Def If G is a group, $H \subset G$ is a subgroup of G

we define $a \equiv b \bmod H$ if $a = bh$ for some $h \in H$

(1) $a = a \cdot 1$, $(1 \in H)$

(2) $a = bh \Rightarrow ah^{-1} = b$ ($h^{-1} \in H$ since H is a group)

(3) $a = bh_1$, $b = ch_2$ then $a = ch_2h_1$, ($h_2h_1 \in H$ since H is a group.)

This is an equivalence relation and the equivalence classes are called cosets

Def let H be a subgroup of G, for every $a \in G$

$aH := \{ah, h \in G\}$ is called a left coset of H in G.

with respect to above equivalence relation we have $aH = [a]$

note $1H = H$

Remark let aH, bH be two left cosets of H in G

Then we have a bijective map $aH \to bH$

$ah \to bh$.

(well defined)

(one to one) $f(ah_1) = f(ah_2) \Rightarrow bh_1 = bh_2 \to h_1 = h_2 \Rightarrow ah_1 = ah_2$

(onto) if $bh \in bH$ then $f(ah) = bh$.

Thm So the order of the subgroup divides the order of the group.

Def Let G be a group, H a subgroup the set of cosets of H in G is denoted by G/H and called Quotient set of G by H.

$$G/H = \{ aH \mid a \in G \}$$

The index of H in G is $[G:H] := |G/H|$ if it's finite

Def Let $a \in G$, G group The order of a in G is

$$ord(a) = \begin{cases} \min\{n \in \mathbb{Z}_+ \mid a^n = 1\}, & \text{if this min is finite} \\ \infty, & \text{otherwise} \end{cases}$$

Corollary Let G be a finite group, $a \in G$ then the order of a divides the order of G

pf: $|\langle a \rangle| \mid |G|$ but $\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\}$ if $|\langle a \rangle| = m$ then we write $n = dm + r$, $0 \le r \le m-1$

$a^n = a^{dm+r} = a^r \Rightarrow \langle a \rangle = \{1, a, \dots a^{ord(a)-1}\}$

those numbers are different since if $a^{r_1} = a^{r_2}$ and $0 \le r_1 < r_2 < ord(a)$ then $a^{r_1 - r_2} = 1$ also $0 \le r_2 - r_1 < ord(a)$, by def of $ord(a)$

$r_2 - r_1 = 0$ so $r_2 = r_1$ so $ord(a) \mid |G|$

Corollary $a^{|G|} = 1$

proof $\text{ord}(a) \mid |G| \Rightarrow |G| = \text{ord}(a) \cdot d,\ d \in \mathbb{Z}_+$

$a^{|G|} = a^{\text{ord}(a)\, d} = \left(a^{\text{ord}(a)}\right)^d = (1)^d = 1$

Corollary G finite group, $|G| = p$ then the only subgroups of G are $\{1\}$ and G itself furthermore G is cyclic

If G is a group, H a subgroup of G and define $(aH).(bH) = abH$

This is a valid definition if it's independent of representitives, thus we need $bH = b'H$ and $aH = a'H \Rightarrow abH = a'b'H$

take $a = 1$, $a' = h \in H$, $b \in G$, $b' = b$ so we want $abH = a'b'H$ that is $bH = hbH$ so $\exists h' \in H$ s.t $bh' = hb$ i.e. $h' = b^{-1}hb$.

condition: $\forall h \in H, b \in G : bhb^{-1} \in H$.

so we have a chance of the definition being valid ☺

Def Let G be a group, H a subgroup of G

H is called a Normal subgroup of G if

$aha^{-1} \in H$, $\forall a \in G, \forall h \in H$.

Ex If G is abelian then all subgroups are normal

Ex $\{1\}$, G are normal subgroup of G (trivial)

Ex $G = S_3$, $H = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$, H is not normal

since $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \notin H$

proposition Let G be a group, H normal subgroup of G then G/H is group under the operation

$$(aH)(bH) = abH$$

Proof Let $a, a', b, b' \in G$ s.t $aH = a'H$, $bH = b'H$

i.e. $a = a'h$ for some $h \in H$, $b = b'h'$ for some $h' \in H$

so $ab = (a'h)(b'h') = a'b'\underbrace{(b')^{-1}hb'}_{h''}h'$

$= a'b'h''h'$ but since $h''h' \in H \Rightarrow abH = a'b'H$.

Associativity is inherited from G

Identity is H, inverse of aH is $a^{-1}H$. ∎

Def Let G, H be groups a homomorphism

$f: G \to H$ is a map with $f(ab) = f(a)\, f(b)$

Ex $f: (\mathbb{Z}, +) \to (\mathbb{Z}, +)$

$n \longrightarrow kn$, for $k \in \mathbb{Z}$

Remark Let $f: G \to H$ be a group homomorphism and e is the identity of G, e' is the identity of H then $f(e) = e'$, $f(a^{-1}) = f(a)^{-1}$, $\forall a \in G$

Remark if $f: G \to H$, $g: H \to L$ are group homomorphisms then $g \circ f: G \to L$ is a group homomorphism.

Def Let $f: G \to H$ be a group homomorphism the kernel of f is $\ker(f) = \{a \in G \mid f(a) = 1\} \subset G$

and $Im(f) = \{f(a) \mid a \in G\} \subset H$

Lemma (1) $Im(f)$ is a subgroup of H

(2) $\ker(f)$ is a normal subgroup of G

Note Normal subgroups are kernels of a homomorphism

Def An isomorphism is a bijective homomorphism

Remark If f is an isomarphism then f^{-1} is an isomorphism

Def G and H are isomorphic if there exist an isomorphism $f: G \to H$. Denoted by $G \cong H$

Remark The class of isomorphisms is an equivalence class. ☺

Note If $f: G \to H$ is a homomorphism then $Im(f)$ is a subgroup of H so we can think of f as a surjective group homomorphism $f: G \to Im(f)$.

Lemma If $f: G \to H$ is a group homomorphism then

f is injective $\iff \ker(f) = \{1\}$.

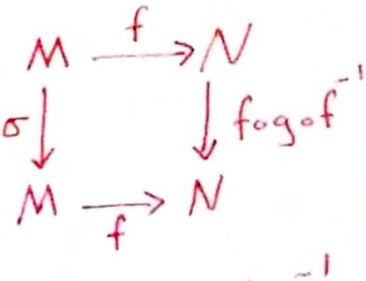
Ex If $f: M \to N$ is a bijection then

$g: S(M) \to S(N)$, $\sigma \to f \circ \sigma \circ f^{-1}$

is an isomorphism, since

$g(\sigma\tau) = f \circ \sigma \circ \tau \circ f^{-1} = f \circ \sigma \circ f^{-1} \circ f \circ \tau \circ f^{-1}$

$= g(\sigma)\, g(\tau)$



and it's inverse $g^{-1}: S(N) \to S(M)$ $\sigma \to f \circ \sigma^{-1} \circ f^{-1}$ is a function. If M finite $S(M) \cong S_{|M|}$

Lemma Let G be a group, N normal subgroup of G. Then the canonical projection (Natural map) $\pi: G \to G/N$, $g \to gN$ is a surjective group homomorphism with kernel N

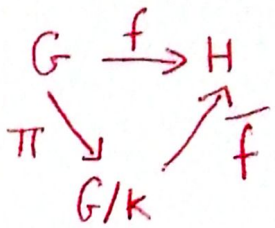
Thm There exist an isomorphism $\bar{f}: G/\ker(f) \to H$ such that $f = \bar{f} \circ \pi$ that is $H \cong G/\ker(f)$.

we want $f = \bar{f} \circ \pi$, $f(a) = \bar{f}(\pi(a)) = \bar{f}(aK)$

So define $\bar{f}(aK) := f(a)$

$aK = bK \Rightarrow a = bk, k \in K$ so

$f(a) = f(bk) = f(b)f(k) = f(b)$



so it's well defined and it's easy to check it's an isomorphism. ☺