4.5 Bayes' Theorem

Often we begin probability analysis with initial or **prior probabilities** (PRIOR PROBABILITY The initial probability based on the present level of information)**.** Then, from a sample, special report, or a product test we obtain some additional information. Given this information, we calculate revised or **posterior probabilities (**POSTERIOR PROBABILITY a revised probability based on additional information).

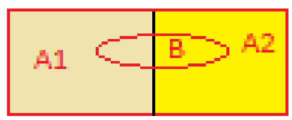
Bayes' theorem provides the means for revising the prior probabilities.

TWO-EVENT CASE

Let A_1, A_2 be two events of a sample space S such that:

$A_1 \cap A_2 = \phi, \; A_1 \cup A_2 = S$

Let B be an event of S, then



$$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) \end{aligned}$$

P (B) is called the total probability.

BAYES' THEOEREM: TWO-EVENT CASE

$$\begin{aligned} P(A_1 | B) &= \frac{P(A_1 \cap B)}{P(B)} \\ &= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} \end{aligned}$$

$$P(A_2 | B) = \frac{P(B|A_2)P(A_2)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}$$

39. The prior probabilities for events A_1 and A_2 are $P(A_1) = .40$ and $P(A_2) = .60$. It is also known that $P(A_1 \cap A_2) = 0$. Suppose $P(B \mid A_1) = .20$ and $P(B \mid A_2) = .05$.

a. Are A_1 and A_2 mutually exclusive? Explain.

b. Compute $P(A_1 \cap B)$ and $P(A_2 \cap B)$.

c. Compute $P(B)$.

d. Apply Bayes' theorem to compute $P(A_1 \mid B)$ and $P(A_2 \mid B)$.

SOLUTION

a. Yes,

$$P(A_1 \cap A_2) = 0$$

b. $P(A_1 \cap B) = P(B \mid A_1)P(A_1) = 0.08,\ P(A_2 \cap B) = P(B \mid A_2)P(A_2) = 0.03$

c. $P(B) = P(A_1 \cap B) + P(A_2 \cap B) = 0.08 + 0.03 = 0.11$

d.

$$P(A_1 \mid B) = \frac{P(B \mid A_1)P(A_1)}{P(B)} = \frac{0.08}{0.11} = 0.73$$

$$P(A_2 \mid B) = \frac{P(B \mid A_2)P(A_2)}{P(B)} = \frac{0.03}{0.11} = 0.27$$

EXAMPLE

Suppose that 30% of M&M Company supplies are from supplier A and 70% are from supplier B. Company A reports that 5% of its products are defective and company B reports that 4% of its products are defective.

SOLUTION

LET D = Defective, N = Not defective

P (A) = 0.3, P (B) = 0.7, P (D|A) = 0.05, P (D|B) = 0.04

P (N|A) =0.95, P (N|B) = 0.96

	A	**B**
%	*30*	*70*
D	*5%*	*4%*
N	*95%*	*96*

a. Find the probability that a product is supplied by company A and it is defective.
P (D and A) = P (D|A) P (A) = (0.05) (0.3) = 0.015.

b. Find the probability that a product is supplied by company A and it is not defective.
P (N and A) = P (N|A) P (A) = (0.95) (0.3) = 0.285.

c. Find the probability that a product is supplied by company B and it is defective.
P (D and B) = P (D|B) P (B) = (0.04) (0.7) = 0.028.

d. Find the probability that a product is supplied by company B and it is not defective.
P (N and B) = P (N|B) P (B) = (0.96) (0.7) = 0.672

e. What is the probability that a selected part is defective?
D = **D and from** A or **D and from B**.
P (D) = P (D|A) P (A) + P (D|B) P (B) = 0.015 + 0.028 = 0.043.

f. What is the probability that a selected part is not defective?
N = **N and from** A or **N and from B**.
P (N) = P (N|A) P (A) + P (N|B) P (B) = 0.285 + 0.672 = 0.957.
(P (N) = 1 – P (D) = 1 - 0.043 = 0.957)

g. If a product found to be defective what is the probability that it was from A? From B?

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D|A)P(A)}{P(D|A)P(A)+P(D|B)P(B)} = \frac{0.015}{0.043} = 0.35$$

$$P(B|D) = \frac{P(B \cap D)}{P(D)} = \frac{P(D|B)P(B)}{P(D|A)P(A)+P(D|B)P(B)} = \frac{0.028}{0.043} = 0.65$$

h. If a product found to be not defective what is the probability that it was from A? From B?
P (A|N) = 0.285/0.975 = 0.2923
P (B|N) = 0.672/0.957 = 0.7022