to \pm\infty} f(x)$ (when each exists).

Using this notation, we can state a <u>sequential characterization of Limits</u> valid for two-sided, one-sided, and infinite limits.

<u>Thm</u> 2 Let a be an extended real number, and let I be a nondegenerate open interval ~~I~~ which either contains a or has a as one of its endpoints. Suppose further that f is a real function defined

on I except possibly at a. Then

$\lim_{\substack{x \to a \\ x \in I}} f(x)$ exists and equals L

if and only if $f(x_n) \longrightarrow L$ for all sequences $x_n \in I$ which satisfy $x_n \neq a$ and $x_n \longrightarrow a$ as $n \longrightarrow \infty$.

Proof. Since we have proved this for two-sided limits, we must show it for the remaining 8 cases which notation (*) represents. Since the proofs are similar, we shall give the details for only one case, namely, $\boxed{\lim_{x \to a} f(x) = \infty}$. Thus, we must prove that.

$$\boxed{\lim_{x \to a} f(x) = \infty \text{ iff } f(x_n) \longrightarrow \infty \text{ for any sequence } x_n \in I \text{ which converges to } a \text{ and satisfies } x_n \neq a \text{ for } n \in \mathbb{N}.}$$

($\Rightarrow$) Suppose that $\lim_{x \to a} f(x) = \infty$. If $x_n \in I$, $x_n \longrightarrow a$ as $n \longrightarrow \infty$, and $x_n \neq a$, then given $M \in \mathbb{R}$, $\exists$ a $\delta > 0$ such that

$0 < |x-a| < \delta \Rightarrow f(x) > M.$

and $\exists$ an $N \in \mathbb{N}$ such that

$n \geq N \Rightarrow |x_n - a| < \delta.$

Consequently, $n \geq N \Rightarrow f(x_n) > M$, i.e.,

$f(x_n) \to \infty$ as $n \to \infty$ as required.

($\Leftarrow$) Conversely, Suppose to the contrary that $f(x_n) \to \infty$ for any sequence $x_n \in I$ which converges to a and satisfies $x_n \neq a$ but $\lim\limits_{x \to a} f(x) \neq \infty$. By the definition of "convergence" to ∞ there are numbers $M_0 \in \mathbb{R}$ and $x_n \in I$ s.t., $|x_n - a| < \frac{1}{n}$ and $f(x_n) \leq M_0 \quad \forall n \geq N.$

Now, $|x_n - a| < \frac{1}{n} \Rightarrow a - \frac{1}{n} < x_n < a + \frac{1}{n}$

this implies, using squeeze thm, $x_n \to a$

but the Condition $f(x_n) \leq M_0$, $\forall n \geq N$ implies that $f(x_n) \not\to \infty$ as $n \to \infty$ which is a contradiction & this proves Thm (2) in the case $\lim\limits_{x \to a} f(x) = \infty$, $a \in I$.

Rmk. Using Thm 2, we can prove limit theorems represented in sec. 3.1. These limits thms can be used to evaluate infinite limits and limits at $\pm\infty$.

Ex. (3) prove that $\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} = -2$.

Proof. Since the limit of a product is the product of the limits, we have by Ex (2)(i) that $\lim_{x\to\infty} \frac{1}{x^m} = 0$, for any $m \in \mathbb{N}$. Multiplying numerator and denominator of the expression above by $\frac{1}{x^2}$, we obtain

$$\lim_{x\to\infty} \frac{2x^2-1}{1-x^2} = \lim_{x\to\infty} \frac{2-\frac{1}{x^2}}{-1+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(2-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(-1+\frac{1}{x^2}\right)} = \frac{2}{-1} = -2$$

H.W's Exercises p. 81

0, 1, 2, 3, 4, 5, 6, 7, 8 (i.e., All)

Also, Give such an example on Q8.

3.3 Continuity

Df 1 Let $\phi \neq E \subseteq \mathbb{R}$ and $f: E \to \mathbb{R}$.

(i) f is said to be continuous at a point $a \in E$ iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ (depends on ε, f, and a) s.t.

$$|x-a| < \delta \text{ and } x \in E \Rightarrow |f(x) - f(a)| < \varepsilon \quad (*)$$

(ii) f is said to be continuous on E iff f is continuous at every $x \in E$.

Rmk. Let I be an open interval which contains a point a and $f: I \to \mathbb{R}$. then f is continuous at $a \in I$ iff $f(a) = \lim_{x \to a} f(x)$.

proof: see the book.

thm (1) [Sequential characterization of Continuity]
Spse that E is a nonempty subset of $\mathbb{R}$, that $a \in E$, and that $f: E \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is cont. at $a \in E$.

(ii) If $x_n \to a$ and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to \infty$

Thm (2). Let E be a nonempty subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$. If f, g are continuous at a point $a \in E$ (resp. continuous on the set E), then so are $f+g$, fg, and αf (for any $\alpha \in \mathbb{R}$). Moreover, f/g is cont. at $a \in E$ when $g(a) \neq 0$ (resp., on E when $g(x) \neq 0$ for all $x \in E$).

Df (2). Spse that A and B are subsets of $\mathbb{R}$, that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If $f(A) \subseteq B$ for every $x \in A$, then the composition of g with f is the function $g \circ f : A \to \mathbb{R}$ defined by

$$(g \circ f)(x) := g(f(x)), \quad x \in A.$$

Thm (3). Spse that A and B are subsets of $\mathbb{R}$, and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, and that

$$f(x) \in B, \ \forall x \in A.$$

(i) If $A := I \setminus \{a\}$, where I is a nondegenerate interval which either contains a or has a as one of its endpoints, if

$$L := \lim_{\substack{x \to a \\ x \in I}} f(x)$$

exists and belongs to B, and if g is cont. at $L \in B$, then

$$\lim_{\substack{x \to a \\ x \in I}} (g \circ f)(x) = g\left(\lim_{\substack{x \to a \\ x \in I}} f(x)\right).$$

(ii) If f is cont. at $a \in A$ and g is cont. at $f(a) \in B$, then $g \circ f$ is cont. at $a \in A$.

Proof (i) Spse that $x_n \in I \setminus \{a\}$ and that $x_n \to a$ as $n \to \infty$. Since $f(A) \subseteq B$. $f(x_n) \in B$. Also, by the sequential characterization of Limits, $f(x_n) \to L$ as $n \to \infty$. Since g is cont. at $L \in B$, it follows from thm(1) that $(g \circ f)(x_n) := g(f(x_n)) \to g(L)$ as $n \to \infty$. Hence, by the sequential characterization of limits again, $(g \circ f)(x) \to g(L)$ as $x \to a$ in I.

(ii) Exercise ∎

Df (3). Let $\phi \neq E \subseteq \mathbb{R}$. A function $f: E \to \mathbb{R}$ is said to be bounded on E iff $\exists$ an $M \in \mathbb{R}$ s.t., $|f(x)| \leq M$, $\forall x \in E$. (f is dominated by M on E)

Rmk: Notice that whether a function f is bounded or not on a set E depends on E as well as on f. For ex, $f(x) = \frac{1}{x}$ is bounded on $[1, \infty)$ but unbounded on $(0, 2)$. Again, $f(x) = x^2$ is bounded on $(-2, 2)$ (dominated by 4) but unbounded on $[0, \infty)$.

Thm (4). [Extreme value thm]

If I is a closed, bounded interval and $f: I \to \mathbb{R}$ is continuous on I, then f is bounded on I. Moreover, if $M = \sup\limits_{x \in I} f(x)$ and $m = \inf\limits_{x \in I} f(x)$, then $\exists$ points $x_m, x_M \in I$ s.t.

$$f(x_M) = M \text{ and } f(x_m) = m.$$

<u>Proof</u>: Spse first that f is not bounded on I. Then $\exists$ $x_n \in I$ s.t.

$$|f(x_n)| > n, \ n \in \mathbb{N}. \qquad (**)$$

Since I is bounded, by the Bolzano–Weierstrass Thm, $\{x_n\}$ has a convergent subsequence, say $x_{n_k} \longrightarrow a$ as $k \to \infty$. Since I is closed, by the Comparison Thm, $a \in I$. In particular, $f(a) \in \mathbb{R}$. On the other hand, substituting n_k for n in $(**)$ and taking the limit as $k \to \infty$, we have $|f(a)| = \infty$, a contradiction. Hence, f is bounded on I.

We have proved that both M and m are finite real numbers. To show that $\exists$ an $x_M \in I$ s.t. $f(x_M) = M$, suppose to the contrary, that $f(x) < M$ for all $x \in I$. Then $g(x) = \dfrac{1}{M - f(x)}$ is cont.; hence bounded on I. In particular, $\exists$ a $C > 0$ such that $|g(x)| = g(x) \le C$.

6

It follows that $f(x) \le M - \frac{1}{C}$, $\forall x \in I$.

It follows that $\sup_{x\in I} f(x) \le \sup_{x \in I} \left(M - \frac{1}{C}\right)$.

This implies $M \le M - \frac{1}{C} < M$,

a contradiction. Hence, $\exists$ an $x_M \in I$ such that $f(x_M) = M$. Similarly, you can prove that $\exists$ an $x_m \in I$ s.t.,

$f(x_m) = m$ (please do it). ▤

Rmk. ① We also call the value M (resp., m) the maximum (resp., the minimum) of f on I.

② The Extreme value Thm (Thm ④) is <u>false</u> if either "closed" or "bounded" is dropped from the hypothesis.

<u>Counterexamples</u> (2) $(0,1)$ is bounded interval but not closed and $f(x) = \frac{1}{x}$ is continuous and unbounded on $(0,1)$.

$[0,\infty)$ is closed but not bounded, and the function $f(x) = x$ is cont. and unbounded on $[0,\infty)$.

<u>Lemma.</u> Spse that $a < b$ and that $f : [a, b) \to \mathbb{R}$. If f is continuous at $x_0 \in [a, b)$ and $f(x_0) > 0$, then $\exists$ an $\varepsilon > 0$ and a point $x_1 \in [a, b)$ such that $x_1 > x_0$ and $f(x) > \varepsilon$, $\forall x \in [x_0, x_1]$.

<u>Proof.</u> Strategy: If $f(x_0) > 0$, then $f(x) > \frac{f(x_0)}{2}$ for x near x_0. the following are the details.

Let $\varepsilon = \frac{f(x_0)}{2}$. Since $x_0 < b$, then

$\delta_0 := \frac{b - x_0}{2} > 0$ and $x \in [a, x_0 + \delta) \Longrightarrow x \in [a, b)$.

Since f is cont. at x_0, then we can choose $0 < \delta < \delta_0$ s.t. $x \in [a, b)$ and

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

Fix $x_1 \in (x_0, x_0 + \delta)$ and spse that $x \in [x_0, x_1]$. By the choice of ε & δ, it is clear that

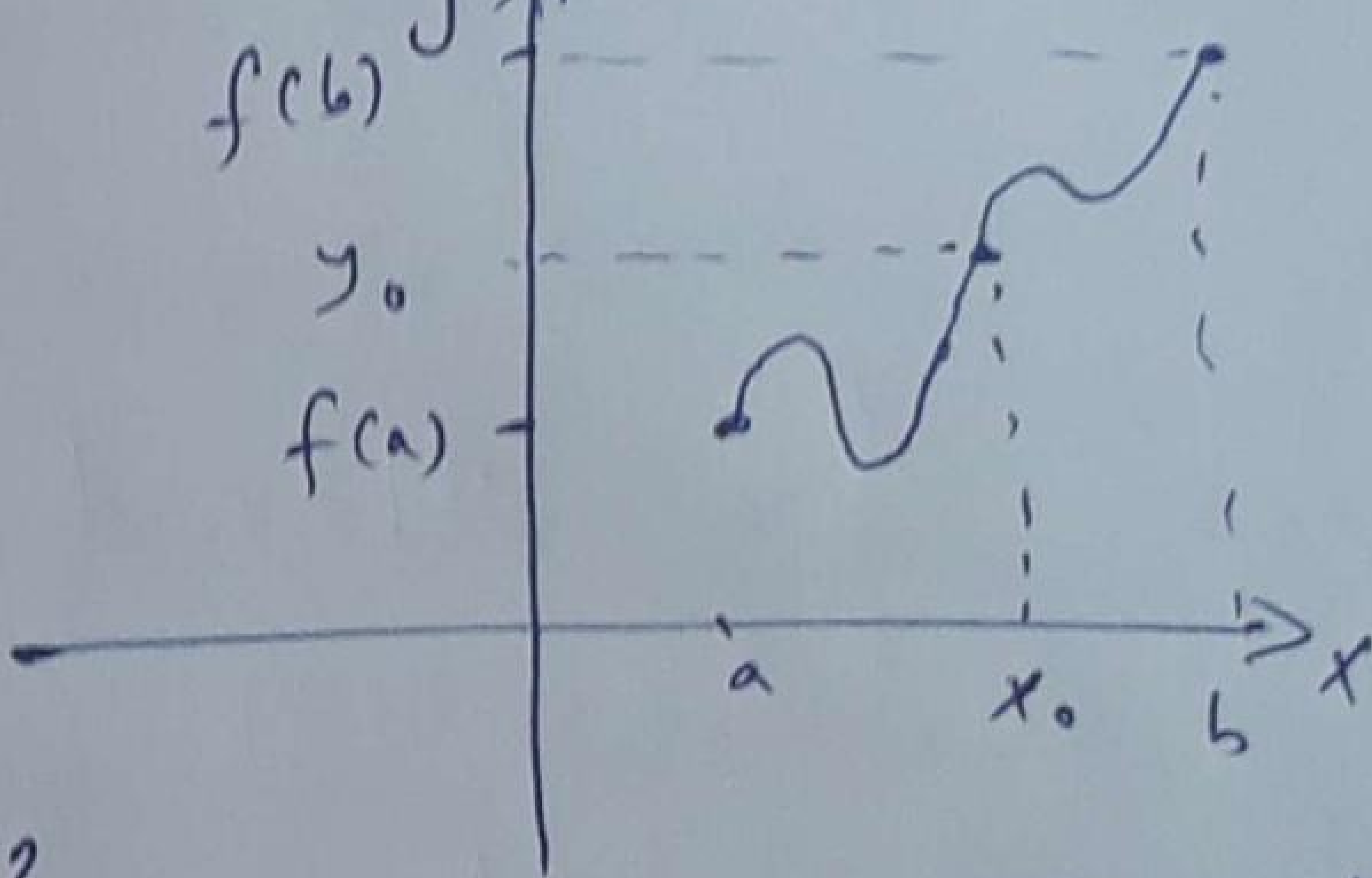
$$-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}.$$

Solving the left-hand ineq. we conclude that

$f(x) > \frac{f(x_0)}{2} = \varepsilon$, as required $\blacksquare$

• A real number y_0 is said to lie between two numbers c and d iff $c < y_0 < d$ or $d < y_0 < c$.

Thm ⑤ [Intermediate Value Theorem]
Spse that $a < b$ and that $f : [a,b] \longrightarrow \mathbb{R}$ is continuous. If y_0 lies between $f(a)$ and $f(b)$, then $\exists$ an $x_0 \in (a,b)$, such that $f(x_0) = y_0$.



proof. We may suppose that $f(a) < y_0 < f(b)$.
Consider the set

$$E = \{ x \in [a,b] : f(x) < y_0 \}.$$ Since $a \in E$ and $E \subseteq [a,b]$, then E is a nonempty, bounded subset of $\mathbb{R}$. Hence by the Completeness Axiom, $x_0 := \sup E$ is a finite real number. It remains to prove that $x_0 \in (a,b)$ and $f(x_0) = y_0$. Since E has a finite sup x_0, by Thm, $\exists$ a sequence $x_n \in E$ such that

$$x_n \longrightarrow x_0 \text{ as } n \longrightarrow \infty.$$

Since $E \subseteq [a,b]$, it follows from the Comparison thm, $x_0 \in [a,b]$. Moreover, by the continuity of f and the def'n of E, we have $f(x_0) = \lim_{n\to\infty} f(x_n) \le y_0$

To show that $f(x_0) = y_0$, suppose to the contrary that $f(x_0) < y_0$. Then $y_0 - f(x)$ is a cont. function on $[a,b)$ whose value at $x = x_0$ is positive. Hence, by previous lemma, we can choose an ε and an $x_1 > x_0$ such that $y_0 - f(x_1) > \varepsilon > 0$. In particular, $x_1 \in E$ and $x_1 > \sup E$, a contradiction. We have proved that $x_0 \in [a,b]$ and $y_0 = f(x_0)$. Since we assumed that $f(a) < y_0 < f(b)$, it follows that x_0 <u>Cannot</u> equal a or b. We conclude that $x_0 \in (a,b)$. ∎

<u>Ex ①</u>. prove that $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$ is

Continuous on $(-\infty, 0)$ and $[0, \infty)$, discont. at $x = 0$, and both $f(0^+)$ and $f(0^-)$ exist.

Pf: Since $f(x) = 1$ for $x \geq 0$, it is clear that $f(0^+) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$ exists and $f(x) \longrightarrow f(a)$ as $x \longrightarrow a$ for any $a > 0$. In particular, f is cont. on $[0, \infty)$. Similarly, $f(0^-) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$ exists and f is continuous on $(-\infty, 0)$. Finally, since $f(0^+) \neq f(0^-)$, then $\lim_{x \to 0} f(x)$ DNE. Therefore, f is not cont. at $x = 0$. $\blacksquare$

Ex. 2 Assume that $\sin x$ is cont. on $(-\infty, \infty)$, prove that $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 1, & x = 0 \end{cases}$ is cont.

on $(-\infty, 0)$ and $(0, \infty)$, discont. at 0, and neither $f(0^+)$ nor $f(0^-)$ exists.

proof: The function $g(x) = \frac{1}{x}$ is cont. for $x \neq 0$. Hence by thm 3, $f(x) = (\sin \circ g)(x) = \sin\left(\frac{1}{x}\right)$ is cont. on $(-\infty, 0) \cup (0, \infty)$. To prove that

(11)

$f(0^+)$ DNE, let $x_n = \frac{2}{(2n+1)\pi}$, and observe that $\sin\left(\frac{1}{x_n}\right) = (-1)^n$, $n \in \mathbb{N}$.

Since $x_n \downarrow 0$ but $(-1)^n$ does not converge, it follows from (the sequential characterization of Continuity Thm) that $f(0^+)$ DNE.

A similar way proves that $f(0^-)$ DNE (please do it as Exercise). $\blacksquare$

Ex③. The Dirichlet function is defined by

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

prove that every point $x \in \mathbb{R}$ is a point of discontinuity of f (i.e. f is nowhere continuous)

proof. By Density of Rationals and Irrationals, given any $a \in \mathbb{R}$ and $\delta > 0$ we can choose $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}^c$ such that $|x_1 - a| < \delta$ and $|x_2 - a| < \delta$.

(12)

Since $f(x_1)=1$ and $f(x_2)=0$, then f cannot be continuous at a.

Ex(4). Prove that $f(x)=\begin{cases} \frac{1}{q}, & x=\frac{p}{q}\in\mathbb{Q} \text{ (in reduced form)} \\ 0, & x\notin\mathbb{Q}. \end{cases}$

is continuous at every irrational in $(0,1)$ but discontinuous at every rational in $(0,1)$.

Proof. First, we shall prove that f is dis cont. at every rational in $(0,1)$. Let a be a rational in $(0,1)$ and spse that f is cont. at a. If x_n is a sequence of irrationals s.t. $x_n \longrightarrow a$, then $f(x_n) \longrightarrow f(a)$ as $n\to\infty$, i.e; $f(a)=0$. But $f(a)\neq 0$ by defn. Hence, f is discontinuous at every rational in $(0,1)$.

Next, we want to prove that f is cont. at every irrational in $(0,1)$. Indeed,

(13)

Let a be an irrational in $(0,1)$. We must show that $f(x_n) \longrightarrow f(a)$ for every sequence $x_n \in (0,1)$ which satisfies $x_n \to a$ as $n \to \infty$.

We may suppose that $x_n \in \mathbb{Q}$. $\forall n \in \mathbb{N}$,

Write $x_n = \frac{p_n}{q_n}$ in reduced form. Since $f(a)=0$, it sufficies to show that $q_n \to \infty$ as $n \to \infty$ $\left(\begin{array}{l} \text{Since we need to prove that} \\ f(x_n) = \frac{1}{q_n} \longrightarrow f(a) = 0 \end{array} \right)$.

Suppose to the contrary that there exist integers $n_1 < n_2 < \cdots$ such that $|q_{n_k}| \le M < \infty$ for $k \in \mathbb{N}$. Since $x_{n_k} \in (0,1)$, it follows that the set $E := \left\{ x_{n_k} = \frac{p_{n_k}}{q_{n_k}} : k \in \mathbb{N} \right\}$

contains only a finite number of pts.

Hence, the limit of any sequence in E must belong to E, a contradiction since a is such a limit and a is irrational $\blacksquare$

Rmk. the composition of two functions $g \circ f$ can be nowhere continuous, even though f is discont. only on $\mathbb{Q}$ and g is discont. at only one point.

proof. Let $f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ (in reduced form)} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

$$g(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

clearly, $(g \circ f)(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

Hence, $g \circ f$ is the Dirichlet function, nowhere continuous by Ex (3).

H.w's (Exercises p. 90; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 i.e., All).

Good Luck.

3.4 Uniform Continuity

Df ①. Let E be a nonempty subset of $\mathbb{R}$ and $f: E \to \mathbb{R}$. Then f is said to be uniformly continuous on E iff $\forall \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$$|x-a| < \delta \text{ and } x, a \in E \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Notice that δ here depends on ε and f, but not on a and x.

Ex ① Prove that $f(x) = x^2$ is uniformly continuous on $(0,1)$.

Proof. Given $\varepsilon > 0$, set $\delta = \frac{\varepsilon}{2}$. If $x, a \in (0,1)$, then $|x+a| \le |x| + |a| \le 2$

Therefore, if $x, a \in (0,1)$ and $|x-a| < \delta$, then

$$|f(x) - f(a)| = |x^2 - a^2| = |x+a||x-a| \le 2|x-a| < 2\delta = \varepsilon$$

$\blacksquare$

Rmk. ① The difference between the def'n of continuity and uniform continuity is that for a continuous function, δ may depend on a, whereas for a uniformly continuous function,

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δ must be chosen independently of a.

(2) Every uniformly continuous function on E is also continuous on E. But the converse is not true. For ex. $f(x) = x^2$ is cont. on $(-\infty, \infty)$ but it is not uniformly cont.

Ex. (2) Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

proof. Spse to the contrary that f is uniformly cont. on $\mathbb{R}$. Then $\exists$ a $\delta > 0$ such that

$$|x-a| < \delta \implies |f(x) - f(a)| < 1, \ \forall x, a \in \mathbb{R}.$$

By the Archimedean Principle, choose $n \in \mathbb{N}$ so large that $n\delta > 1$. Set $a = n$ and $x = n + \frac{\delta}{2}$. Then $|x-a| = |n + \frac{\delta}{2} - n| = \frac{\delta}{2} < \delta$

but
$$1 > |f(x) - f(a)| = |x^2 - a^2| = |x-a||x+a|$$
$$= \frac{\delta}{2} \cdot \left(2n + \frac{\delta}{2}\right)$$
$$= n\delta + \frac{\delta^2}{4} > n\delta > 1$$

this implies $1 > 1$ which is a contradiction. Hence f is not uniformly cont. on $\mathbb{R}$ ∎

<u>lemma</u> Suppose that $E \subseteq \mathbb{R}$ and that $f: E \to \mathbb{R}$ is uniformly continuous. If $x_n \in E$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

<u>proof</u>. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|x - a| < \delta$, $x, a \in E \Rightarrow |f(x) - f(a)| < \varepsilon$.

Since $\{x_n\}$ is Cauchy. choose $N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow |x_n - x_m| < \delta$.

Then $n, m \geq N \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$.

This means $\{f(x_n)\}$ is Cauchy ∎.

<u>Thm</u> ① Suppose that I is a closed, bounded interval. If $f: I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

<u>proof</u>. Spse to the contrary that f is continuous but not uniformly continuous on I.

Then $\exists$ a $\varepsilon_0 > 0$ and $x_n, y_n \in I$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $n \in \mathbb{N}$.

By the Bolzano–Weierstrass Thm and the Comparison Thm, $\{x_n\}$ has a convergent subseq. Say $x_{n_k} \longrightarrow x \in I$ as $k \to \infty$.

Similarly, the sequence $\{y_{n_k}\}_{k=1}^{\infty}$ has a convergent subsequence Say $y_{n_{k_j}} \longrightarrow y \in I$ as $j \to \infty$. Since $x_{n_{k_j}} \longrightarrow x$ as $j \to \infty$ and f is continuous, it follows that

$$|f(x) - f(y)| \geq \varepsilon_0, \text{ i.e } f(x) \neq f(y).$$

But $|x_n - y_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$ so by the Squeeze Thm implies $x = y$ Therefore, $f(x) = f(y)$, a contradiction. ▤

<u>Rmk.</u> Thm(1) might not hold if "closed" replaced by "open".

<u>Ex.</u> $f(x) = \frac{1}{x}$ is continuous on $(0,1)$ but not uniformly cont. on $(0,1)$. (see the book p. 93).

<u>Thm (2)</u>. Suppose that $a<b$ and that $f:(a,b)\longrightarrow\mathbb{R}$. Then f is <u>uniformly continuous</u> on (a,b) iff f can be continuously extended to $[a,b]$, i.e., iff there is a continuous function $g:[a,b]\longrightarrow\mathbb{R}$ which satisfies

$$f(x)=g(x), \quad x\in(a,b).$$

Proof. See the book p. 94.

<u>Ex.</u> Prove that $f(x)=\dfrac{x-1}{\ln x}$ is uniformly continuous on $(0,1)$.

<u>Proof</u>. $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\dfrac{x-1}{\ln x}=0.$

. $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\dfrac{x-1}{\ln x}=\lim\limits_{x\to 1^-}\dfrac{1}{1/x}=1$

Let $g(x)=\begin{cases}\dfrac{x-1}{\ln x}, & x\in(0,1)\\ 0, & x=0\\ 1, & x=1\end{cases}$

Notice that $g : [0,1] \to \mathbb{R}$ is a continuous function on $[0,1]$ and $g(x) = f(x)$, $\forall x \in (0,1)$

Hence f is continuously extendable to $[0,1]$, So by thm (2), f is uniformly continuous on $(0,1)$.

<u>Rmk.</u> Let f be cont. on a bounded, open, nondegenerate interval (a,b). Notice that f is continuously extendable to $[a,b]$ iff $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist.

Indeed, when they exist, we define g at $x = a$ and $x = b$ as

$$g(a) = \lim_{x \to a^+} f(x), \quad g(b) = \lim_{x \to b^-} f(x)$$

<u>H.W's</u> <u>Exercises p.95</u> [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]

(i.e. All)

Birzeit University
Mathematics Department
Math3331
H.W#3 (Chapter 3)

Instructor: Dr. Ala Talahmeh **Second Semester 2019/2020**
Name:... **Date: 27/04/2020**

Exercise#1 [10 marks]. Let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{R}$ is called **Lipschitz** if there exists a constant $\alpha > 0$ such that

$$|f(x) - f(y)| \leq \alpha |x - y|,$$

for all $x, y \in E$.

a. Give two examples of Lipschitz functions.

b. Prove that every Lipschitz function is uniformly continuous.

c. Let $g : [0, 1] \to \mathbb{R}$, $g(x) = \sqrt{x}$. Prove that g is uniformly continuous but not Lipschitz.

Exercise#2 [5 marks]. Let $f : E \to \mathbb{R}$. Let $a \in E$ such that $\lim_{x\to a} f(x)$ exists. Show that $\lim_{x\to a} |f(x)|$ exists and the following identity holds:

$$\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)|.$$

Exercise#3 [5 marks]. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each $x \in I$ there exists $y \in I$ such that $|f(x)| > 2|f(y)|$. Prove there exists a point $c \in I$ such that $f(c) = 0$.

Exercise#4 [5 marks]. Using $(\varepsilon - \delta)$ definition of limit show that

$$\lim_{x\to -1} \frac{x + 5}{2x + 3} = 4.$$

Exercise#5 [10 marks].

a. Let a be a real number such that $a > 0$. Show that the function $f : [a, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is uniformly continuous.

b. Show that if f and $g : E \to \mathbb{R}$ are uniformly continuous and bounded, then fg is uniformly continuous.

Exercise#6 [10 marks]. Let a and b two real numbers such that $a < b$ and $f : [a, b] \to [a, b]$.

a. Suppose that for every $x, y \in [a, b] : |f(x) - f(y)| \leq |x - y|$. Show that f is continuous. Deduce that there exists $c \in [a, b]$ such that $f(c) = c$.

b. Suppose that for every x, y such that $x \neq y$ we have$|f(x) - f(y)| < |x - y|$. Show that there exists one and only one $c \in [a, b]$ such that $f(c) = c$.

Exercise#7 [15 marks]. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$.

a. Compute $f(0)$ and show that $f(-x) = -f(x)$.

b. Prove that for every $x \in \mathbb{R}$ and $n \in \mathbb{Z} : f(nx) = nf(x)$.

c. Prove that for every $x \in \mathbb{R}$ and q rational: $f(qx) = qf(x)$.

d. Prove that for every $x \in \mathbb{R}$ and λ real: $f(\lambda x) = \lambda f(x)$.

e. Find $f(x)$.

Good Luck

CH4 Differentiability on $\mathbb{R}$

4.1 the Derivative

Df (1). A real function f is said to be **differentiable** at a point $a \in \mathbb{R}$ iff f is defined on some open interval I containing a and

$$\boxed{f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}} \quad (*)$$

In this case $f'(a)$ is called the derivative of f at a.

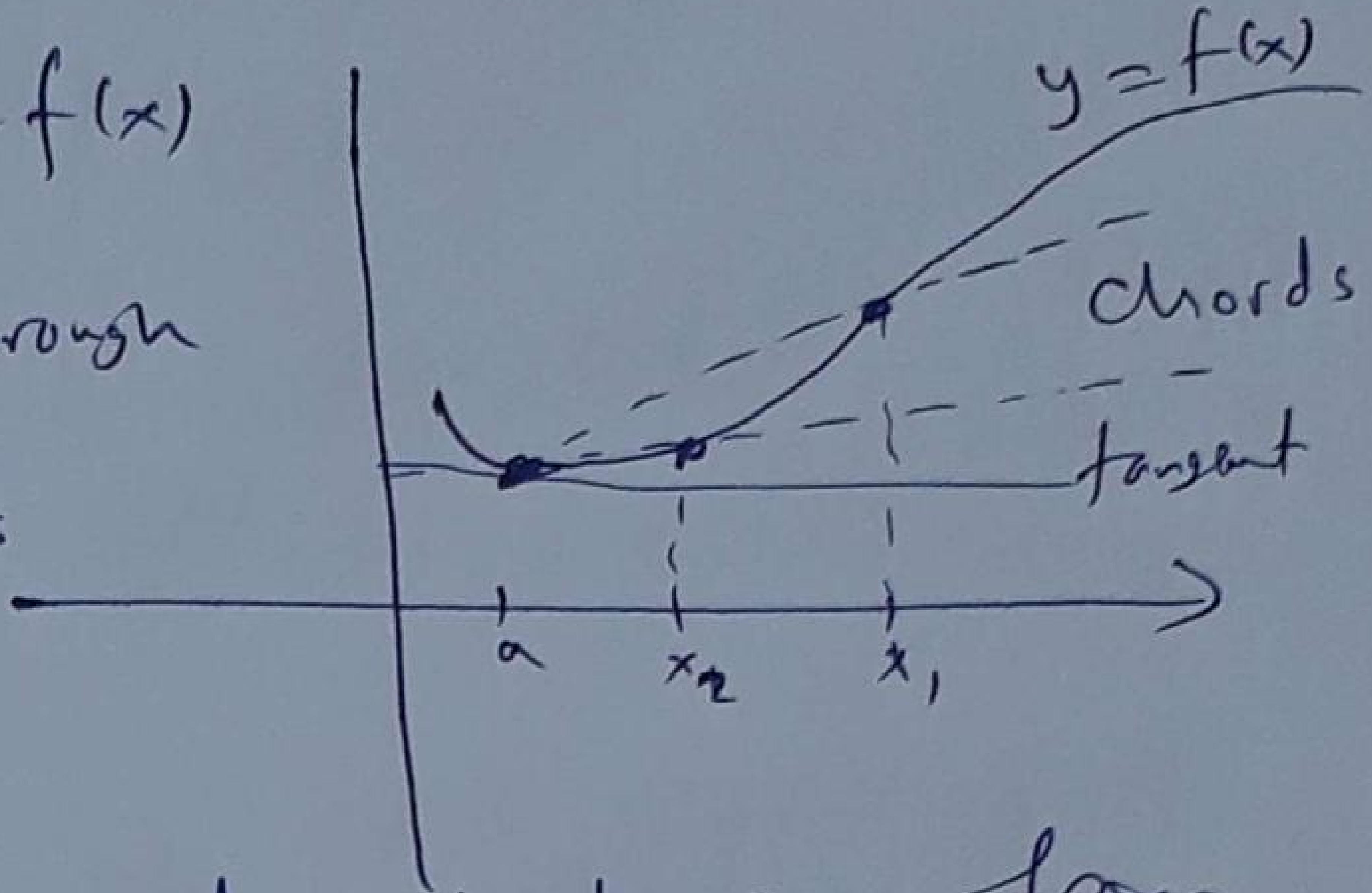
Rmk. (1) the assumption that f be defined on an open interval containing a is made so that the quotients in $(*)$ are defined for all $h \neq 0$ sufficiently small.

(2) the graph of $y = f(x)$ has a non-vertical tangent line at $(a, f(a))$ iff $f'(a)$ exists, in this case the slope of the tangent line is $f'(a)$.

Let us consider a geometric interpretation of $(*)$

②

Spse that f is diffble at a. A secant line of the graph $y=f(x)$ is a line passing through at least two points on the graph, and



a chord is a line segment which runs from one point on the graph to another.

Let $x = a+h$, the slope of the chord passing through $(x, f(x))$, $(a, f(a))$ is

$$\frac{f(x)-f(a)}{x-a}$$

Since $x=a+h$, (*) becomes

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Hence, as $x \longrightarrow a$, the slopes of the chords through $(x, f(x))$ and $(a, f(a))$ approximate the slope of the tangent line of $y = f(x)$ at $x=a$

(3)

Thus, the slope of the tangent line to $y=f(x)$ at $x=a$ is $f'(a)$.

- $y=f(x)$ has a unique tangent line at $(a, f(a))$ iff $f'(a)$ exists.

- If f is diffble at each point in E, then f' is a function on E.

Notations.. $D_x f = \frac{df}{dx} = f^{(1)}(x) = f'(x) = y' = \frac{dy}{dx}$.

when $y=f(x)$.

- Higher order derivatives are defined as $f^{(n+1)}(a) := \left(f^{(n)}\right)'(a)$, $n \in \mathbb{N}$

provided these derivatives exist.

Notation $D_x^n f$, $\frac{d^n f}{dx^n}$, $f^{(n)}$, and $\frac{d^n y}{dx^n}$, $y^{(n)}$

when $y=f(x)$

<u>Thm (1)</u>. A real function f is diffble at $x = a \in \mathbb{R}$ iff $\exists$ an open interval I and a function $F : I \longrightarrow \mathbb{R}$ such that $a \in I$, f is defined on I, F is continuous at a, and

$$f(x) = F(x)(x-a) + f(a).$$

holds $\forall x \in I$, in which case $F(a) = f'(a)$.

<u>Proof</u>. $(\Longrightarrow)$ suppose that f is diffble at a. Then f is defined on some open interval I containing a, and the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}.$$

Define F on I by

$$F(x) := \begin{cases} \dfrac{f(x) - f(a)}{x-a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

Then $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$, and F is continuous at a. since $f'(a)$ exists.

($\Leftarrow$) conversely, suppose that $\exists$ an open interval I and $F: I \longrightarrow \mathbb{R}$ s.t $a \in I$, f is defined on I, F is continuous at a and $f(x) = F(x)(x-a) + f(a)$, $\forall x \in I$. then

$$F(x) = \frac{f(x) - f(a)}{x-a}, \quad x \neq a.$$

The continuity of F implies that

$$F(a) = \lim_{x \to a} F(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \text{ exists.}$$

therefore, f is diff'ble at a and

$$f'(a) = F(a).$$

$\blacksquare$

<u>Thm (2)</u> A real function f is diff'ble at $x=a$ iff $\exists$ a function T of the form $T(x) := mx$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0.$$

6

<u>proof.</u> ($\Rightarrow$) suppose that f is diffble at a, and define T as $T(x) = mx$, where $m = f'(a)$. Then by (*),

$$\frac{f(a+h) - f(a) - T(h)}{h} = \frac{f(a+h) - f(a)}{h} - f'(a)$$

$$\longrightarrow 0 \quad \text{as } h \to 0.$$

Conversely, suppose that $\exists$ a function T of the form $T(x) = mx$ s.t

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0,$$

then for $h \neq 0$,

$$\frac{f(a+h) - f(a)}{h} = m + \frac{f(a+h) - f(a) - mh}{h}$$

$$= m + \underbrace{\frac{f(a+h) - f(a) - T(h)}{h}}_{\to 0}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = m + 0 \quad (\text{assumption})$$

$$= m$$

that is, $f'(a)$ exists and equals m

therefore, f is diffble at $x = a$ ▪

(7)

Thm (3). If f is diffble at a, then f is continuous at a.

Proof. Suppose that f is diffble at a. By Thm (1), $\exists$ an open interval I and a function F, continuous at a, such that

$$f(x) = F(x)(x-a) + f(a), \quad \forall x \in I.$$

Taking the limit as $x \to a$, we see that

$$\lim_{x \to a} f(x) = F(a) \cdot 0 + f(a) = f(a).$$

In particular, $f(x) \to f(a)$ as $x \to a$; i.e., f is continuous at a $\blacksquare$

Rmk. The converse of Thm (3) is false

example. Show that $f(x) = |x|$ is continuous at 0 but not diffble there.

proof. Since $x \to 0 \Rightarrow |x| \to 0$, f is continuous at 0. On the other hand,

$$f'_{+}(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h| - 0}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

Since $f'_+(0) \neq f'_-(0)$, it follows that $f'(0)$ doesnot exist. Therefore, f is not diffble at 0. $\boxed{}$

<u>Df ②</u>. Let I be a nondegenerate interval

(i) A function $f : I \to \mathbb{R}$ is said to be diffble on I if and only if

$$f'_I(a) := \lim_{\substack{x \to a \\ x \in I}} \frac{f(x) - f(a)}{x - a} \quad \text{exists}$$

and is finite $\forall a \in I$.

(ii) f is said to be continuously diffble on I if and only if f'_I exists and is continuous on I.

<u>Rmk.</u> When a is not an endpoint of I, $f'_I(a)$ is the same as $f'(a)$.

If is diffble on $[a,b]$. then

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$ and

$$f'(b) = \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h}.$$

Ex. show that $f(x) = x^{3/2}$ is diffble on $[0,\infty)$ and $f'(x) = \frac{3\sqrt{x}}{2}, \forall x \in [0,\infty)$.

Pf. By the power Rule, $f'(x) = \frac{3}{2} x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$ $\forall x \in (0,\infty)$. And by def'n,

$$f'(0) = \lim_{h\to 0^+} \frac{h^{3/2}-0}{h} = \lim_{h\to 0^+} \sqrt{h} = 0.$$

$$\therefore f'(x) = \frac{3\sqrt{x}}{2}, \forall x \in [0,\infty). \quad \boxtimes$$

Notation $C^n(I)$.

Let I be anondegenerate interval. For $n \in \mathbb{N}$, we define the collection of functions $C^n(I)$ by

$$C^n(I) := \left\{ f : f : I \longrightarrow \mathbb{R} \text{ and } f^{(n)} \text{ exists and is continuous on } I \right\}$$

- When $f \in C^n(I)$, $\forall n \in \mathbb{N}$, we shall denote it by $f \in C^\infty(I)$.

- Notice that $C^1(I)$ is precisely the collection of real functions which are continuously diff'ble on I.

- $C^n([a,b]) = C^n[a,b]$.

- $C^\infty(I) \subset C^m(I) \subset C^n(I)$, for all integers $m > n > 0$

- Not every function which is diff'ble on $\mathbb{R}$ belongs to $C^1(\mathbb{R})$.

<u>Ex.</u> $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

is diff'ble on $\mathbb{R}$ but not continuously diff'ble on any interval contains the origin.

<u>proof.</u> By def'n.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

and $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right),\ x \neq 0.$

Thus, f is diffble on $\mathbb{R}$ but $\lim_{x\to 0} f'(x)$ does not exist. In particular f' is not continuous on any interval which contains the orgin. ∎

<u>Rmk.</u> A function which is diffble on two sets is <u>not necessarily</u> diffble on their union.

<u>example</u> $f(x) = |x|$ is diffble on $[0,1]$ and on $[-1,0]$ but not on $[-1,1]$.

<u>proof.</u> Since $f(x) = \begin{cases} x & \text{when } x > 0 \\ -x & \text{when } x < 0 \end{cases}$, it is clear that f is diffble on $[-1,0)\cup(0,1]$

$\left(\text{with } f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \right)$

we proved that f is not diffble at $x=0$.

However, $f'_{[0,1]}(0) = \lim_{h \to 0^+} \frac{|h|}{h} = 1$

$$f'_{[-1,0]}(0) = \lim_{h \to 0^-} \frac{|h|}{h} = -1.$$

Therefore, f is diffble on $[0,1]$ and on $[-1,0]$ but not on $[-1,1]$ ∎

<u>Exercises</u> (H.W's) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 (All).

4.2 Differentiability Theorems

Thm (9). Let $I \subseteq \mathbb{R}$ be an interval, let $a \in I$, $\alpha \in \mathbb{R}$ and let $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ be functions that diff'ble at a, then $f+g$, αf, $f \cdot g$ and [when $g(a) \neq 0$] $\frac{f}{g}$ are all diff'ble at a. In fact,

(i) $(f+g)'(a) = f'(a) + g'(a)$

(ii) $(\alpha f)'(a) = \alpha f'(a)$.

(iii) $(f \cdot g)'(a) = g(a) f'(a) + f(a) g'(a)$

(iv) $\left(\frac{f}{g}\right)'(a) = \dfrac{g(a) f'(a) - f(a) g'(a)}{g^2(a)}$

Proof. We shall prove (iii) and (iv), leaving (i), (ii) as exercises.

(iii) Let $p := fg$, then for $x \in I$, $x \neq a$ we have

$$\frac{p(x) - p(a)}{x-a} = \frac{f(x) g(x) - f(a) g(a)}{x-a}$$

$$= \frac{f(x) g(x) - f(a) g(x) + f(a) g(x) - f(a) g(a)}{x-a}$$

$$= \frac{f(x)-f(a)}{x-a} \cdot g(x) + f(a) \cdot \frac{g(x)-g(a)}{x-a},$$

Since g is continuous at a, by thm (3), then $\lim_{x\to a} g(x) = g(a)$. Since f and g are diffble at $x=a$, we deduce that

$$\lim_{x\to a} \frac{p(x)-p(a)}{x-a} = f'(a)\cdot g(x) + f(a)\, g'(a)$$

Hence $p := fg$ is diffble at a and (iii) holds.

(iv) Let $q := \frac{f}{g}$. Since g is diffble at a, it is continuous at that point (Thm). Therefore, since $g(a) \neq 0$, then (by thm), $\exists$ an interval $J \subseteq I$ with $a \in J$ s.t. $g(x) \neq 0$, $\forall x \in J$.

For $x \in J$, $x \neq a$, we have

$$\frac{q(x)-q(a)}{x-a} = \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a}$$

$$= \frac{f(x)\,g(a) - f(a)\,g(x)}{(x-a)\,g(x)\,g(a)}$$

$$= \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)\,g(x)}{(x-a)\ g(x)\,g(a)}$$

$$= \frac{1}{g(x)g(a)}\left[\frac{f(x)-f(a)}{x-a}\cdot g(a) - f(a)\cdot\frac{g(x)-g(a)}{x-a}\right]$$

Using the continuity of g at a and the differentiability of f and g at a, then we get

$$q'(a) = \lim_{x \to a} \frac{q(x)-q(a)}{x-a} = \frac{f'(a)\,g(a) - g'(a)\,f(a)}{g^2(a)}$$

Thus, $q = \frac{f}{g}$ is diffble at a and (iv) holds. ∎

<u>Rmk.</u> Formula in (i) is called the <u>Sum Rule</u>, in (iii) is called the <u>Product Rule</u>, in (ii) the <u>homogeneous Rule</u> and in (iv) is called the <u>Quotient Rule</u>.

<u>Corollary.</u> If $f_1, f_2, \dots, f_n$ are functions on an interval I to $\mathbb{R}$ that are diff'ble at $a \in I$, then:

(i) The function $f_1 + f_2 + \dots + f_n$ is diff'ble at a and $(f_1 + f_2 + \dots + f_n)'(a) = f_1'(a) + f_2'(a) + \dots + f_n'(a)$

(ii) the function $f_1 f_2 \dots f_n$ is diff'ble at a, and

$$(f_1 f_2 \dots f_n)'(a) = f_1'(a) f_2(a) \dots f_n(a)$$
$$+ f_1(a) f_2'(a) \dots f_n(a) + \dots + f_1(a) f_2(a) \dots f_n'(a).$$

<u>Proof:</u> Use Mathematical Induction.

<u>Thm 5</u> [<u>Chain Rule</u>]

Let f and g be real functions. If f is diff'ble at a and g is diff'ble at $f(a)$, then $g \circ f$ is diff'ble at a with

$$(g \circ f)'(a) = g'(f(a))\, f'(a).$$

<u>Proof:</u> By thm 1, $\exists$ open intervals I and J, and functions $F : I \to \mathbb{R}$, continuous at a,

and $G : J \longrightarrow \mathbb{R}$, continuous at $f(a)$,

such that $F(a) = f'(a)$, $G(f(a)) = g'(f(a))$,

$$\boxed{f(x) = F(x)(x-a) + f(a), \quad x \in I} \quad (A)$$

and $$\boxed{g(y) = G(y)\left(y - f(a)\right) + g(f(a)), \; y \in J.} \quad (B)$$

Since f is continuous at a, we may assume that $f(x) \in J$, $\forall x \in I$.

Fix $x \in I$. Apply (B) to $y = f(x)$ and (A) to x to write

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= G(f(x))\underbrace{\left(f(x) - f(a)\right)} + g(f(a)) \quad [\text{using (B)}] \\ &= G(f(x))\, F(x)(x-a) + (g \circ f)(a) \quad [\text{using (A)}]\end{aligned}$$

Set $H(x) = G(f(x))\, F(x)$ for $x \in I$.

Since F is continuous at a and G is continuous at $f(a)$, it is clear that H is continuous at a. Moreover,

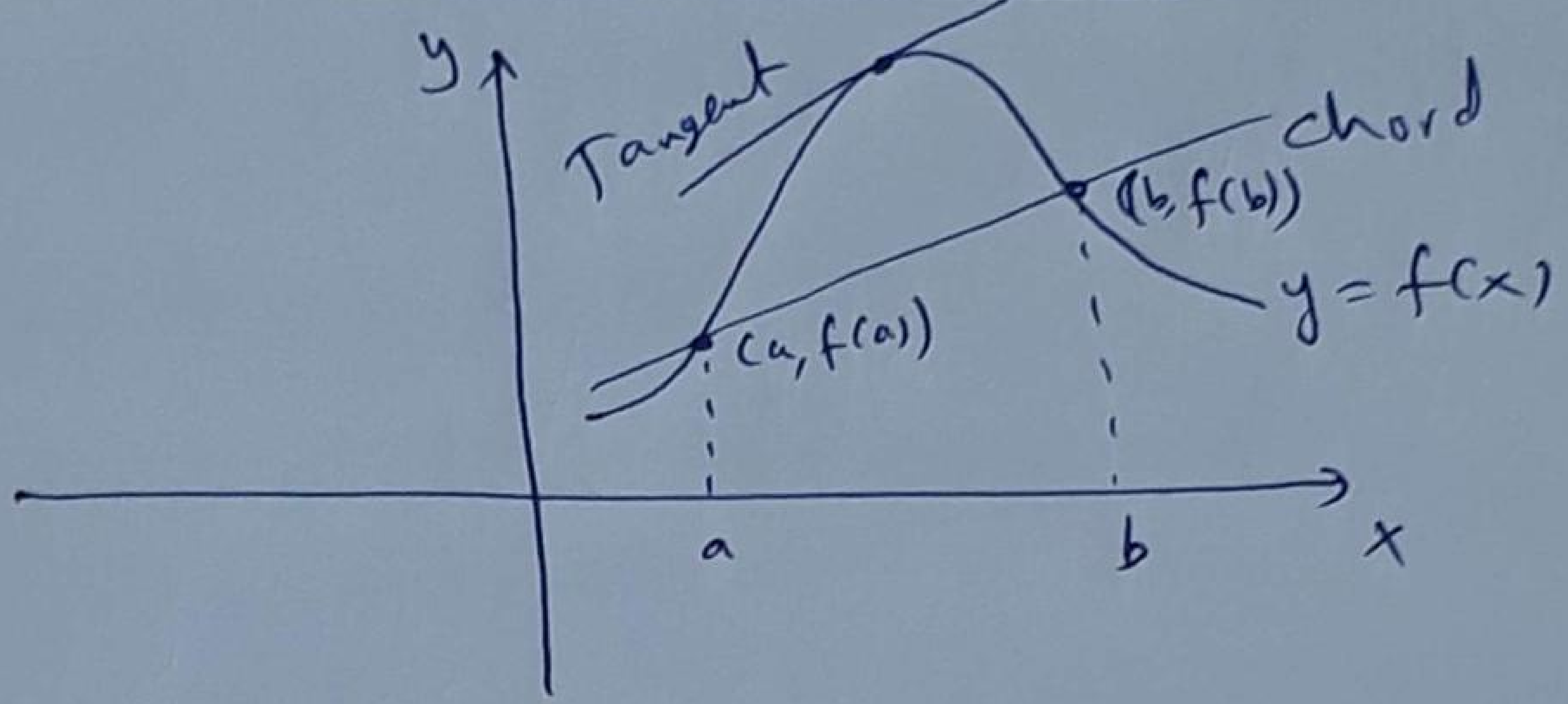
$$H(a) = G(f(a))\,F(a) = g'(f(a))\,f'(a)$$

It follows from Thm 1, $(g\circ f)'(a) = H(a)$, i.e.,

$$(g\circ f)'(a) = g'(f(a))\,f'(a).$$

Exercises p.106 (H.w's) 0, 1, 2, 3, 4, 5, 8, 9.

4.3 The Mean value theorem



Lemma. [Rolle's thm].

Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, diffble on (a,b), and if $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a,b)$.

Proof. By the Extreme value thm, f has a finite maximum M and a finite minimum m on $[a,b]$.

If $M = m$, then f is constant on (a,b) and $f'(x) = 0$, $\forall x \in (a,b)$.

Suppose that $M \neq m$. Since $f(a) = f(b)$, f must assume one of the values M or m at some $c \in (a,b)$. We may suppose that $f(c) = M$ (similar proof when $f(c) = m$)

Since M is the max. of f on $[a,b]$, we have $f(c+h)-f(c) \le 0$

for all h satisfy $c+h \in (a,b)$. In the case $h>0$ this implies

$$f'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$$

and the case $h<0$,

$$f'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0.$$

It follows that $f'(c)=0$ ▤

Rmk. (1) the continuity hypothesis in Rolle's Thm cannot be relaxed at even one point in $[a,b]$.

proof. $f(x) = \begin{cases} x, & x\in[0,1) \\ 0, & x=1 \end{cases}$ is continuous

on $[0,1)$, diff'ble on $(0,1)$, and $f(0)=f(1)=0$

but $f'(x)$ is never zero.

(2) the differentiability hypothesis in Rolle's Thm cannot be relaxed at even

one point in (a,b).

<u>proof</u>: $f(x) = |x|$ is cont. on $[-1,1]$, diffble on $(-1,1)\setminus\{0\}$, and $f(-1) = f(1) = 1$ but $f'(x)$ is never zero.

<u>Thm 6</u> Suppose that $a, b \in \mathbb{R}$ with $a<b$.

(i) [<u>Generalized Mean value Thm</u>]. If f, g are continuous on $[a,b]$ and diffble on (a,b), then there is a $c \in (a,b)$ such that

$$g'(c)\big(f(b) - f(a)\big) = f'(c)\big(g(b) - g(a)\big).$$

(ii) [<u>Mean value Thm</u>] If f is continuous on $[a,b]$, and diffble on (a,b), then there is a $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

<u>proof</u>. (i) Set

$$g(x) = f(x)\big(g(b) - g(a)\big) - g(x)\big(f(b) - f(a)\big).$$

Since $h'(x) = f'(x)\big(g(b)-g(a)\big) - g'(x)\big(f(b)-f(a)\big)$

it is clear that h is cont. on $[a,b]$,

diff'ble on (a,b), and $h(a) = h(b) = 0$.

Thus, by Rolle's Thm, $h'(c) = 0$ for some

$c \in (a,b)$. That is, there is a $c \in (a,b)$ s.t.

$$g'(c)\big(f(b)-f(a)\big) = f'(c)\big(g(b)-g(a)\big).$$

(ii) Set $g(x) = x$ and apply part (i). Then,

$\exists$ a $c \in (a,b)$ s.t

$$f(b)-f(a) = f'(c)(b-a). \quad \blacksquare$$

<u>Rmk.</u> ① the Generalized Mean Value Thm

is also called <u>Cauchy's Mean Value Thm</u>

② For a geometric interpretation of (ii),

see the opening graph p.19 (Note).

③ the Mean Value Thm is most often used

to extract information about f from f'

as follows.

Df: Let E be a nonempty subset of $\mathbb{R}$ and $f: E \longrightarrow \mathbb{R}$.

(i) f is said to be increasing (resp., strictly increasing) on E iff

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \; [\text{resp. } f(x_1) < f(x_2)]$$

(ii) f is said to be decreasing (resp., strictly decreasing) on E iff

$$x_1, x_2 \in E \text{ and } x_1 < x_2 \Longrightarrow f(x_1) \ge f(x_2) \; [\text{resp., } f(x_1) > f(x_2)].$$

(iii) f is said to be monotone (resp., strictly monotone) on E iff f is either decreasing or increasing (resp., either strictly decreasing or strictly increasing) on E.

ex. $f(x) = x^2$ is strictly monotone on $[0, 1]$, and on $[-1, 0]$, it is not monotone on $[-1, 1]$.

Thm(7). Suppose that $a, b \in \mathbb{R}$, with $a < b$, that f is continuous on $[a, b]$, and that f is diff'ble on (a, b).

i) If $f'(x) > 0$ [resp., $f'(x) < 0$], $\forall x \in (a,b)$, then f is strictly increasing [resp, strictly decreasing] on $[a,b]$.

ii) If $f'(x) = 0$, $\forall x \in (a,b)$, then f is constant on $[a,b]$.

iii) If g is continuous on $[a,b]$ and diffble on (a,b), and if $f'(x) = g'(x)$, $\forall x \in (a,b)$, then $f-g$ is constant on $[a,b]$.

Proof: (i) Let $a \leq x_1 < x_2 \leq b$. By the Mean Value thm, $\exists$ a $c \in (a,b)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus, $f(x_2) > f(x_1)$ when $f'(c) > 0$ and $f(x_2) < f(x_1)$, when $f'(c) < 0$.

(ii) If $f'(x) = 0$, then by the proof of part (i), f is both increasing and decreasing, and hence constant on $[a,b]$.

(iii) Follows from part (ii) applied to $h = f - g$. ($h'(x) = f'(x) - g'(x) = 0 \Rightarrow h(x) = f(x) - g(x)$ is constant on $[a,b]$) ∎

Thm 8. Suppose that f is increasing on $[a,b]$.

(i) If $c \in [a,b)$, then $f(c^+)$ exists and
$$f(c) \le f(c^+) = \lim_{x \to c^+} f(x)$$

(ii) If $c \in (a,b]$, then $f(c^-)$ exists and
$$f(c^-) = \lim_{x \to c^-} f(x) \le f(c)$$

Proof. See the textbook p. 112.

Thm 9. If f is monotone on an interval I, then f has at most countably many points of discontinuity on I.

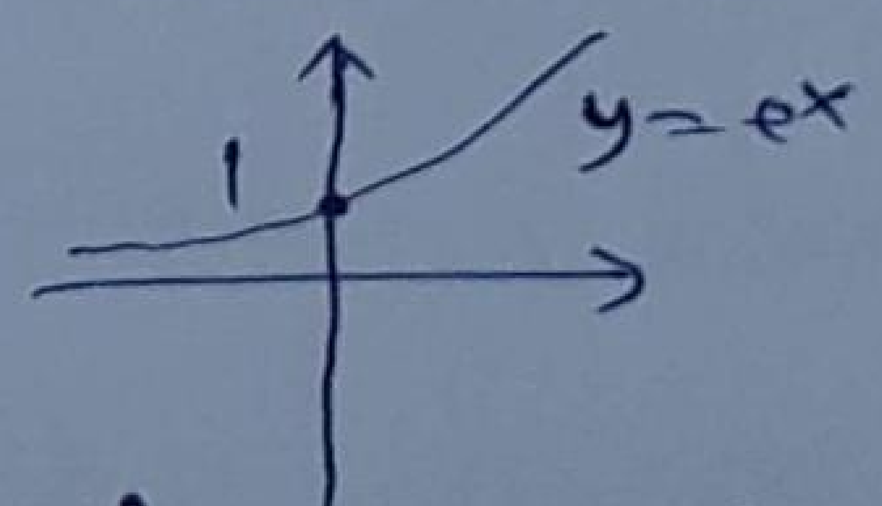
Proof. See the textbook p. 113.

Application (Thm 7 (i)).

ex. Prove that $1 + x < e^x$, $\forall x > 0$.

Proof. Let $f(x) = e^x - x$, and observe that

$$f'(x) = e^x - 1 > 0, \ \forall x > 0$$



It follows from Thm 7 (i), that $f(x)$ is strictly increasing on $(0,\infty)$. Thus, As $x > 0$, then

$e^x - x = f(x) > f(0) = 1$. In particular, $e^x > x + 1$, $\forall x > 0$ ∎

<u>thm</u> 10. [Bernoulli's Inequality].

Let α be a positive real number.

If $0 < \alpha \leq 1$, then $(1+x)^{\alpha} \leq 1 + \alpha x$, $\forall x \geq -1$,

and if $\alpha \geq 1$, then $(1+x)^{\alpha} \geq 1+\alpha x$, $\forall x \geq -1$.

<u>Proof.</u> <u>Case 1 $0 < \alpha \leq 1$.</u>

Fix $x \geq -1$ and let $f(t) = t^{\alpha}$, $t \in [0, \infty)$.

Since $f'(t) = \alpha t^{\alpha - 1}$, it follows from the Mean value thm (applied to $a = 1$, and $b = 1+x$)

that $f(1+x) - f(1) = f'(c)(1+x-1)$

(*) $\boxed{f(1+x) - f(1) = \alpha x c^{\alpha-1}}$, for some c between 1 and $1+x$.

Sub<u>Case 1.1</u> $x > 0$. Then $c > 1$. Since $0 < \alpha \leq 1$ implies $\alpha - 1 \leq 0$, it follows that $c^{\alpha-1} \leq 1$, hence $x c^{\alpha-1} \leq x$. therefore, we have

by (*) that $(1+x)^{\alpha} = f(1+x) = f(1) + \alpha x c^{\alpha-1}$

$$\leq f(1) + \alpha x = 1 + \alpha x$$

as required.

Sub Case 1.2. $-1 \leq x \leq 0$. Then $c \leq 1$ so

$c^{\alpha-1} \geq 1$. But since $x \leq 0$, it follows that

$x c^{\alpha-1} \leq x$ as before, we have by (*) that

$$(1+x)^\alpha = f(1+x) = f(1) + \alpha x c^{\alpha-1}$$
$$\leq f(1) + \alpha x = 1 + \alpha x$$

Case 2. $\alpha \geq 1$. (Exercise) $\boxed{}$

Ex. prove that the sequence $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing and $\lim_{n\to\infty} x_n = L$ satisfies $2 < L \leq 3$ $\left[\lim_{n\to\infty} x_n = e \text{ as you know } = 2.718281828459\ldots\right]$

proof: $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing, since by Bernoulli's Inequality,

$$x_n = \left(1+\frac{1}{n}\right)^n = \left[\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}}\right]^{n+1}$$
$$\leq \left[1 + \left(\frac{1}{n}\right)\left(\frac{n}{n+1}\right)\right]^{n+1}$$
$$= \left(1+\frac{1}{n+1}\right)^{n+1} = x_{n+1}.$$

- To prove that this sequence is bounded above, observe that by the Binomial Formula

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k \cdot (1)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

Now, $\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{k!\,(n-k)!\cdot n^k}$

$$= \frac{n(n-1)(n-2)\cdots(n-(k-1))\,\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}\,n^k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k}\cdot\frac{1}{k!}$$

$$\leq 1\cdot\frac{1}{k!} \leq \frac{1}{2^{k-1}},$$

for all $k \in \mathbb{N}$. Next,

$$2 = \left(1+\frac{1}{1}\right) < \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 3 - \frac{1}{2^{n-1}}$$

$$< 3$$

for $n > 1$. Hence, by the Monotone Convergence thm, the limit L exists and satisfies

$$2 < L \leq 3 .$$

Thm 11 [Intermediate value theorem for derivatives] . [or Darboux's Thm].

Suppose that f is diffble on $[a,b]$ with $f'(a) \neq f'(b)$. If y_0 is a real number which lies between $f'(a)$ and $f'(b)$, then there is an $x_0 \in (a,b)$ such that $f'(x_0) = y_0$.

Proof. Suppose that y_0 lies between $f'(a)$ and $f'(b)$. By symmetry, we may assume that $f'(a) < y_0 < f'(b)$. Set $F(x) = f(x) - y_0 x$, for $x \in [a,b]$, and observe that F is diffble on $[a,b]$. Hence, by the Extreme Value Thm, F has an absolute minimum, say $F(x_0)$, on $[a,b]$. Now, $F'(a) = f'(a) - y_0 < 0$, so $F(a+h) - F(a) < 0$ for $h > 0$ sufficiently small. Hence, $F(a)$ is NOT the absolute minimum of F on $[a,b]$. Similarly, $F(b)$ is not the absolute minimum of F on $[a,b]$. Hence, the absolute minimum $F(x_0)$

must occur on (a,b), i.e., $x_0 \in (a,b)$ and $0 = F'(x_0) = y_0 - f'(x_0)$. Hence, $f'(x_0) = y_0$.

Ex. The function $g : [-1,1] \longrightarrow \mathbb{R}$ defined by

$$g(x) := \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0, \end{cases}$$

clearly fails to satisfy the intermediate value property on $[-1,1]$. Therefore, by Darboux's Thm, there does not exist a function f s.t. $f'(x) = g(x)$, $\forall x \in [-1,1]$. In otherwords, g is NOT the derivative on $[-1,1]$ of any function.

H.W's Exercises 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 (i.e., All).

Chapter 5 Integrability on $\mathbb{R}$

5.1 The Riemann Integral.

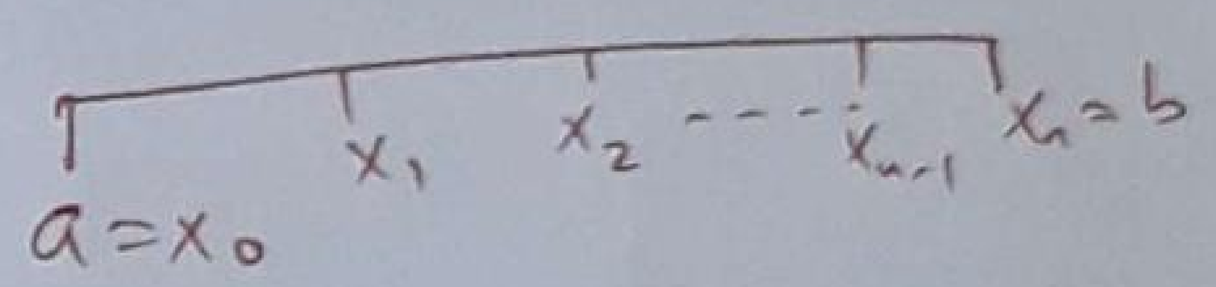
Df 1. let $a, b \in \mathbb{R}$, with $a < b$.

(i) A *partition* of $[a,b]$ is a set of subintervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n],$$

where $a = x_0 < x_1 < \cdots < x_n = b$. $(*)$

Thus any set of $(n+1)$ points satisfying $(*)$ defines a partion P of $[a,b]$, which we denote by



$$P = \{x_0, x_1, \ldots, x_n\}.$$

(ii) the *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

$$\|P\| := \max\{x_1 - x_0,\ x_2 - x_1,\ \ldots,\ x_n - x_{n-1}\}$$

$$= \max_{1 \le j \le n} |x_j - x_{j-1}|$$

(iii) A *refinement* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is a partition Q of $[a,b]$ which satisfies $Q \supseteq P$. And we say that Q is finer than P.

Ex. Prove that for each $n \in \mathbb{N}$,

$$P_n = \left\{ \frac{j}{2^n} : j = 0, 1, \dots, 2^n \right\}$$ is a partition

of $[0,1]$ and P_m is finer than P_n when $m > n$.

proof: Fix $n \in \mathbb{N}$. If $x_j = \frac{j}{2^n}$, then

$$0 = x_0 < x_1 < x_2 < \dots < x_{2^n} = 1.$$ Thus,

P_n is a partition of $[0,1]$.

Next, we need to show that $P_m \supseteq P_n$ when $m > n$.

Let $m > n$ and set $p = m - n$. If $0 \le j \le 2^n$,

then $\frac{j}{2^n} = \frac{j \cdot 2^p}{2^m}$ and $0 \le j 2^p \le 2^m$.

thus P_m is finer than P_n. $\blacksquare$

Rmk ① If P and Q are partitions of $[a,b]$, then

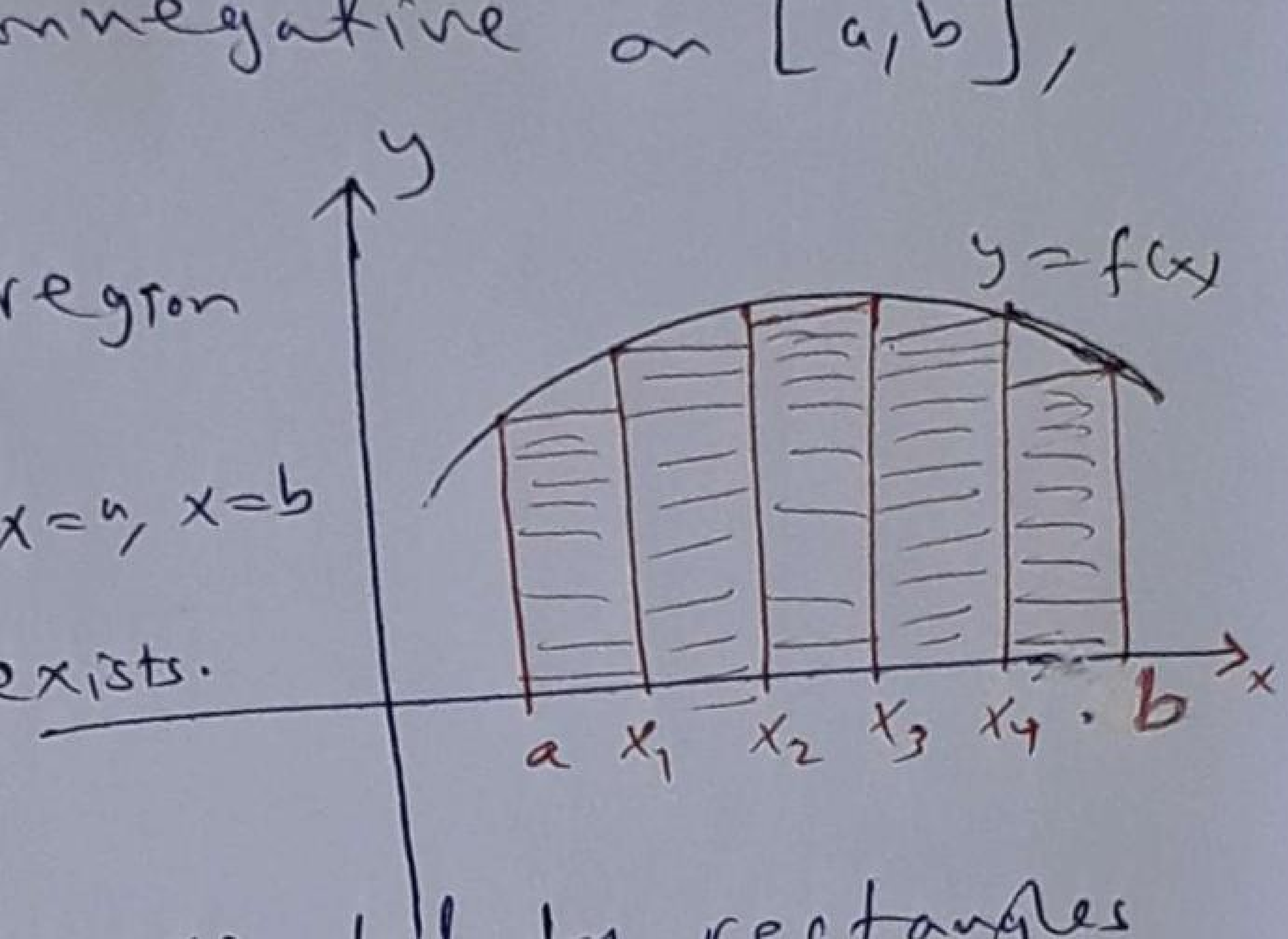
$P \cup Q$ is finer than both P and Q.

② If Q is a refinement of P (i.e., $Q \supseteq P$),

then $\|Q\| \le \|P\|$

Recall, let f be nonnegative on $[a,b]$,

$\int_a^b f(x)dx$ = Area of the region bounded by $y=f(x)$, $y=0$, $x=a$, $x=b$ when this integral exists.



this Area can be approximated by rectangles whose base lie in $[a,b]$ and whose heights approximate f. If the tops of these rectangles lie above $y=f(x)$, then $A_{approximate} > A_{exact}$. If the tops of these rectangles lie below $y=f(x)$, then $A_{approximate} < A_{exact}$.

Df (2). Let $a, b \in \mathbb{R}$ with $a<b$. Let $P = \{x_0, x_1, ---, x_n\}$ be a partition of $[a,b]$, set $\Delta x_j := x_j - x_{j-1}$, for $j = 1, 2, ---, n$ and suppose that $f: [a,b] \longrightarrow \mathbb{R}$ is bounded. The upper Riemann sum of f over P is

(i) the upper Riemann sum

$$\boxed{U(f,P) := \sum_{j=1}^{n} M_j(f)\Delta x_j}, \text{ where}$$

$$\boxed{M_j(f) = \sup_{x_{j-1} \le t \le x_j} f(t)}$$

(ii) the lower Riemann sum of f over P is $\boxed{L(f,P) = \sum_{j=1}^{n} m_j(f)\,\Delta x_j}$,

where $\boxed{m_j(f) = \inf_{x_{j-1} \le t \le x_j} f(t)}$

<u>Note</u>: Since f is <u>bounded</u>, then $M_j(f)$ and $m_j(f)$ exist and are finite.

<u>Rmk.</u> If $g : \mathbb{N} \longrightarrow \mathbb{R}$, then

$$\sum_{k=m}^{n} \big(g(k+1) - g(k)\big) = g(n+1) - g(m).$$

<u>Proof</u>. Use induction on n. (see the textbook).

by this Rmk., If $P = \{x_0, x_1, ---, x_n\}$ is a partition of $[a,b]$, then

$$\sum_{j=1}^{n} \Delta x_j = \sum_{j=1}^{n} x_j - x_{j-1}$$

$$= x_n - x_0 = b - a.$$

Rmk. If $f(x) = \alpha$ is constant on $[a,b]$, then $U(f,P) = L(f,P) = \alpha(b-a)$, for all partitions P of $[a,b]$.

Proof.

Since $M_j(f) = \sup\limits_{t\in[x_{j-1},x_j]} f(t) = \sup\limits_{t\in[x_{j-1},x_j]}(\alpha) = \alpha$,

then
$$U(f,P) = \sum_{j=1}^{n} M_j(f)\,\Delta x_j = \sum_{j=1}^{n} \alpha\,\Delta x_j = \alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a).$$

Similarly,
$$L(f,P) = \sum_{j=1}^{n} m_j(f)\,\Delta x_j = \sum_{j=1}^{n} \alpha\,\Delta x_j = \alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a)$$
$\square$

Rmk. $L(f,P) \le U(f,P)$ for all partitions P and all bounded functions f.

Proof. By definition, $m_j(f) \le M_j(f)$ for all j, then $L(f,P) \le U(f,P)$ $\square$

Rmk: If P is any partition of $[a,b]$. and Q is a refinement of P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Pf (see textbook).

Rmk. If P and Q are any partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.

proof: Since $P \cup Q$ is a refinement of P and Q, it follows from last remark that

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,Q)$$

for any pair of partitions P, Q whether Q is a refinement of P or not ∎

Df (3) Let $a,b \in \mathbb{R}$ with $a<b$. A function $f: [a,b] \to \mathbb{R}$ is said to be (Riemann) integrable on $[a,b]$ iff f is bounded on $[a,b]$ and $\forall \varepsilon > 0$, $\exists$ a partion P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

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Thm (1). Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a,b]$, $\exists$ a $\delta > 0$ such that

$$\boxed{|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad (1)}$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ which satisfies $\|P\| < \delta$. Fix an index j. Then, by Extreme value thm, there are points x_m and x_M in $[x_{j-1}, x_j]$ such that $f(x_m) = m_j(f)$ and $f(x_M) = M_j(f)$.

Since $\|P\| < \delta$, we also have $|x_M - x_m| < \delta$.

Hence by (1),

$$M_j(f) - m_j(f) = |M_j(f) - m_j(f)|$$
$$= |f(x_M) - f(x_m)| < \frac{\varepsilon}{b-a}.$$

In particular,

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} [M_j(f) - m_j(f)] \Delta x_j < \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \varepsilon$$

(8)

Ex. The Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$ is not Riemann integrable on $[0,1]$.

proof. clearly f is bounded on $[0,1]$.
The supremum of f over any nondegenerate interval is 1 and $\inf = 0$ (أفتح نقطة).
Therefore $U(f,P) - L(f,P) = 1 - 0 = 1$ for any partition P of $[0,1]$ (i.e., $\exists\, \varepsilon_0 = 1 > 0$ s.t for any partition P of $[0,1]$,
$U(f,P) - L(f,P) = \varepsilon_0 = 1$), that is,
f is not integrable on $[0,1]$ $\blacksquare$

Ex. Show that the function

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$ is integrable on $[0,1]$.

proof. let $\varepsilon > 0$, choose $0 < x_1 < \frac{1}{2} < x_2 < 1$
such that $x_2 - x_1 < \varepsilon$.

(9)

The set $P := \{0, x_1, x_2, 1\}$ is a partition of $[0,1]$. Since $m_1(f) = 0 = M_1(f)$,

$m_2(f) = 0 < 1 = M_2(f)$ and $m_3(f) = M_3(f) = 1$,

then $U(f,P) - L(f,P)$

$$= \sum_{j=1}^{3} M_j(f)\,\Delta x_j - \sum_{j=1}^{3} m_j(f)\,\Delta x_j$$

$$= \left(\cancelto{0}{M_1(f)\Delta x_1} + \cancelto{1}{M_2(f)}\,\Delta x_2 + \cancelto{1}{M_3(f)}\,\Delta x_3\right)$$

$$- \left(\cancelto{0}{m_1(f)\Delta x_1} + \cancelto{0}{m_2(f)}\,\Delta x_2 + \cancelto{1}{m_3(f)}\,\Delta x_3\right)$$

$$= \Delta x_2 + \Delta x_3 - \Delta x_3 = \Delta x_2 = x_2 - x_1 < \varepsilon$$

$$\Rightarrow U(f,P) - L(f,P) = x_2 - x_1 < \varepsilon.$$

therefore, f is integrable on $[0,1]$ ▤

Df ④. Let $a, b \in \mathbb{R}$ with $a<b$, and $f:[a,b] \to \mathbb{R}$ be bounded.

(i) the upper integral of f on $[a,b]$ is

$$(U)\int_a^b f(x)\,dx := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

(ii) the lower integral of f on $[a,b]$ is

$$(L)\int_a^b f(x)\,dx := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

(iii) If $(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx$, then

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

Rmk. ① we define the integral of any bounded function f on $[a,a]$ to be zero, i.e.,

$$\int_a^a f(x)\,dx := 0.$$

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(11)

(2) A bounded function might not be integrable

Ex. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ (Dirichlet function).

Rmk. If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded, then $(U)\int_a^b f(x)\,dx$ and $(L)\int_a^b f(x)\,dx$ exist and are finite, and satisfy

$$(L)\int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

Proof. We know $L(f,P) \le U(f,Q)$ for all partitions P and Q of $[a,b]$. Taking the sup over all partitions P of $[a,b]$, we have

$$\boxed{(L)\int_a^b f(x)\,dx \le U(f,Q)}, \text{ i.e.,}$$

$(L)\int_a^b f(x)\,dx$ exists and is finite.

Taking the inf over all partitions P of $[a,b]$, we conclude that $(U)\int_a^b f(x)\,dx$ is also finite and $(U)\int_a^b f(x)\,dx \geq (L)\int_a^b f(x)\,dx$. ▨

<u>Thm (2)</u> Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ be bounded. Then f is <u>integrable</u> on $[a,b]$

iff $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$

<u>Proof</u>: suppose that f is integrable. Let $\varepsilon > 0$ and choose a partion P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

By def'n, $(U)\int_a^b f(x)\,dx \leq U(f,P)$ and

$$(L)\int_a^b f(x)\,dx \geq L(f,P).$$ therefore,

$$\left|(U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx\right|$$

$$= (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx$$ since $(L)\int_a^b f \leq (U)\int_a^b f$

$$\leq U(f,P) - L(f,P) < \varepsilon.$$

(13)

Since $\left| (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \right| < \varepsilon, \ \forall \varepsilon > 0,$

this implies $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$

Conversely, suppose that $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$.

Let $\varepsilon > 0$ and choose, by the Approximation property, partitions P_1 and P_2 of $[a,b]$ s.t.

$$U(f,P_1) < (U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

and $$L(f,P_2) > (L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Since P is a refinement of P_1 and P_2, it follows that

$$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_2)$$

Given

$$\le \cancel{(U)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2} - \cancel{(L)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

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Thm (3). If $f(x) = \alpha$ is constant on $[a,b]$, then

$$\int_a^b f(x)\,dx = \alpha\,(b-a).$$

Proof. By thm 1, f is integrable on $[a,b]$ since it is continuous on $[a,b]$. Hence, it follows from Thm (2) and Rmk $\left(U(f,P) = L(f,P) = \alpha(b-a)\right)$ that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$$

$$= \inf\{U(f,P) : P \text{ is a partion of } [a,b]\}$$

$$= \inf\{\alpha(b-a) : \text{" " "}\}$$

$$= \alpha(b-a)$$

$\blacksquare$

H.w's (Exercise p.138)

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (All).

Chapter 5 Integrability on $\mathbb{R}$

5.1 The Riemann Integral.

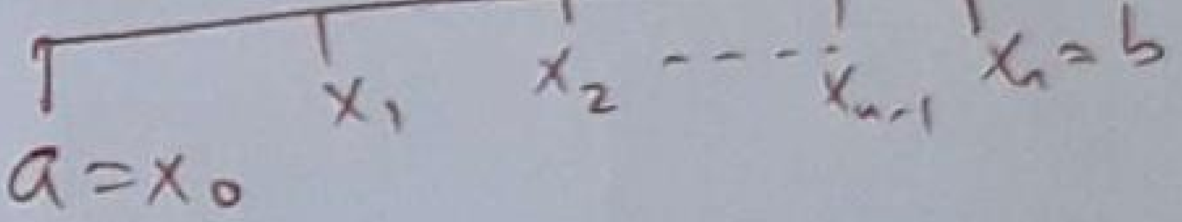
Df 1. Let $a, b \in \mathbb{R}$, with $a < b$.

(i) A *partition* of $[a,b]$ is a set of subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n],$$

where $a = x_0 < x_1 < \cdots < x_n = b$. $(*)$

Thus any set of $(n+1)$ points satisfying $(*)$ defines a partion P of $[a,b]$, which we denote by



$$P = \{x_0, x_1, \ldots, x_n\}.$$

(ii) the *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is the number

$$\|P\| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$$

$$= \max_{1 \le j \le n} |x_j - x_{j-1}|$$

(iii) A *refinement* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is a partition Q of $[a,b]$ which satisfies $Q \supseteq P$. And we say that Q is finer than P.

②

Ex. Prove that for each $n \in \mathbb{N}$,

$$P_n = \left\{ \frac{j}{2^n} : j = 0, 1, ---, 2^n \right\}$$ is a partition of $[0,1]$ and P_m is finer than P_n when $m > n$.

proof: Fix $n \in \mathbb{N}$. If $x_j = \frac{j}{2^n}$, then

$$0 = x_0 < x_1 < x_2 < ---- < x_{2^n} = 1.$$ Thus,

P_n is a partition of $[0,1]$.

Next, we need to show that $P_m \supseteq P_n$ when $m > n$.

Let $m > n$ and set $p = m - n$. If $0 \le j \le 2^n$,

then $\frac{j}{2^n} = \frac{j \cdot 2^p}{2^m}$ and $0 \le j 2^p \le 2^m$.

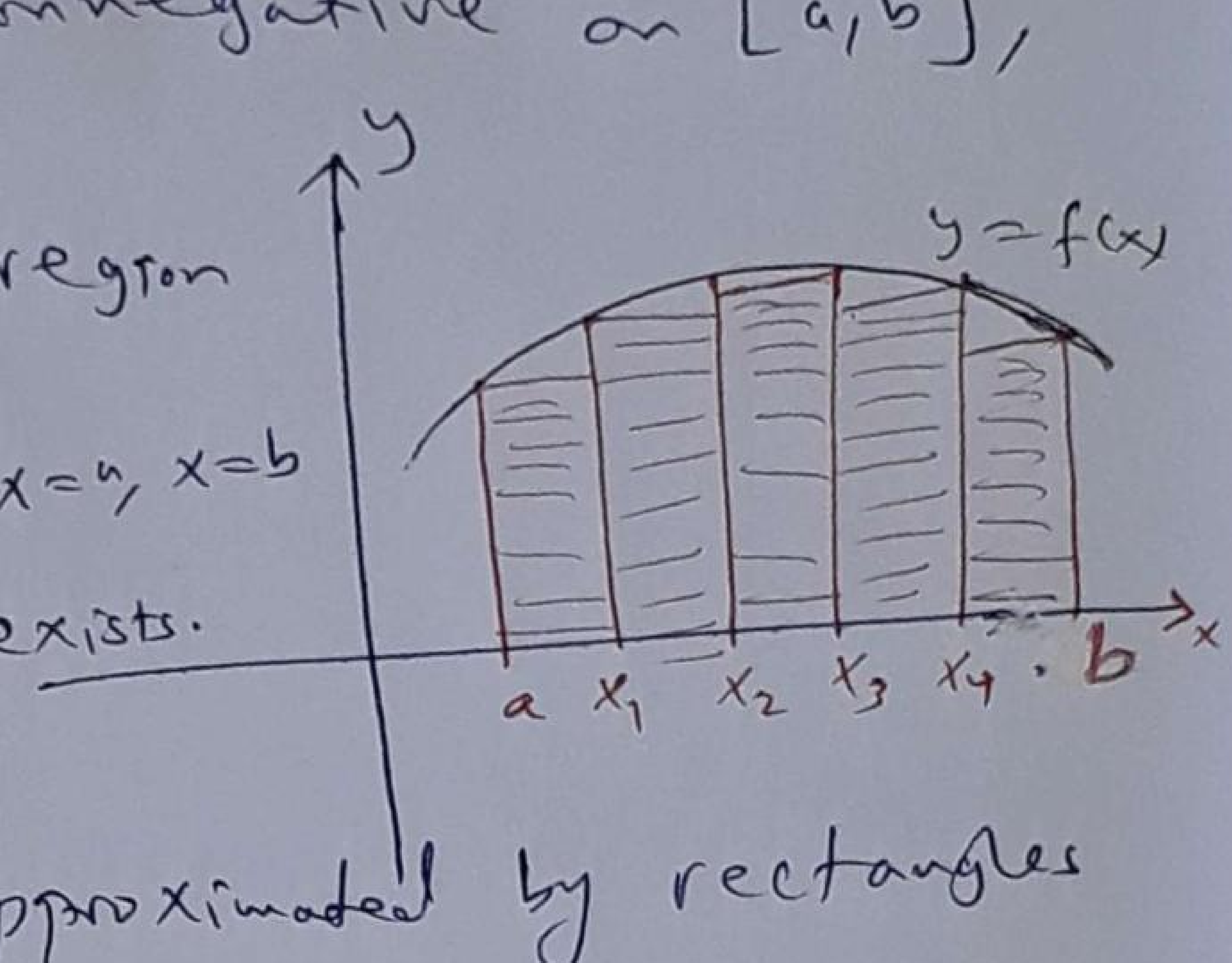
thus P_m is finer than P_n. ∎

Rmk ①. If P and Q are partitions of $[a,b]$, then $P \cup Q$ is finer than both P and Q.

② If Q is a refinement of P (i.e., $Q \supseteq P$), then $\|Q\| \le \|P\|$

Recall, let f be nonnegative on $[a,b]$,

$\int_a^b f(x)dx$ = Area of the region bounded by $y=f(x)$, $y=0$, $x=a$, $x=b$ when this integral exists.



this Area can be approximated by rectangles whose base lie in $[a,b]$ and whose heights approximate f. If the tops of these rectangles lie above $y=f(x)$, then $A_{approximate} > A_{exact}$. If the tops of these rectangles lie below $y=f(x)$, then $A_{approximate} < A_{exact}$.

Df (2). let $a,b \in \mathbb{R}$ with $a<b$. Let $P = \{x_0, x_1, ---, x_n\}$ be a partition of $[a,b]$, set $\Delta x_j := x_j - x_{j-1}$, for $j = 1, 2, ---, n$ and suppose that $f: [a,b] \longrightarrow \mathbb{R}$ is bounded. The upper Riemann sum of f over P is

(i) $\boxed{U(f,P) := \sum_{j=1}^{n} M_j(f)\, \Delta x_j}$, where

$$\boxed{M_j(f) = \sup_{x_{j-1} \le t \le x_j} f(t)}$$

(ii) the lower Riemann sum of f over P is $\boxed{L(f,P) = \sum_{j=1}^{n} m_j(f)\, \Delta x_j}$,

where $\boxed{m_j(f) = \inf_{x_{j-1} \le t \le x_j} f(t)}$

<u>Note:</u> Since f is <u>bounded</u>, then $M_j(f)$ and $m_j(f)$ exist and are finite.

<u>Rmk.</u> If $g : \mathbb{N} \longrightarrow \mathbb{R}$, then

$$\sum_{k=m}^{n} \big(g(k+1) - g(k)\big) = g(n+1) - g(m).$$

<u>Proof.</u> Use induction on n. (see the textbook). ☺

by this Rmk., If $P = \{x_0, x_1, ---, x_n\}$ is apartition of $[a,b]$, then

$$\sum_{j=1}^{n} \Delta x_j = \sum_{j=1}^{n} x_j - x_{j-1}$$
$$= x_n - x_0 = b - a.$$

Rmk. If $f(x) = \alpha$ is constant on $[a,b]$, then $U(f,P) = L(f,P) = \alpha(b-a)$, for all partitions P of $[a,b]$.

Proof.
Since $M_j(f) = \sup\limits_{t\in[x_{j-1},x_j]} f(t) = \sup\limits_{t\in[x_{j-1},x_j]}(\alpha) = \alpha$,

then
$$U(f,P) = \sum_{j=1}^{n} M_j(f)\,\Delta x_j = \sum_{j=1}^{n} \alpha\,\Delta x_j = \alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a).$$

Similarly,
$$L(f,P) = \sum_{j=1}^{n} m_j(f)\,\Delta x_j = \sum_{j=1}^{n}\alpha\,\Delta x_j = \alpha\sum_{j=1}^{n}\Delta x_j = \alpha(b-a)$$
$\blacksquare$

Rmk. $L(f,P) \le U(f,P)$ for all partitions P and all bounded functions f.

Proof. By definition, $m_j(f) \le M_j(f)$ for all j, then $L(f,P) \le U(f,P)$ $\blacksquare$

Rmk. If P is any partition of $[a,b]$ and Q is a refinement of P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Pf (see textbook).

Rmk. If P and Q are any partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$.

proof. Since $P \cup Q$ is a refinement of P and Q, it follows from last remark that

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P\cup Q) \leq U(f,Q)$$

for any pair of partitions P, Q whether Q is a refinement of P or not $\blacksquare$

Df (3) Let $a,b \in \mathbb{R}$ with $a<b$. A function $f: [a,b] \to \mathbb{R}$ is said to be (Riemann) integrable on $[a,b]$ iff f is bounded on $[a,b]$ and $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Thm ①. Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on $[a,b]$, $\exists$ a $\delta > 0$ such that

$$\boxed{|x-y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad (1)}$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ which satisfies $\|P\| < \delta$. Fix an index j. Then, by Extreme value thm, there are points x_m and x_M in $[x_{j-1}, x_j]$ such that $f(x_m) = m_j(f)$ and $f(x_M) = M_j(f)$.

Since $\|P\| < \delta$, we also have $|x_M - x_m| < \delta$.

Hence by (1),

$$M_j(f) - m_j(f) = |M_j(f) - m_j(f)|$$
$$= |f(x_M) - f(x_m)| < \frac{\varepsilon}{b-a}.$$

In particular,

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} [M_j(f) - m_j(f)] \Delta x_j < \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \varepsilon$$

Ex. The Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$ is not Riemann

integrable on $[0,1]$.

proof. clearly f is bounded on $[0,1]$.
The supremum of f over any nondegenrate (أقصى نقطة) .
interval is 1 and $\inf = 0$
therefore $U(f,P) - L(f,P) = 1 - 0 = 1$ for
any partition P of $[0,1]$ (i.e., $\exists\, \varepsilon_0 = 1 > 0$
s.t for any partion P of $[0,1]$,
$U(f,P) - L(f,P) = \varepsilon_0 = 1$), that is,
f is not integrable on $[0,1]$ $\blacksquare$

Ex. show that the function

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$$ is integrable on $[0,1]$.

proof. let $\varepsilon > 0$, choose $0 < x_1 < \frac{1}{2} < x_2 < 1$
such that $x_2 - x_1 < \varepsilon$.

9

The set $P := \{0, x_1, x_2, 1\}$ is a partition of $[0,1]$. Since $m_1(f) = 0 = M_1(f)$, $m_2(f) = 0 < 1 = M_2(f)$ and $m_3(f) = M_3(f) = 1$,

then $U(f,P) - L(f,P)$

$$= \sum_{j=1}^{3} M_j(f)\Delta x_j - \sum_{j=1}^{3} m_j(f)\Delta x_j$$

$$= \left(\cancelto{0}{M_1(f)\Delta x_1} + \cancelto{1}{M_2(f)}\Delta x_2 + \cancelto{1}{M_3(f)}\Delta x_3\right)$$

$$- \left(\cancelto{0}{m_1(f)\Delta x_1} + \cancelto{0}{m_2(f)}\Delta x_2 + \cancelto{1}{m_3(f)}\Delta x_3\right)$$

$$= \Delta x_2 + \Delta x_3 - \Delta x_3 = \Delta x_2 = x_2 - x_1 < \varepsilon$$

$$\Rightarrow U(f,P) - L(f,P) = x_2 - x_1 < \varepsilon .$$

therefore, f is integrable on $[0,1]$ ▤

Df ④. Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ be bounded.

(i) the upper integral of f on $[a,b]$ is

$$(U)\int_a^b f(x)\,dx := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

(ii) the lower integral of f on $[a,b]$ is

$$(L)\int_a^b f(x)\,dx := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$$

(iii) If $(U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx$, then

$$\int_a^b f(x)\,dx := (U)\int_a^b f(x)\,dx = (L)\int_a^b f(x)\,dx.$$

Rmk. ① we define the integral of any bounded function f on $[a,a]$ to be zero, i.e.,

$$\int_a^a f(x)\,dx := 0.$$

2. A bounded function might not be integrable

Ex. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ (Dirichlet function).

<u>Rmk.</u> If $f : [a,b] \longrightarrow \mathbb{R}$ is bounded, then $(U)\int_a^b f(x)dx$ and $(L)\int_a^b f(x)dx$ exist and are finite, and satisfy

$$(L)\int_a^b f(x)dx \le (U)\int_a^b f(x)dx.$$

<u>Proof.</u> we know $L(f,P) \le U(f,Q)$ for all partitions P and Q of $[a,b]$. Taking the sup over all partitions P of $[a,b]$, we have

$$\boxed{(L)\int_a^b f(x)dx \le U(f,Q)}, \text{ i.e.,}$$

$(L)\int_a^b f(x)dx$ exists and is finite.

Taking the inf over all partitions P of $[a,b]$,

(12)

we conclude that $(U)\int_a^b f(x)\,dx$ is also finite and $(U)\int_a^b f(x)\,dx \geq (L)\int_a^b f(x)\,dx$. ▨

<u>Thm (2)</u> Let $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to \mathbb{R}$ be bounded. Then f is <u>integrable</u> on $[a,b]$

iff $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$

<u>Proof</u>: suppose that f is integrable. Let $\varepsilon > 0$ and choose a partion P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

By def'n, $(U)\int_a^b f(x)\,dx \leq U(f,P)$ and

$$(L)\int_a^b f(x)\,dx \geq L(f,P).$$ therefore,

$$\left|(U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx\right|$$

$= (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx$ since $(L)\int_a^b f \leq (U)\int_a^b f$

$\leq U(f,P) - L(f,P) < \varepsilon.$

Since $\left| (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \right| < \varepsilon, \ \forall \varepsilon > 0,$

this implies $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$

Conversely, suppose that $(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx$.

Let $\varepsilon > 0$ and choose, by the Approximation property, partitions P_1 and P_2 of $[a,b]$ s.t.

$$U(f,P_1) < (U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

and $$L(f,P_2) > (L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

Set $P = P_1 \cup P_2$. Since P is a refinement of P_1 and P_2, it follows that

$$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_2)$$

Given

$$\le \cancel{(U)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2} - \cancel{(L)\int_a^b f(x)\,dx} + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \ \blacksquare$$

Thm (3). If $f(x) = \alpha$ is constant on $[a,b]$, then

$$\int_a^b f(x)\,dx = \alpha\,(b-a).$$

Proof. By thm 1, f is integrable on $[a,b]$ since it is continuous on $[a,b]$. Hence, it follows from Thm (2) and Rmk $\big(U(f,P) = L(f,P) = \alpha(b-a)\big)$ that

$$\begin{aligned}
\int_a^b f(x)\,dx &= (U)\int_a^b f(x)\,dx \\
&= \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \\
&= \inf\{\alpha(b-a) : \text{〃} \quad \text{〃} \quad \text{〃}\} \\
&= \alpha(b-a)
\end{aligned}$$

$\blacksquare$

H.w's (Exercise P.138)

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (All).

5.2 Riemann Sums

Df ①. Let $f : [a,b] \longrightarrow \mathbb{R}$.

(i) A Riemann sum of f with respect to a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is

$$\boxed{S(f, P, t_j) := \sum_{j=1}^{n} f(t_j)\,\Delta x_j}.$$

(ii) The Riemann sums of f are said to converge to $I(f)$ as $\|P\| \longrightarrow 0$ iff $\forall \varepsilon > 0$, $\exists$ a partition P_ε of $[a,b]$ such that

$$P = \{x_0, \dots, x_n\} \supseteq P_\varepsilon \Longrightarrow |S(f, P, t_j) - I(f)| < \varepsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, \dots, n$.

In this case we use the notation

$$I(f) = \lim_{\|P\| \to 0} S(f, P, t_j)$$

$$:= \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\,\Delta x_j.$$

Thm(1) Let $a, b \in \mathbb{R}$ with $a < b$ and spse that $f : [a,b] \to \mathbb{R}$. Then

f is Riemann integrable on $[a,b]$ iff

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\,\Delta x_j \quad \text{exists.}$$

in which case $I(f) = \int_a^b f(x)\,dx$.

Proof. Spse that f is integrable on $[a,b]$ and $\varepsilon > 0$. By the Approximation Property, there is a partition P_ε of $[a,b]$ such that

$$\boxed{L(f, P_\varepsilon) > \int_a^b f(x)\,dx - \varepsilon \quad \text{and} \quad U(f, P_\varepsilon) < \int_a^b f(x)\,dx + \varepsilon} \quad (1)$$

Let $P = \{x_0, x_1, \ldots, x_n\} \supseteq P_\varepsilon$. Then (1) holds with P in place of P_ε. But $m_j(f) \le f(t_j) \le M_j(f)$ for any choice of $t_j \in [x_{j-1}, x_j]$. Hence,

$$\int_a^b f(x)\,dx - \varepsilon < L(f,P) \le \sum_{j=1}^{n} f(t_j)\,\Delta x_j \le U(f,P) < \int_a^b f(x)\,dx + \varepsilon$$

i.e, $-\varepsilon < \sum_{j=1}^{n} f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx < \varepsilon$.

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We conclude that

$$\left| \sum_{j=1}^{n} f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx \right| < \varepsilon .$$

for all partitions $P \supseteq P_\varepsilon$ and all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$.

Conversely, Spse that the Riemann sums of f converge to $I(f)$. Let $\varepsilon > 0$ and choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that

$$\left| \sum_{j=1}^{n} f(t_j)\,\Delta x_j - I(f) \right| < \frac{\varepsilon}{3} \qquad (2)$$

for all choices of $t_j \in [x_{j-1}, x_j]$. Since f is bounded on $[a,b]$ (Exercise 11), use the Approximation property to choose t_j, $u_j \in [x_{j-1}, x_j]$ such that

$$f(t_j) - f(u_j) > M_j(f) - m_j(f) - \frac{\varepsilon}{3(b-a)}.$$

By (2) and telescoping, we have

(18)

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} \left(M_j(f) - m_j(f) \right) \Delta x_j$$

$$< \sum_{j=1}^{n} \left(f(t_j) - f(u_j) \right) \Delta x_j + \frac{\varepsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j$$

$$\leq \left| \sum_{j=1}^{n} f(t_j) \Delta x_j - I(f) \right|$$

$$+ \left| I(f) - \sum_{j=1}^{n} f(u_j) \Delta x_j \right|$$

$$+ \frac{\varepsilon}{3(b-a)} \sum_{j=1}^{n} \Delta x_j$$

$$< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3(b-a)} (b-a) = \varepsilon.$$

Therefore, f is integrable on $[a,b]$.

Thm (2). If f, g are integrable on $[a,b]$ and $\alpha \in \mathbb{R}$, then $f+g$ and αf are integrable on $[a,b]$. In fact,

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \qquad (3)$$

and

$$\int_a^b (\alpha f(x))\, dx = \alpha \int_a^b f(x)\, dx \qquad (4)$$

<u>Proof.</u> Let $\varepsilon > 0$ and choose P_ε such that for any partition $P = \{x_0, x_1, --, x_n\} \supseteq P_\varepsilon$ of $[a,b]$ and any choice of $t_j \in [x_{j-1}, x_j]$, we have

$$\left| \sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{2}$$

and $$\left| \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{2}.$$

By the Triangle inequality,

$$\left| \sum_{j=1}^{n} f(t_j)\Delta x_j + \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \right|$$

$$< \left| \sum_{j=1}^{n} f(t_j)\Delta x_j - \int_a^b f(x)\,dx \right| + \left| \sum_{j=1}^{n} g(t_j)\Delta x_j - \int_a^b g(x)\,dx \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for any choice of $t_j \in [x_{j-1}, x_j]$.

Hence (3) follows directly from thm ①.

To prove (4), we may assume that $\alpha \neq 0$.
choose P_ε such that if $P = \{x_0, \dots, x_n\}$
is finer than P_ε, then

$$\left| \sum_{j=1}^{n} f(t_j)\, \Delta x_j - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{|\alpha|},$$

for any choice of $t_j \in [x_{j-1}, x_j]$.
Multiply this inequality by $|\alpha|$, we obtain

$$\left| \sum_{j=1}^{n} \alpha f(t_j)\, \Delta x_j - \alpha \int_a^b f(x)\,dx \right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon,$$

for any choice of $t_j \in [x_{j-1}, x_j]$. We conclude
by Thm(1) that (4) holds. $\blacksquare$

<u>Thm ③</u>. If f is integrable on $[a,b]$, then f
is integrable on each subintervals $[c,d]$ of
$[a,b]$. Moreover,

$$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx} \quad (5)$$

proof. See the textbook p. 144.

Thm 4. If f, g are integrable on $[a,b]$ and $f(x) \le g(x)$, $\forall x \in [a,b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

In particular, if $m \le f(x) \le M$, $\forall x \in [a,b]$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

proof. Let P be a partition of $[a,b]$. By hypothesis, $M_j(f) \le M_j(g)$ whence $U(f,P) \le U(g,P)$.

It follows that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx \le U(f,P) \le U(g,P).$$

for all partitions P of $[a,b]$. Taking the infimum of this inequality over all partition P of $[a,b]$, we obtain

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

If $m \le f(x) \le M$, then (by what we proved)

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$$

$\blacksquare$

(22)

Thm (5). If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx .$$

Proof. let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$.

claim $M_j(|f|) - m_j(|f|) \leq M_j(f) - m_j(f)$ (6)

holds for $j = 1, 2, \dots, n$.

pf(claim). let $x, y \in [x_{j-1}, x_j]$. If $f(x), f(y)$ have the same sign, say both are nonnegative,

then $|f(x)| - |f(y)| = f(x) - f(y) \leq M_j(f) - m_j(f)$.

If $f(x), f(y)$ have opposite signs, say,
$f(x) \geq 0 \geq f(y)$, then $m_j(f) \leq 0$ and

hence $|f(x)| - |f(y)| = f(x) + f(y) \leq M_j(f) + 0$
$\leq M_j(f) - m_j(f)$.

Thus, in either case,

$$\boxed{|f(x)| \leq M_j(f) - m_j(f) + |f(y)|}$$

Taking the sup of this inequality for $x \in [x_{j-1}, x_j]$

$$\sup_{x \in [x_{j-1}, x_j]} |f(x)| \le M_j(f) - m_j(f) + |f(y)|$$

$$\Rightarrow \quad M_j(|f|) \le M_j(f) - m_j(f) + |f(y)|$$

Next, taking the inf as $y \in [x_{j-1}, x_j]$,

$$M_j(|f|) \le M_j(f) - m_j(f) + m_j(|f|)$$

$$\Rightarrow \quad M_j(|f|) - m_j(|f|) \le M_j(f) - m_j(f).$$

We see that (6) holds, as promised.

Let $\varepsilon > 0$ and choose a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Since (6) implies $U(|f|,P) - L(|f|,P) \le U(f,P) - L(f,P)$

it follows that

$$U(|f|,P) - L(|f|,P) < \varepsilon.$$

Thus, $|f|$ is integrable on $[a,b]$.

Since $-|f(x)| \le f(x) \le |f(x)|$ holds, $\forall x \in [a,b]$

we conclude by Thm 4 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx. \quad \blacksquare$$

Corollary. If f and g are (Riemann) integrable on $[a,b]$, then so is fg.

Proof: claim the square of any integrable function is integrable.

pf (claim). We need to prove that f^2 is integrable on $[a,b]$. Let P be a partition of $[a,b]$.

Since $M_j(f^2) = \left(M_j(|f|)\right)^2$ and $m_j(f^2) = \left(m_j(|f|)\right)^2$,

it is clear that

$$M_j(f^2) - m_j(f^2) = \left(M_j(|f|)\right)^2 - \left(m_j(|f|)\right)^2$$
$$= \left(M_j(|f|) + m_j(|f|)\right)\left(M_j(|f|) - m_j(|f|)\right)$$
$$\leq 2M\left(M_j(|f|) - m_j(|f|)\right)$$

where $M = \sup\limits_{x \in [a,b]} |f(x)|$, i.e, $|f(x)| \leq M$, $\forall x \in [a,b]$.

$$\Rightarrow \sum_{j=1}^{n}\left(M_j(f^2) - m_j(f^2)\right)\Delta x_j \leq 2M \sum_{j=1}^{n}\left(M_j(|f|) - m_j(|f|)\right)\Delta x_j$$

$$\Rightarrow U(f^2, P) - L(f^2, P) \leq 2M\left(U(|f|, P) - L(|f|, P)\right)$$
$$< 2M \cdot \frac{\varepsilon}{2M} \quad \text{since } f \text{ is integrable}$$
$$= \varepsilon. \Rightarrow f^2 \text{ is integrable on } [a,b]$$

Then, by claim, f^2, g^2, and $(f+g)^2$ are integrable on $[a,b]$. Since

$$fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}f^2 - \frac{1}{2}g^2$$

it follows from (thm ② p.4) that fg is integrable on $[a,b]$. $\blacksquare$

Thm ⑥. [First Mean value Thm for integrals]
Spse that f and g are integrable on $[a,b]$ with $g(x) \geq 0$, $\forall x \in [a,b]$. If

$$m = \inf_{x \in [a,b]} f(x), \quad M = \sup_{x \in [a,b]} f(x),$$

then there is a number $c \in [m, M]$ such that

$$\int_a^b f(x) g(x)\, dx = c \int_a^b g(x)\, dx.$$

In particular, if f is continuous on $[a,b]$, then there is an $x_0 \in [a,b]$ which satisfies

$$\int_a^b f(x) g(x)\, dx = f(x_0) \int_a^b g(x)\, dx.$$

Proof. Since $g \geq 0$ on $[a,b]$, then

$$m\,g(x) \leq f(x)\,g(x) \leq M\,g(x).$$

Thm (4) implies

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

If $\int_a^b g(x)\,dx = 0$, then $\int_a^b f(x)g(x)\,dx = 0$.
and there is nothing to prove.

If $\int_a^b g(x)\,dx \neq 0$, then we have

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

set $c = \dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$ and note that $c \in [m, M]$.

If f is continuous then by the intermediate value thm, $\exists\ x_0 \in [a,b]$ such that

$$f(x_0) = c$$

<u>Thm 7</u>. If f is (Riemann) integrable on $[a,b]$, then $F(x)=\int_a^x f(t)\,dt$ exists and is continuous on $[a,b]$.

<u>Proof</u>. (Exercise).

<u>Thm 8</u> [<u>Second mean value thm for integrals</u>]

Spse that f, g are integrable on $[a,b]$, that $g \geq 0$ on $[a,b]$, and that $m, M \in \mathbb{R}$ which satisfy $m \leq \inf_{x\in[a,b]} f(x)$ and $M \geq \sup_{x\in[a,b]} f(x)$.

Then $\exists$ an $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = m\int_a^c g(x)\,dx + M\int_c^b g(x)\,dx.$$

In particular, if f is also nonnegative on $[a,b]$, then $\exists$ an $c \in [a,b]$ which satisfies

$$\int_a^b f(x)g(x)\,dx = M\int_a^c g(x)\,dx.$$

<u>proof</u>. To prove the first statement, set

$$F(x) = m\int_a^x g(t)\,dt + M\int_x^b g(t)\,dt, \quad \forall x \in [a,b]$$

Observe that by Thm 7 that F is continuous on $[a,b]$. Since $g \geq 0$, we also have

$$m g(t) \leq f(t) g(t) \leq M g(t), \quad \forall t \in [a,b].$$

Hence it follows from thm (4) that

$$F(b) = m \int_a^b g(t)\,dt \leq \int_a^b f(t) g(t)\,dt \leq M \int_a^b g(t)\,dt = F(a).$$

Since F is continuous and $\int_a^b f(t) g(t)\,dt$ lies between $F(b)$ and $F(a)$, we conclude by the intermediate value thm that $\exists$ an $c \in [a,b]$ such that

$$F(c) = \int_a^b f(t) g(t)\,dt.$$

$$\left(\text{i.e.,} \quad m \int_a^c g(x)\,dx + M \int_c^b g(x)\,dx = \int_a^b f(t) g(t)\,dt \right).$$

The second statement follows from the first statement since we may use $m = 0$ when $f \geq 0$. That is $\exists$ an $c \in [a,b]$ s.t

$$\int_a^b f(x) g(x)\,dx = M \int_a^b g(x)\,dx$$ $\blacksquare$

H.W's (Exercises p. 150 (0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 11).

Birzeit University
Mathematics Department
Math3331
Quiz I&II

Instructor: Dr. Ala Talahmeh — **Time: 20 minutes**
Second Semester 2019/2020 — **Date: 05/03/2020**
Name:... — **Number:**...............

Exercise#1 [5 points].

a) Find all real numbers x that satisfy the inequality $x^2 > \frac{1}{x}$.

b) Show that

$$\max\{\alpha,\ \beta\} = \frac{\alpha + \beta + |\alpha - \beta|}{2},\ \ \forall\ \alpha, \beta \in \mathbb{R}.$$

Exercise#2 [5 points]. Let

$$E = \Big\{ r |\ r \text{ is a rational and } r^2 < 2 \Big\}.$$

Show that E has no rational supremum.

Exercise#3 [5+5 points]. Let A and B be bounded nonempty subsets of $\mathbb{R}$.

a) Show that $A \cup B$ is bounded.

b) Prove that $\sup\big(A \cup B\big) = \max\big\{\sup A,\ \sup B\big\}$.

Good Luck

Birzeit University
Mathematics Department
Math3331
H.W#2

Instructor: Dr. Ala Talahmeh **Second Semester 2019/2020**
Name:.. **Date: 06/04/2020**

Exercise#1 [10 points].

a. Use the definition of a limit to prove:
$$\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}.$$

b. Show that the sequence defined by
$$x_n = \frac{1}{3}\sin\left(n^3 - \frac{1}{n}\right) - 3\cos\left(\frac{1}{n} - n^3\right)$$
has a convergent subsequence.

Exercise#2 [10 points]. We say that a sequence $\{x_n\}$ of real numbers is **contractive** if $\exists$ a constant $C > 0,\ 0 < C < 1$, such that
$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|$$
for all $n \in \mathbb{N}$. Answer the following:

a. Show that every contractive sequence is convergent.

b. Let $\{x_n\}$ be a sequence defined by
$$x_1 > 0, \quad x_{n+1} = \frac{1}{2 + x_n} \text{ for } n \geq 1.$$
Show that $\{x_n\}$ is a contractive sequence. Find the limit.

Exercise#3 [10 points].

a. Show that the sequence $\{x_n\}$ defined by
$$x_n = \int_1^n \frac{\cos t}{t^2} dt$$
is Cauchy.

b. Let $0 < \beta < 1$ and x_1, x_2 be two real numbers such that $x_1 < x_2$ and
$$x_n = (1-\beta)x_{n-1} + \beta x_{n-2} \text{ for } n > 2.$$
Show that the sequence $\{x_n\}$ is convergent. What its limit?

Exercise#4 [10 points]. Let $\Big\{I_n = [a_n, b_n] : n \in \mathbb{N}\Big\}$ be a sequence of closed bounded intervals in $\mathbb{R}$, that is nested. If $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ and $\beta = \inf\{b_n : n \in \mathbb{N}\}$, show that

$$\bigcap_{n\geq 1}[a_n, b_n] = [\alpha, \beta].$$

Exercise#5 [15 points]. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_0 = \frac{3}{2}, \quad x_{n+1} = (x_n - 1)^2 + 1.$$

a. Prove that for each $n \in \mathbb{N},\ 1 < x_n < 2$.

b. Prove that the sequence $\{x_n\}$ is strictly monotone.

c. Deduce that $\{x_n\}$ is convergent and compute its limit.

Exercise#6 [15 points]. Let $\{x_n\}$ be a bounded sequence of real numbers. Let us define

$$y_n = \sup\{x_k : k \geq n\} \quad \text{and} \quad z_n = \inf\{x_k : k \geq n\}.$$

a. Show that the sequence $\{y_n\}$ is decreasing and $\{z_n\}$ is increasing.

b. Deduce that $\{y_n\}$ and $\{z_n\}$ are convergent sequences.

c. Prove that the sequence $\{x_n\}$ is convergent if and only if $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n$.

Good Luck

Birzeit University
Mathematics Department
Math3331
H.W#3 (Chapter 3)

Instructor: Dr. Ala Talahmeh **Second Semester 2019/2020**
Name:... **Date: 27/04/2020**

Exercise#1 [10 marks]. Let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{R}$ is called **Lipschitz** if there exists a constant $\alpha > 0$ such that
$$|f(x) - f(y)| \leq \alpha|x - y|,$$
for all $x, y \in E$.

a. Give two examples of Lipschitz functions.

b. Prove that every Lipschitz function is uniformly continuous.

c. Let $g : [0, 1] \to \mathbb{R}$, $g(x) = \sqrt{x}$. Prove that g is uniformly continuous but not Lipschitz.

Exercise#2 [5 marks]. Let $f : E \to \mathbb{R}$. Let $a \in E$ such that $\lim_{x\to a} f(x)$ exists. Show that $\lim_{x\to a} |f(x)|$ exists and the following identity holds:
$$\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)|.$$

Exercise#3 [5 marks]. Let $I := [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function on I such that for each $x \in I$ there exists $y \in I$ such that $|f(x)| > 2|f(y)|$. Prove there exists a point $c \in I$ such that $f(c) = 0$.

Exercise#4 [5 marks]. Using $(\varepsilon - \delta)$ definition of limit show that
$$\lim_{x\to -1} \frac{x + 5}{2x + 3} = 4.$$

Exercise#5 [10 marks].

a. Let a be a real number such that $a > 0$. Show that the function $f : [a, +\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is uniformly continuous.

b. Show that if f and $g : E \to \mathbb{R}$ are uniformly continuous and bounded, then fg is uniformly continuous.

Exercise#6 [10 marks]. Let a and b two real numbers such that $a < b$ and $f : [a, b] \to [a, b]$.

a. Suppose that for every $x, y \in [a, b] : |f(x) - f(y)| \leq |x - y|$. Show that f is continuous. Deduce that there exists $c \in [a, b]$ such that $f(c) = c$.

b. Suppose that for every x, y such that $x \neq y$ we have$|f(x) - f(y)| < |x - y|$. Show that there exists one and only one $c \in [a, b]$ such that $f(c) = c$.

Exercise#7 [15 marks]. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$.

a. Compute $f(0)$ and show that $f(-x) = -f(x)$.

b. Prove that for every $x \in \mathbb{R}$ and $n \in \mathbb{Z} : f(nx) = nf(x)$.

c. Prove that for every $x \in \mathbb{R}$ and q rational: $f(qx) = qf(x)$.

d. Prove that for every $x \in \mathbb{R}$ and λ real: $f(\lambda x) = \lambda f(x)$.

e. Find $f(x)$.

Good Luck

Birzeit University
Mathematics Department
Math3331
H.W#4 (Chapter 4)

Instructor: Dr. Ala Talahmeh **Second Semester 2019/2020**
Name:... **Date: 09/05/2020**

Exercise#1 [10 marks]. Let $f:\mathbb{R}\to\mathbb{R}$ be such that
$$|f(x)-f(y)|\leq|\sin x-\sin y|,$$
for all $x,y\in\mathbb{R}$.

a. Show that f is $2\pi-$periodic (A function f is $2\pi-$periodic if $\forall x\in\mathbb{R},\ f(x+2\pi)=f(x)$).

b. Show that f is continuous.

c. Show that f is differentiable at $\frac{\pi}{2}$ and compute $f'(\frac{\pi}{2})$.

Exercise#2 [5 marks]. Prove that for $x>1$,
$$\frac{x-1}{x}<\ln x<x-1.$$

Exercise#3 [5 marks]. Let $g:\mathbb{R}\to\mathbb{R}$ be a positive function such that $g(0)=1$ and $g(x+y)=g(x)g(y)$, $\forall x,y\in\mathbb{R}$. Show that if g is continuous at $x=0$, then g is continuous at every point of $\mathbb{R}$.

Exercise#4 [5 marks]. Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that f is differentiable on (a,b). Assume that $f(x)>0$, for every $x\in[a,b]$. Show that there exists $c\in(a,b)$ such that
$$\frac{f(b)}{f(c)}=\exp\Big((b-a)\frac{f'(c)}{f(c)}\Big).$$

Exercise#5 [5 marks]. Prove that if the function $f:I\to\mathbb{R}$ has a bounded derivative on I, then f is uniformly continuous on I. Is the converse true? Justify.

Exercise#6 [5 marks]. Show that the equation $e^x=1-x$ has exactly one solution in $\mathbb{R}$. Find this solution.

Exercise#7 [5 marks]. Let $f:\mathbb{R}\to\mathbb{R}$. Assume that for any $x,y\in\mathbb{R}$, we have
$$|f(x)-f(y)|\leq|x-y|^{1+\alpha},$$
where $\alpha>0$. Show that f is constant.

Exercise#8 [10 marks]. Let $f:[0,\infty)\to\mathbb{R}$ be differentiable everywhere. Assume that
$$\lim_{x\to\infty}\big(f(x)+f'(x)\big)=0.$$
Show that $\lim\limits_{x\to\infty}f(x)=0$.

Good Luck

Birzeit University
Mathematics Department
Math3331
HW#5

Instructor: Dr. Ala Talahmeh **Second Semester 2019/2020**
Name:.. **Date: 19/05/2020**

Exercise#1 [10 marks]. Is the function

$$f(x) = \begin{cases} 1, & x \neq 1 \\ 0, & x = 1. \end{cases}$$

integrable over the interval $[0, 2]$? Justify.

Exercise#2 [5 marks]. Show that if f is Riemann integrable on $[a, b]$, then it is bounded. What about the converse? Justify.

Exercise#3 [10 marks].

(a) Suppose that $a > 0$ and that f is Riemann integrable on $[-a, a]$. If f is even show that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(b) Let f be a continuous function on $[a, b]$. Show that there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a}\int_{a}^{b} f(x)dx.$$

Good Luck