CHAPTER 6
Continuous Random Variable

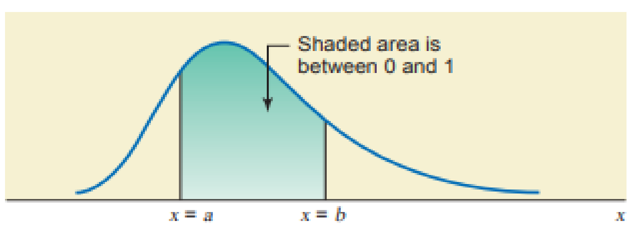
- A continuous random variable X is a random variable that assume any value over an **interval or union of intervals**.

- The probability that X assumes a value in any interval lies in the range 0 to 1. The total probability of all the (mutually exclusive) intervals within which x can assume a value is 1.

- The probability that a continuous random variable X assumes a value within a certain interval is given by the area under the curve between the two limits of the interval.

- For a continuous probability distribution, the probability is always calculated for an interval. The probability that a continuous random variable x assumes a single value is always zero.

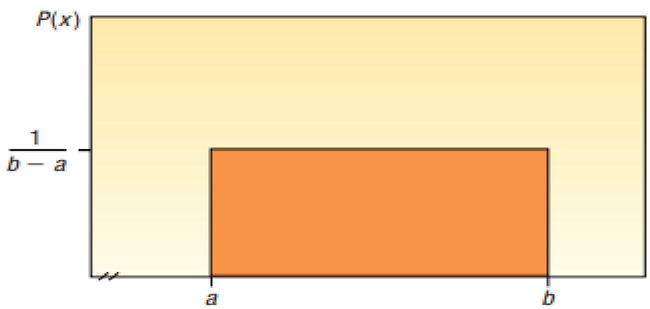
- $P(a \leq X \leq b) = P(a < X < b)$ = **Area under the curve** from x = a to x= b.



6.1 The Uniform Distribution

A **uniform random variable** X over an interval [a, b], denoted by X = U [a, b] is a continuous random variable with the following probability (density) function:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



Let X = U [a, b], then

- $P(x_1 < X < x_2) = \dfrac{1}{b-a}.(x_2 - x_1) = \dfrac{x_2 - x_1}{b-a}$
- $E(X) = \dfrac{a+b}{2}$
- $Var(X) = \dfrac{(b-a)^2}{12}$
- $\sigma(X) = \dfrac{b-a}{\sqrt{12}}$

Example

The time needed to finish STAT 2361 final exam is uniformly distributed between 60 and 100 minutes

1. Write the probability function for this distribution.
   X = U [60, 100]

$$f(x) = \begin{cases} \dfrac{1}{40} & 60 \le x \le 100 \\ 0 & \text{elsewhere} \end{cases}$$

2. What is the mean time for this exam? What is the standard deviation?

$$E(X) = \frac{a+b}{2} = \frac{60+100}{2} = 80$$

$$\sigma(X) = \frac{b-a}{\sqrt{12}} = \frac{100-60}{\sqrt{12}} 11.55$$

3. Find $P(70 < X < 85)$

$$P(70 < X < 85) = \frac{85-70}{40} = 0.375$$

4. Find $P(X > 90)$

$$P(X > 90) = \frac{100-90}{40} = 0.25$$

5. Find $P(50 < X < 65)$

$$P(50 < X < 65) = \frac{65-50}{40} = 0.125$$

6.2 Normal Distribution

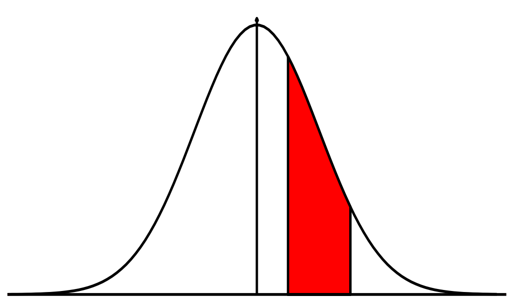
Data obtained from many experiments and studies often follow a common distribution called a **normal distribution**. The normal distribution is regarded as the most important and most widely used of all distributions.

Properties of a Normal Distribution

1. The normal distribution is a **bell-shaped distribution.**
2. The normal curve is symmetrical about the mean (The mean is at the middle and divides the area into two halves)
3. The total area under the curve is equal to 1.
4. The distribution is completely described by its mean μ and standard deviation σ.
5. If X is a normal random variable with mean μ and standard deviation σ, then X is denoted by $X = N(\mu, \sigma)$
6. The probability density function for $X = N(\mu, \sigma)$ is given by

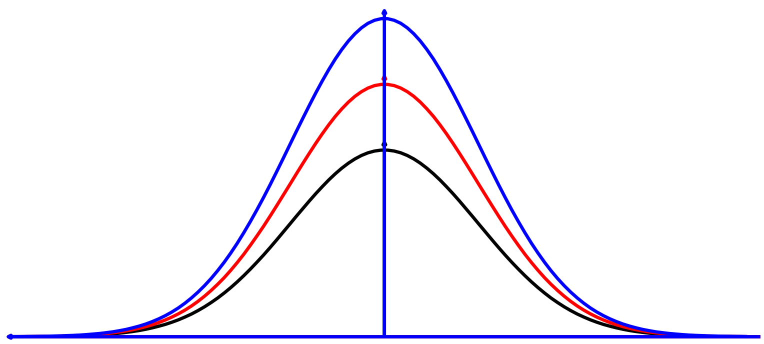
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2 / 2\sigma^2}$$

7. $P(a \le X \le b$ ($P(a < X < b$) is the area from x = a to x = b under the normal curve.



P (a < x < b)

8. The **standard deviation** of a normal curve gives the spread of the curve. As the standard deviation increases the spread of the curve increases. The following graph shows three normal curves with the same mean and different standard deviations.



9. Recall : The Empirical Rule:

- **68%** of data values are within **1 standard deviation** of the mean (-1 to +1).
- **95%** of data values are within **2 standard deviations** of the mean (-2 to +2).
- **99.7%** of data values are within **3 standard deviations** of the mean (-3 to +3).

10. Standard Normal Curve

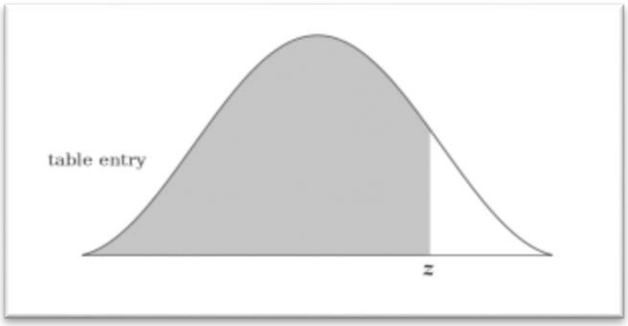
A normal distribution curve is determined by two factors, the mean μ and the standard deviation σ. That is, normal distributions with different means or different standard deviations give different normal curves. In such cases it is difficult to calculate the areas under the curves, or even to construct separate tables for areas under a normal curve for every value of μ and of σ. Fortunately, we are able to transform all normal distributions into one distribution called **standard normal distribution.**

- The **standard normal** curve is a normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$, **N (0, 1).**
- To transform from a normal curve into a standard normal curve we use the standardized value (z-score). The **z-score** is given by:

$$z = \frac{x - \mu}{\sigma}$$

- To find the area under a standard normal curve we use the standard normal distribution table, also called the **Z table**.

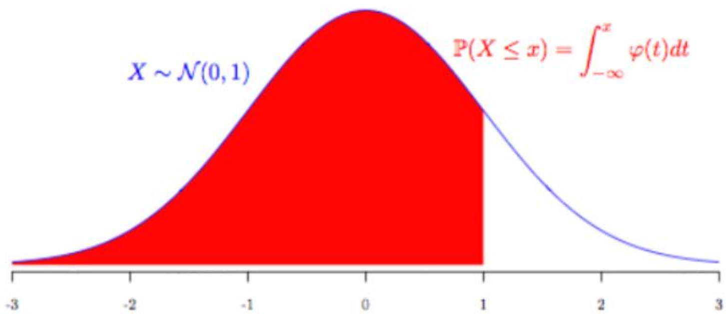
- The Z table lists the areas under the standard normal curve to the left of z that is, $P(Z < z)$ for $z \geq 0$.



- We use the Z- table to find values on the right of the mean ($\mu = 0$).

- Table entries for z represent the area under the bell curve to the left of z. (it is at least 0.5)

- To find areas to the right of z and to the left (right) negative value(s) we will use the **symmetric** of the curve and the fact that the **total area** under the curve is 1.



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

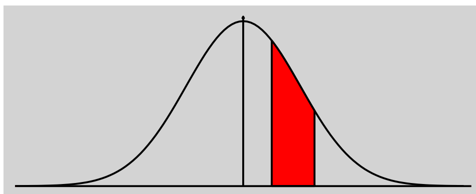
Example

Find the area under the standard normal curve for each of the following cases.

(a) $P(Z < 1.25)$, $P(Z < 1)$

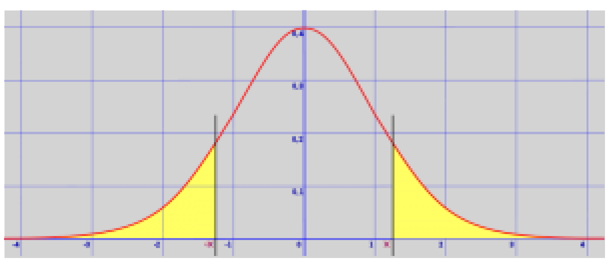
From the table directly: $P(Z < 1.25) = 0.8994$, $P(Z < 1) = 0.8413$.

(b) $P(1 < Z < 1.25)$



$$P(1 < Z < 1.25) = P(Z < 1.25) - P(Z < 1)$$
$$= 0.8994 - 0.8413$$
$$= 0.0581$$

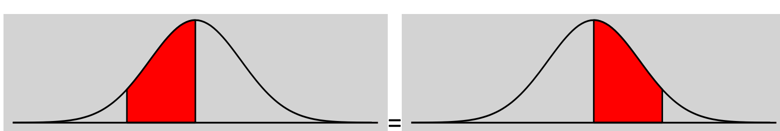
(c) $P(Z > 1.25) = 1 - P(Z < 1.25) = 1 - 0.8994 = 0.1006$



(d) $P(Z < -1.25) = P(Z > 1.25) = 0.1006$

(e) $P(-1.25 < Z < 0) = P(0 < Z < 1.25)$

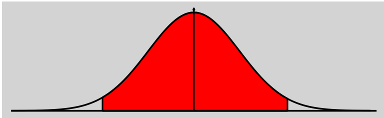
$$= 0.8994 - 0.5$$



(f) $P(-1 < Z < 1.25) =$

$P(-1 < Z < 1.25) = P(Z < 1.25) - P(Z < -1) = P(Z < 1.25) - [1 - P(Z < 1]$

$= P(Z < 1.25) + P(Z < 1) - 1$



Exercise

1. Find z from z – table such that the area to the right of z is 0.05. That is, find z such that: P (Z > z) = 0.05
2. Find z from z – table such that the area to the right of z is 0.025
3. Find z from z – table such that the area to the right of z is 0.02

Example

The IQ scores of 5000 students have a normal distribution with $\mu = 100$ and $\sigma = 15$**.**

(a) How many students have an IQ between 115 and 130?
(b) How many students have an IQ between 100 and 130?
(c) How many students have an IQ more than 124?
(d) How many students have an IQ less than 85?
(e) Find the IQ score for which 3 % of the scores exceed.

Solution:

The IQ scores are normally distributed $\Rightarrow$ X = N (100, 15)

(a) P(115 < X < 130)

x = 115: $z_1 = \frac{x - \mu}{\sigma} = \frac{115 - 100}{15} = 1$

x = 130: $z_2 = \frac{x - \mu}{\sigma} = \frac{130 - 100}{15} = 2$

P (115 < X < 130) = P (1 < Z < 2) = P (Z < 2) – P (Z < 1)
= 0.9772 – 0.8413 = 0.1341

The number of students with IQ scores between 115 and 130 is

(5000)(0.1341) = 671 students

(b) x = 100: $z_1 = 0$

x = 130: $z_2 = 2$

The percentage of students with IQ scores between 100 and 130 is,

$$P(0 < Z < 2) = 0.4772$$

The number of students with IQ scores between 100 and 130 is

(5000)(0.4772) = 2386 students.

(c) P(X > 124) = P(Z >1.6)

[x = 124: $z = \frac{124-100}{15} = 1.6$ **]**

P (Z > 1.6) = 1 – 0.9452

= 0.0448

The number of students with IQ scores more than 124 is

(5000)(0.0448) = 224 students

(d) P(X > 85) = P (Z > -1) = p (Z < 1) = 0.8413

[x = 85: $z = \frac{85-100}{15} = -1$]

The number of students with IQ scores less than 85 is,

(5000)(0.8413) = 4207 students.

(e) Find x-value so that the area to the right of x under the normal curve is 0.03.

First, we find the z value that corresponds to the required x value.

P (Z > z) = 0.03 $\Rightarrow$ P (Z< z) = 1 – 0.03 = 0.97

The value of z corresponds to the area 0.97 is 1.88

Now, use the formula for the z-score to find x. that is

$$x = \mu + z\,\sigma$$

$$= 100 + (1.88)(15) = 128.2$$

The IQ score for which 3 % of the scores exceed is 128.2.

6.4 The Exponential Distribution

- An exponential distribution is a continuous probability distribution that is usually describes times between two occurrences.
- Because time is never negative, an exponential random variable is always positive.
  - The service times in a system (how long it takes to serve a customer at a bank or a governmental department).
  - The time between "hits" on a website.
  - The lifetime of an electrical component.
  - The time until the next phone call arrives in a customer service center.

- The probability density function for the exponential random variable X is given by

EXPONENTIAL PROBABILITY DENSITY FUNCTION

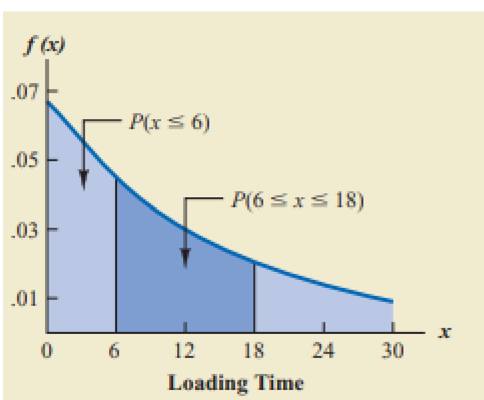
$$f(x) = \frac{1}{\mu} e^{-x/\mu} \quad \text{for } x \geq 0 \qquad (6.4)$$

where μ = expected value or mean

Example

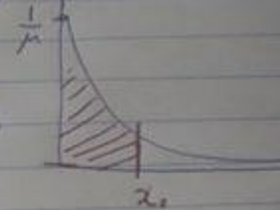
Suppose that X represents the loading time for a truck which follows an exponential distribution. If the mean, or average, loading time is 15 minutes (m = 15), the appropriate probability density function for x is

$$f(x) = \frac{1}{15} e^{\frac{-x}{15}}$$



Exponential Probability Distribution

Let X be an exponential random variable with $f(x) = \frac{1}{\mu} e^{-x/\mu}$ $\quad x \geq 0, \; \mu > 0$



* $P(X \leq x_0) = \int_0^{x_0} \frac{1}{\mu} e^{-x/\mu} dx$

$= \frac{1}{\mu} \left(-\mu e^{-x/\mu}\right)\Big]_0^{x_0}$

$= -e^{-x/\mu}\Big]_0^{x_0} = -e^{x_0/\mu} + 1$

$= \boxed{1 - e^{x_0/\mu}}$

* $P(x_1 < X < x_2) = \int_{x_1}^{x_2} \frac{1}{\mu} e^{-x/\mu} dx$

$= -e^{x/\mu}\Big]_{x_1}^{x_2} = \boxed{e^{-x_1/\mu} - e^{-x_2/\mu}}$

$P(x_1 < X < x_2) = P(X < x_2) - P(X < x_1)$

$= \left(1 - e^{-x_2/\mu}\right) - \left(1 - e^{-x_1/\mu}\right)$

$= \boxed{e^{-x_1/\mu} - e^{-x_2/\mu}}$

* $P(X > x_1) = 1 - P(X < x_1)$

$= 1 - \left(1 - e^{-x_1/\mu}\right) = \boxed{e^{-x_1/\mu}}$