4.1 Experiments and Counting Rules

* **Experiment** is an process that generates well-define outcomes

- **Sample Space (S)** for an experiment is the set of all experimental outcomes

Example: What is the sample space for the following experiments

[a] Toss a coin $S = \{Head, Tail\} = \{H, T\}$

[b] Roll a die $S = \{1, 2, 3, 4, 5, 6\}$

[c] Play a game $S = \{win, lose, tie\}$

* Experimental outcomes are also called **sample points**

Counting Rules

[1] Multiple-step experiment [2] Combinations [3] Permutations

[1] Multiple step experiment:

If an experiment has K steps with
n_1 possible outcomes for step 1,
n_2 " " " step 2,
⋮
n_K " " " step K

then the total number of experimental outcomes is $n_1 n_2 \cdots n_K$

Example: Consider the experiment of tossing two coins. How many experimental outcomes are possible for this experiment

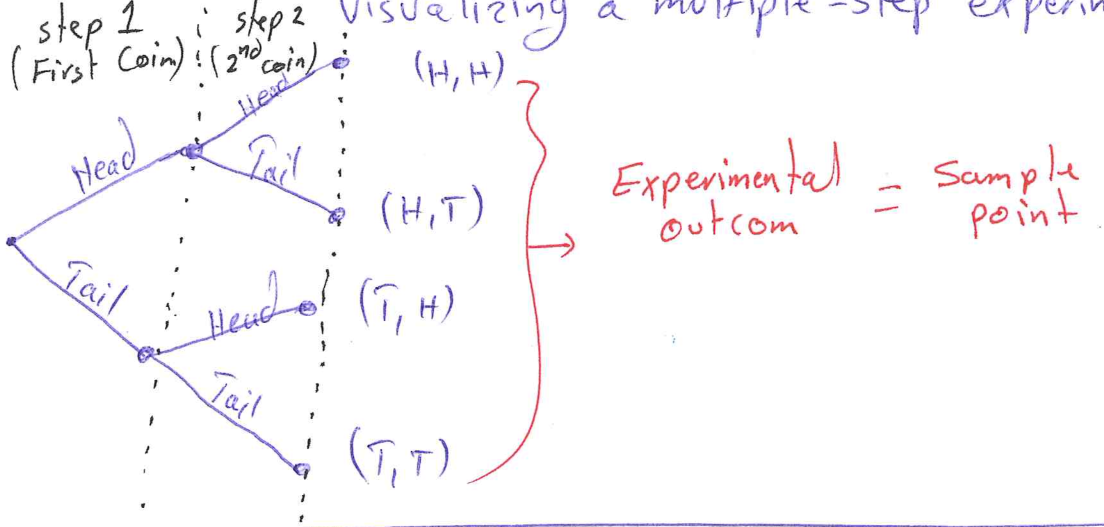
Tossing one coin $(n_1 = 2)$ and tossing the other coin $(n_2 = 2)$

The number of experimental outcomes $= (n_1)(n_2) = (2)(2) = 4$.

$S = \{(H, H), (H, T), (T, H), (T, T)\}$

If we toss 5 coins, then the number of experimental outcomes is $(2)(2)(2)(2)(2) = 32$.

* **Tree diagram:** A graphical representation that helps in visualizing a multiple-step experiment.



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2 Combinations (when the experiment involves slecting n objectis from a set of N objectis, $N \geq n$)

(The order is not important)

$$C_n^N = \binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

where

we use combination to find the number of different samples of size n that can be slected.

$N! = N(N-1)(N-2)\cdots(2)(1)$.

$n! = n(n-1)(n-2)\cdots(2)(1)$

$0! = 1$

$4! = 4\times3\times2\times1 = 24$ * see an example on the back?!

$5! = 5\times4\times3\times2\times1 = 120$

Example (Q2 page 150): How many ways can three items be slected from a group of six items? Use the letters A, B, C, D, E, F to identify them, and list each of the different combinations of the three item.

$C_3^6 =$

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6!}{3!\,3!}$$

$$= \frac{\cancel{6\times5\times4\times3\times2\times1}}{\cancel{3\times2\times1\times3\times2\times1}} = 20$$

ABC	ACD	ADF	BCF	CDE
ABD	ACE	AEF	BDE	CDF
ABE	ACF	BCD	BDF	CEF
ABF	ADE	BCE	BEF	DEF

Example: How many ways 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7, 9 which are even without repeating the digits

$$(2)(5)(4) = 40$$

Example: A hotel surveyed 100 guests with the following data:

	satisfied	unsatisfied
Female	42	2
Male	40	16

If two guests are randomly selected, what is the prob. that both are unsatisfied?

$$\frac{18}{100} \times \frac{17}{99} = 0.03 \quad \text{or} \quad \frac{\binom{18}{2}}{\binom{100}{2}} = 0.03$$

Example In how many ways can the letters of the word (Formula) be rearranged.

$$^{7}P_{7} = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 7! = 5040$$

46.2

3 Permutations: (when the experiment involves selecting n objects from a set of N objects $N \geq n$, where the order of selection is important.)

$$P_n^N = \frac{N!}{(N-n)!}$$

Example (Q3 page 150) How many permutations of three items can be selected from a group of six?

$$P_3^6 = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 120$$

ABC, ACB, BAC, BCA, CAB, CBA are different outcomes here

~~see the back for one more example:-~~

Assigning Probabilities

- Basic requirements for assigning probabilities:
  1. Each experimental outcome E_i must have $0 \leq P(E_i) \leq 1$
  2. Considering all experimental outcomes, we must have $P(E_1) + P(E_2) + \cdots + P(E_n) = 1$

- Three methods for assigining probabilites:

  1 Classical method: when all the experimental outcomes are equally likely. ($\frac{1}{n}$)

  Example ① Toss a fair coin $\Rightarrow S = \{H, T\}$ $n=2$

  $P(H) = P(T) = \frac{1}{2}$ "equally likely" ($\frac{1}{2}$)

  ② Roll a die $\Rightarrow S = \{1, 2, 3, 4, 5, 6\}$ $n=6$

  $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$ ($\frac{1}{6}$)

[2] Relative frequency method (when data are available to estimate the proportion of the time the experimental outcome will occure if the experiment is repeated a large number of items.

Example:

Number of patients wating	outcome Number of days
0	2
1	5
2	6
3	4
4	3
	20 days → number of sample points

2 days were 0 patients watinig
5 days wrer one " "
6 days were 2 " "
4 days were 3
3 days were 4 " "

Using the relative frequency method:

the probability of 0 patients were wating $= \frac{2}{20}$

" " " 1 " " " $= \frac{5}{20}$

" " " 2 " " " $= \frac{6}{20}$

" " " 3 " " " $= \frac{4}{20}$

" " " 4 " " " $= \frac{3}{20}$

[3] Subjective Method (when we can not assume that the experimental outcomes are equally likely

- This method express the person's degree of belief (scal 0-1)

Example: Suppose that student A and student B gave an excuse to there teach about their absance.

E_1: the excuse is accepted

E_2: the excuse is rejected

student A ⇒ $P(E_1) = 0.8$
⇒ $P(E_2) = 0.2$

student B ⇒ $P(E_1) = 0.6$ $P(E_2) = 0.4$

~~4.3~~ + 4.2 Events and their Probability

Event: is a collection of sample points.

Probability of an event: is the sum of the probabilities of the sample points in the event.

Example (Q14 page 154) An experiement has four equally likely outcomes E_1, E_2, E_3, E_4.

[a] What is the prob. that E_2 occures? $\frac{1}{4}$

[b] " " " " " any two of the outcomes occure? يعني i.e. (E_1 or E_3)

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

[c] " " " " " " three " " " " ? يعني (E_1 or E_2 or E_4)

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Example (Q21 page 156)

Assume a person is randomly choosen.

Age	Number (million of people)
≤ 19	80.5
20 – 24	19.0
25 – 34	39.9
35 – 44	45.2
45 – 54	37.7
55 – 64	24.3
≥ 65	35.0
	281.6 million

(a) what is the prob. the person is 20–24 years old?

$$\frac{19}{281.6} = 0.067$$

(b) What is the prob. the person is 20 – 34 years old?

$$\frac{19 + 39.9}{281.6} = \frac{58.9}{281.6} = 0.21$$

(c) what is the prob. the person is 45 years or older?

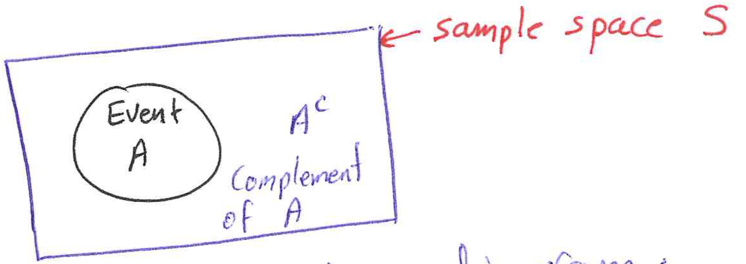
$$\frac{37.7 + 24.3 + 35}{281.6} = \frac{97}{281.6} = 0.34$$

4.3 Some Basic Relationships of Probability

Given an event A. The complement of A, denoted by A^c, consists of all sample points that are not in A.

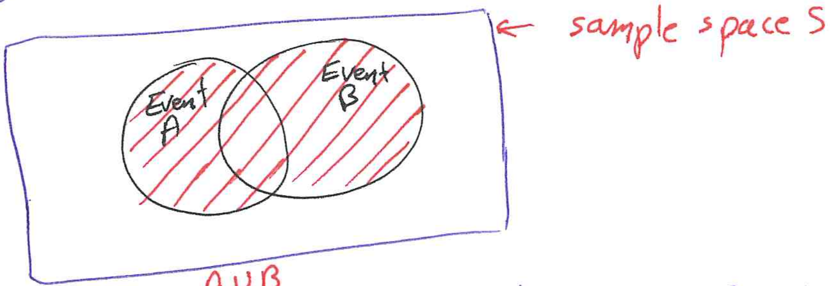
$P(S) = 1$

$P(A) + P(A^c) = 1$



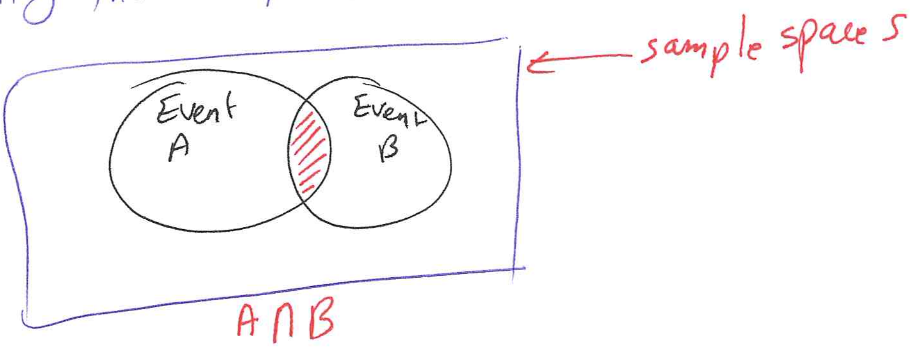
The figure above is called Venn diagram.

- Given the events A and B.
  - The union of A and B is the event containing all sample points belonging to A or B or Both. Denoted by $A \cup B$.



$A \cup B$

  - The intersection of A and B, denoted by $A \cap B$, is the event containing the sample points belonging to both A and B.



$A \cap B$

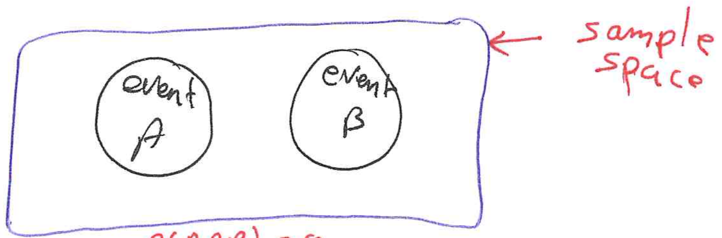
- Addition law:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

* Two events A and B are said to be متنافين mutually exclusive if the events A and B have no sample points in common ($P(A \cap B) = 0$).

* Adition law for mutually exclusive events:

$$P(A \cup B) = P(A) + P(B)$$



$P(A \cap B) = 0$

Example: (Q23 page 161) Suppose we have a sample space

$S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ where E_i are the sample points.

Given $P(E_1) = P(E_7) = 0.05$, $P(E_2) = P(E_3) = 0.20$

$P(E_4) = 0.25$, $P(E_5) = 0.15$, $P(E_6) = 0.10$.

Let $A = \{E_1, E_4, E_6\}$, $B = \{E_2, E_4, E_7\}$, $C = \{E_2, E_3, E_5, E_7\}$.

a) Find $P(A)$, $P(B)$, $P(C)$

$P(A) = P(E_1) + P(E_4) + P(E_6) = 0.05 + 0.25 + 0.1 = 0.4$

$P(B) = P(E_2) + P(E_4) + P(E_7) = 0.2 + 0.25 + 0.05 = 0.50$

$P(C) = P(E_2) + P(E_3) + P(E_5) + P(E_7) = 0.20 + 0.20 + 0.15 + 0.05 = 0.60$

b) Find $A \cup B$ and $P(A \cup B)$

$A \cup B = \{E_1, E_2, E_4, E_6, E_7\}$

$P(A \cup B) = P(E_1) + P(E_2) + P(E_4) + P(E_6) + P(E_7) = 0.05 + 0.20 + 0.25 + 0.10 + 0.05 = 0.65$

c) Find $A \cap B$ and $P(A \cap B)$

$A \cap B = \{E_4\}$, $P(A \cap B) = P(E_4) = 0.25$

d) Are the events A and C mutually exclusive?

Yes, they are mutually exclusive because $P(A \cap C) = 0$

e) Find B^c and $P(B^c)$

$B^c = \{E_1, E_3, E_5, E_6\} \Rightarrow P(B^c) = 1 - P(B) = 1 - 0.5 = 0.5$

4.4 Conditional Probabilities

Let A and B be two events:

- The conditional probability of A given B is

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

- The conditional probability of B given A is

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

- The events A and B are independent if

$$P(A\cap B) = P(A)\ P(B)$$

otherwise, they are dependent.

- If A and B are independent, then

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

$$P(B|A) = P(B)$$

- The events A and B are mutually exclusive if $P(A\cap B) = 0$

Example: suppose that we have two events A and B, with $P(A) = 0.2$, $P(B) = 0.4$, $P(A \cap B) = 0.08$

(a) Find $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.08}{0.4} = 0.2$

(b) Find $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.08}{0.2} = 0.4$

(c) Are A and B independent? why or why not?

Yes Because $P(A|B) = P(A)$
$P(B|A) = P(B)$

Example (Q32 page 168) Sample data representative of the national health insurance coverage are shown here:

		Health Insurance Yes	Health Insurance No	Total
Age	18-34	750	170	920
	35 and older	950	130	1080
Total		1700	300	2000

(a) Develop a joint probability table for these data

		Health insurance Yes	Health insurance No	Total
Age	18-34	$\frac{750}{2000} = 0.375$	0.085	0.46
	35 and older	0.475	0.065	0.54
Total		0.850	0.150	1.00

joint probabilities (arrow to the inner cells)

Marginal probabilities (arrow to the Total row and column)

(54)

(b) what do the mariginal probabilities tell you about the age of the US population?

46 % of the population are within age 18-34

54 % of the population are 35 and older.

(c) what is the probability that a randomly selected individual does not have health insurance?

0.15

(d) If the induvidual is between the ages 18-34, what is the probability that the individual does not have health insurance?

$$P(No \mid 18\text{-}34) = \frac{0.085}{0.46} = 0.1848$$

(e) If the individual age is 35 or older, what is the probability that the individual does not have health insurance.

$$P(No \mid 35 \text{ or older}) = \frac{0.065}{0.54} = 0.1204$$

(f) If the individual does not have health insurance, what is the prob. that the individual is in 18-34 age?

$$P(18\text{-}34 \mid No) = \frac{0.085}{0.150} = 0.567$$

(g) what does the prob. information tell you about the health insurance in United state?

From d and e, we see higher probability of No for 18-34