Chapter 5
Discrete Probability Distributions

5.1 – 5.3 Discrete Random Variables, Expected value, and Variance

Consider a random experiment with a sample space $S = \{s_1, s_2, ..., s_n\}$. A **random variable** is a rule (or a function) that assign a **numerical value** for each outcome of the experiment (describing the experiment outcome by a number).
A random variable is a **numerical description** of the outcome of an experiment.

Random variables are denoted by X, Y, R, …

Two types of random variables

Discrete Random Variable (DRV): A variable that may assume a finite number of values $x_1, x_2, ..., x_n$ or an infinite sequence of values $x_1, x_2, ..., x_n, ...$

Continuous Random Variable (CRV): A random variable that may assume a value in an interval or collection of intervals.

Discrete random Variables

Example

In the experiment of tossing a coin twice, the number of sample points is (2) (2) = 4, the sample space is;

S = {HH, HT, TH, TT}

Define the random variable **X** to be the number of heads observed, then

HH $\Rightarrow$2, HT $\Rightarrow$1, TH $\Rightarrow$1, TT $\Rightarrow$0.

X can assume the values 0, 1, and 2

X = {0, 1, 2}, X is a discrete random variable.

Example 4

In the experiment of rolling a die twice, the sample space is given by the following 36 ordered pairs;

$S = \{(1, 1), \ldots, (6, 6)\}$

- The outcomes are equally likely. That is, all outcomes have the same probability 1/36.

- Define the random variable **X** to be the **sum** of the two faces observed, then

$(1, 1) \Rightarrow 2, (1, 2) \Rightarrow 3, \ldots, (6, 6) \Rightarrow 12.$

X can assume the values: 2, 3, …, 12

$X = \{2, 3, \ldots, 12\}$, X is a discrete random variable

- Define the random variable **Y** to be the **absolute difference** of the two faces observed, then

$(1, 1) \Rightarrow 0, (1, 2) \Rightarrow 1, (1, 6) \Rightarrow 5.$

Y can assume the values 0, 1, …, 5

$Y = \{0, 1, \ldots, 5\}$, Y is a discrete random variable

Example

Three students are to be selected randomly from your class. You are interested in the gender (male or female) of the selected student. The sample space for the experiment is

S = {FFF, FFM, …, MMM} (chapter 4)

The outcomes are equally likely. That is, all outcomes have the same probability 1/8.

Define the random variable **X** to be the number of females, then

$X = 0, 1, 2, 3$

Now let us construct a table for values of X with the associated probability of occurring of each value

x	P(X = x)
0	1/8
1	3/8
2	3/8
3	1/8
Total	1

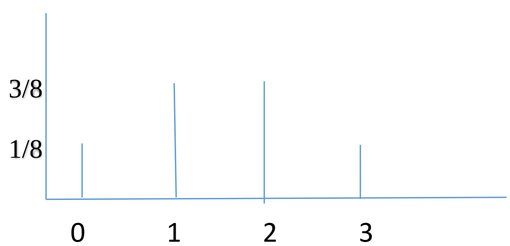
This table is called the **probability distribution** for the random variable X (it describes how probabilities are distributed over the random variable).

The function f(x) = (P(X = x) is called the **probability function.** The probability function satisfied the two requirements:

$$1. f(x) \geq 0$$

$$2. \sum f(x) = 1$$

The following is a graphical representation of the random variable X,



Example 6

Consider the following probability distribution for the random variable X

X	f(x)
-4	0.05
-2	0.15
-1	0.2
0	
3	0.35

1. Find f(0), that is, find P(X = 0))
   f (0) = 1 – (0.05 + 0.15 + 0.2 + 0.35) = 0.25
2. Find $P(X \leq 0)$

$$P(X \le 0) = f(0) + f(-1) + f(-2) + f(-4)$$
$$= 1 - P(X > 0) =$$
$$= 1 - f(3) = 1 - 0.35$$
$$= 0.65$$

3. Find $P(-4 < X \le 0)$

$$P(-4 < X \le 0) = f(-2) + f(-1) + f(0)$$
$$= 0.15 + 0.2 + 0.25 = 0.6$$

Expected value and Variance

Consider a discrete random variable $X = \{x_1, x_2, \ldots, x_n\}$ with probability function f(x) [$f(x) \ge 0, \sum f(x) = 1$], then

1. The **expected value** (the mean) of X, $E(X) = \mu$, is define by
$$E(X) = \mu = \sum x f(x)$$
2. The **variance of X**, $Var(X) = \sigma^2$ is define by
$$Var(X) = \sigma^2 = \sum (x - \mu)^2 f(x)$$
3. The standard deviation of X, $\sigma(X)$ is
$$\sigma(X) = \sqrt{Var(X)} = \sqrt{\sum (x - \mu)^2 f(x)}$$

Example 6

Find the expected value, the variance and the standard deviation for the random variable X = number of heads in tossing 3 coins.

X	P(X = x)	x*f(x)	$x - E(x)$	$(x - E(x))^2$	$(x - E(x))^2$ f(x)
0	1/8	0	-3/2	9/4	9/32
1	3/8	3/8	-1/2	1/4	3/32
2	3/8	6/8	½	1/4	3/32
3	1/8	3/8	3/2	9/4	9/32
Total		**12/8**	**0**		**24/32**

The expected value, $E(X) = \sum x f(x)$ =12/8 = 1.5

(The expected number of heads is 1.5).

The variance, $\text{Var}(X) = \sum (x-\mu)^2 f(x)$ =24/32 = 0.75

The standard deviation $\sigma = \sqrt{\text{Var}(x)} = \sqrt{0.75}$ = 0.8866

<u>Example 7</u>

A discrete random variable X has the following probability function:

$$f(x) = \frac{x-1}{10} \qquad x = 3, 4, 6$$

1. Is f(x) a valid probability function?
   f(3) = 2/10, f(4) = 3/10, f(6) = 5/10
   $\Rightarrow f(x) \geq 0, \ \sum f(x) = 0.2 + 0.3 + 0.5 = 1$
2. Find E(X)
   $E(X) = \sum xf(x) = (3)(0.2) + (4)(0.3) + (6)(0.5) = 4.8$
3. Find the variance of X
   $\text{Var}(X) = \sum (x-\mu)^2 f(x) =$
   $= (-1.8)^2(0.2) + (-0.8)^2(0.3) + (1.2)^2(0.5) = 1.56$

<u>Example</u>

Let X be a discrete random variable with probability function

$f(x) = \frac{1}{n}$, n = number of values of the random variable X.

- X is called a discrete uniform probability distribution
- For example, rolling a die, S= {1, 2, 3, 4, 5, 6}, f(x) = 1/6
- $f(x) = \frac{1}{n} > 0, \ \sum f(x) = n.\frac{1}{n} = 1$. So, f(x) is a valid probability function.

Example (Exercise17 page 197)

A volunteer ambulance service handles 0 t0 5 service calls on any given day. The probability distribution for the number of service calls is as follows.

Number of service Calls	Probability
0	0.1
1	0.15
2	0.30
3	0.20
4	0.15
5	0.10

1. What is the expected number of service calls?
2. What is the variance number of service calls? What is the standard deviation?

Example 10

The following distribution shows the number of smart phones soled per day on ABC store over a period of 250 days.

Number of units sold X	Number of days frequency	f(x)
0	40	
1	70	
2	60	
3	50	
4	20	
5	10	
TOTAL		

P(x)

1. What is the probability that the store will not sell any phone?
2. What is the probability that the store will sell at least one phone?
3. What is the expected number of phones sold per day? What is the standard deviation? → σ
4. If the profit on each phone is \$25, what is the expected profit per day?

$25 \times 40/250$

$25 \times 70/250$

$\vdots$

or

$25 \times 1{,}88 = 47$

$E(x) = 1{,}88$

5.4 The Binomial Probability Distribution

The **binomial probability distribution** is discrete probability distribution used when there are **exactly two mutually exclusive outcomes** of a trial. These outcomes are called "**success**" and "**failure**".

The **binomial experiment** is an experiment satisfies the following four properties;

1. The experiment consists of a sequence of n identical trials.
2. Two outcomes, success and failure, are possible on each trial.
3. The probability of success p, and the probability of failure 1 – p do not change from trial to trial.
4. The trials are independent.

Example

1. Tossing a coin 15 times. Head means success.
2. Rolling a die 4 times, we are interested on the face shown up to be 6 (probability of success is 1/6 and probability of failure is 5/6).
3. Insurance salesperson visits 10 randomly selected families. A success means a family purchases an insurance policy (the salesperson knows the probability/ past experience).
4. Select three item for inspection from a production line. Defective means success (probability of defective is known).

The binomial distribution is used to obtain the probability of observing ***x* successes in *n* trials**, with the probability of success on a single trial denoted by *p*. The binomial distribution assumes that *p* is fixed for all trials.

Let us consider a binomial experiment with **n** trials and probability of success **p**, define the random variable X to be the number of successes in the n trials.
The random variable X can assume the values 0, 1, 2, …, n. That is,

$$X = \{0, 1, 2, \dots, n\}.$$

X is a discrete random variable and is called a **binomial random variable**, it is denoted by

$$\mathbf{X = B\ (n,\ p).}$$

For example, toss a coin 10 times and define X to be the number of heads observed, X is a binomial random variable with 10 trials and probability of success 0.5. That is X = B (10, 0.5)

Assume that **X = B (n, p),** then

- The probability function for X is given by the following formula
  $f(x) = (nCx)(p)^{x}(1-p)^{n-x}$

Where, f(x) probability of x success in n trials

$nCx = \dfrac{n!}{(n-x)!x!}$,

x = number of successes, x = 0, 1, 2, …. , n

p = P (success), 1 – p = P (failure)

- The expected value of X , E(X), is given by

$E(X) = np$

- The Variance of X, Var (x), is given

$Var\ (X) = np(1-p)$

The standard deviation is

$\sigma(X) = \sqrt{np(1-p)}$

<u>Example</u>

A STAT 2361 test has 12 multiple choice questions with four choices and one correct answer each. If you just randomly guess on each of the 12 questions.

1. Define the random variable X to be the number of correct answers. How X is distributed?

X is a binomial random variable with n = 12 and probability of success equals ¼. That is,

$X = B\ (12, 0.25)\ x = 0, 1, \ldots, 12$

2. Find the expected number of correct answers. Find the variance.

$E(X) = np = (12)\ (0.25) = 3$

$Var\ (X) = np\ (1 - p) = (12)\ (0.25)\ (0.75) = 2.25$

3. What is the probability that you get exactly 3 correct questions?

$$f(3) = (12C3)(0.25)^3 (0.75)^9$$
$$= 0.2581$$

4. What is the probability that you get no more than 2 correct questions?

$$P(X \leq 2) = f(2) + f(1) + f(0)$$
$$= 0.2323 + 0.1267 + 0.0317$$
$$= 0.3907$$

5. What is the probability that you get at least 10 false questions?

$P(X \geq 10)$, x = B (12, 0.75)

$$P(X \geq 10) = f(12) + f(11) + f(10)$$
$$= 0.0317 + 0.1267 + 0.2323$$
$$= 0.3907$$

Example

According to a survey 70% of BZU students attend the online lectures (Corona Time).

1. In a randomly selected sample of 1000 BZU students, what is the expected number of students who attend the online lecture? What is the standard deviation?

$$X = B\,(1000,\ 0.7)$$
$$E(X) = np = 700$$
$$\sigma(X) = \sqrt{np(1-p)} = 14.49$$

2. In a randomly selected sample of 400 BZU students, what is the expected number of students who do not attend the online lecture? What is the standard deviation?

$$X = B\,(400,\ 0.3)$$
$$E(X) = np = 120$$
$$\sigma(X) = \sqrt{np(1-p)} = 9.17$$

3. In a randomly selected sample of 10 BZU students, what is the probability that exactly 5 students attend the online lecture?

$$X = B\,(10,\ 0.7)$$
$$f(6) = 10C6(0.7)^6(0.3)^4$$
$$= 0.2$$

4. In a randomly selected sample of 8 BZU students, what is the probability that at least 3 students attend the online lecture?

$$X = B\,(8,\ 0.7)$$
$$P(X \geq 3) = 1 - P(X < 3)$$
$$= 1 - [f(2) + f(1) + f(0)]$$
$$=$$

5. In a randomly selected sample of 12 BZU students, what is the probability that at most 10 students do not attend the online lecture?

$$X = B\,(12,\ 0.3)$$
$$P(X \leq 10) = 1 - P(X > 10)$$
$$= 1 - [f(12) + f(11)]$$
$$=$$

single value f(x)

5.5 The Poisson probability distribution

The ***Poisson distribution***, named after French mathematician Siméon Denis ***Poisson***, is a discrete probability ***distribution*** that expresses **the probability of a given number of events occurring in a fixed interval of time, space, area, …..**

Examples

- The number of arrivals at a car wash in one hour,
- The number of customers using Arab Bank ATM per day.
- The number of telemarketing phone calls received by a household during a given day.
- The number of defective items in the next 100 items manufactured on a machine.

PROPERTIES Of A POISSON EXPERIMENT

- The probability of an occurrence is the same for any two intervals of equal length.
- The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.
- Independence of occurrences means that one occurrence (or nonoccurrence) of an event does not influence the successive occurrences or nonoccurrence of that event.

POISSON PROBABILITY fUNCTION

$$f(x) = \frac{\mu^x e^{-\mu}}{x!} \qquad x = 0, 1, 2, \ldots..$$

Where: f(x) = the probability of x occurrences in an interval

μ = expected value or mean number of occurrences.

e = 2.71828

NOTE: A property of the Poisson distribution is that the mean and variance are equal.

Example

Consider a Poisson distribution with a mean of two occurrences per time period.

1. Write the appropriate Poisson probability function.

   $$f(x) = \frac{2^x e^{-2}}{x!}$$

2. What is the expected number of occurrences in three time periods?

   $\mu = (3)(2) = 6$ Occurrences

3. Write the appropriate Poisson probability function of x occurrences in three time periods.

   $$f(x) = \frac{6^x e^{-6}}{x!}$$

4. Compute the probability of two occurrences in one time period.

   $$f(2) \text{ with } \mu = 2 \Rightarrow f(2) = \frac{2^2 e^{-2}}{2!} = 0.2707$$

5. Compute the probability of six occurrences in three time periods.

   $$f(6) \text{ with } \mu = 6 \Rightarrow f(6) = \frac{6^6 e^{-6}}{6!} = 0.1606$$

6. Compute the probability of five occurrences in two time periods.

   $$f(5) \text{ with } \mu = 2.2 = 4 \Rightarrow f(5) = \frac{4^5 e^{-4}}{5!}$$

7. Compute the probability of at least two occurrences in one time period.

   $$\begin{aligned} \mu = 2.2 = 4, \quad P(X \geq 2) &= f(2) + f(3) + \ldots.. \\ &= 1 - P(X < 2) \\ &= 1 - [f(0) + f(1)] \end{aligned}$$

<u>Example</u>

Phone calls arrive at the rate of 48 per hour at the reservation desk for Regional Airways.

1. Compute the probability of receiving three calls in a 5-minute interval of time.

   $$f(3) \text{ with } \mu = \frac{5 \times 48}{60} = 4 \Rightarrow f(3) = \frac{4^3 e^{-4}}{3!}$$

2. Compute the probability of receiving exactly 10 calls in 15 minutes.

   $$f(10) \text{ with } \mu = \frac{15 \times 48}{60} = 12 \Rightarrow f(10) = \frac{12^{10} e^{-12}}{10!}$$

3. Suppose no calls are currently on hold. If the agent takes 5 minutes to complete the current call, how many callers do you expect to be waiting by that time? What is the probability that none will be waiting?

   $$f(0) \text{ with } \mu 4 \Rightarrow f(0) = \frac{4^0 e^{-4}}{0!}$$

4. If no calls are currently being processed, what is the probability that the agent can take 3 minutes for personal time without being interrupted by a call?

   $$f(0) \text{ with } \mu = \frac{3 \times 48}{60} = 2.4 \Rightarrow f(0) = \frac{2.4^0 e^{-2.4}}{0!}$$

$$f(0) \text{ with } \mu = \frac{3 \times 48}{60} = 2.4 \Rightarrow f(0) = \frac{2.4^0 e^{-2.4}}{0!}$$