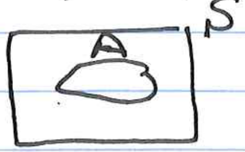
4.2 Events and Their Probabilities

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a sample space for a random experiment.



An event A is a subset of S

$p(A) \equiv$ The probability that event A will occur

Ex: Roll 2 dice

$\#(S) = (6)(6) = 36$ ordered pairs

$$S = \{(1,1), (1,2), \ldots, (6,6)\}$$

Define $A \equiv$ Sum of 2 faces shown up is 10

$$A = \{(4,6), (6,4), (5,5)\}$$

$$P(A) = 3/36$$

② $B \equiv$ The two faces are identical

$$B = \{(1,1), \ldots, (6,6)\}$$

$$P(B) = 6/36 = 1/6$$

③ $C \equiv$ Sum of the 2 faces is 13

$$C = \{\ \} = \phi$$

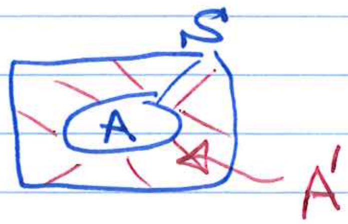
$$p(C) = p(\phi) = 0$$

Note that $p(S) = 1$

4.3: Some Basic Relationships of Probability

Consider a sample space S of a random experiment
Suppose A, B are two events of S.

1- The Complement of A (A^c, A')



$p(A')$ = The probability that A will NOT occur.

$$p(A') = 1 - p(A)$$

Ex: Toss a Coin 6 times, find the probability that at least one head is shown up

$\#(S) = (2)(2)(2)(2)(2)(2) = 64$

let $A \equiv$ at least one head is shown up

$A' \equiv$ No head is shown up $= \{TTTTTT\}$

$p(A') = \frac{1}{64} \Rightarrow p(A) = 1 - \frac{1}{64} = 1/63$

2- The intersection of A and B

$A \cap B$ = A and B

$P(A \cap B) \equiv$ The Probability that BOTH A and B will occur

If $A \cap B = \phi$, A, B are Called disjoint (sets)

A, B disjoint $\Rightarrow p(A \cap B) = p(\phi) = 0$

$p(A \cap B) = 0 \Rightarrow$ The two events Can't occur toghether

A, B are Called mutually exclusive

3- The union of A and B

$A \cup B \equiv A$ or B

$P(A \cup B) = P(A \text{ or } B) \equiv$ At least one event will occur.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A, B are mutually exclusiven, then

$P(A \cup B) = P(A) + P(B)$

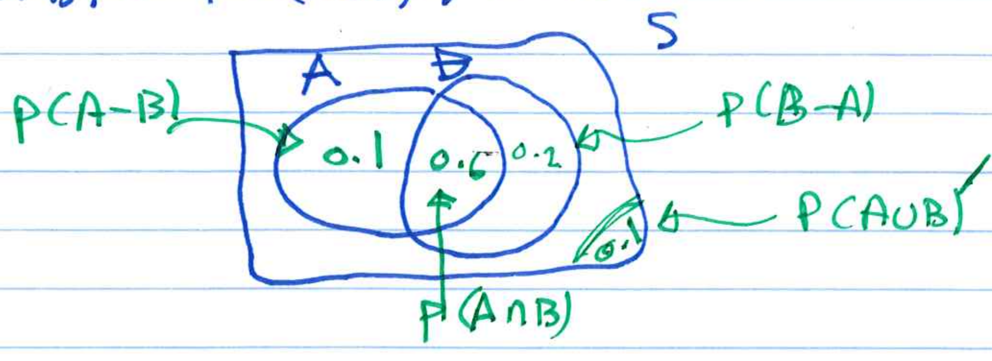
Notes (1) $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B)$

(2) $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

(3) $P(A - B) = P(A) - P(A \cap B)$

Ex: If $P(A) = 0.7$, $P(B) = 0.8$, $P(A \cap B) = 0.6$

- $P(A \cap B) = 0.6 \neq 0 \Rightarrow$ A, B are not mutually exl.
- $P(B') = 1 - P(B) = 0.2$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  $= 0.7 + 0.8 - 0.6$
  $= 0.9$
- $P(A - B) = P(A) - P(A \cap B)$
  $= 0.7 - 0.6 = 0.1$
- $P(B - A) = P(B) - P(A \cap B)$
  $= 0.8 - 0.6 = 0.2$
- $P(A \cup B)' = 1 - 0.9 = 0.1$
- $P(A' \cap B') = P((A \cup B)') = 0.1$



Example: 1 Roll 2 dice, Define the following events

$A \equiv$ Sum of 2 faces is 10 $A = \{(5,5), (6,4), (4,6)\}$

$B \equiv$ The two faces are identical $B = \{(1,1), \dots, (6,6)\}$

$C \equiv$ Sum of 2 faces is 11 $C = \{(5,6), (6,5)$

$P(A) = 3/36$, $P(B) = 6/36$, $P(C) = 2/36$

- A, B are not mutually exclusive $A \cap B \neq \phi$

$P(A \cap B) = P\{(5,5)\} = 1/36$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 3/36 + 6/36 - 1/36 = 8/36$

- B, C are mutually exclusive $B \cap C = \phi$

$P(B \cup C) = P(B) + P(C)$

$= 6/36 + 2/36 = 8/36$

$P(A' \cap B') = P(A \cup B)' = 1 - 8/36 = 24/36$

2 If $P(A) = 0.68$, $P(A \cup B)' = 0.09$

$P(A \cap B) = 0.35$. Find $P(B)$

$P(A \cup B) = 1 - P(A \cup B)' = 1 - 0.09 = 0.91$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.91 = 0.68 + P(B) - 0.35$

$\Rightarrow \boxed{P(B) = 0.58}$

4.4 Conditional Probability

The conditional probability of an event A given an event B, denoted by $P(A|B)$ is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Ex: If $P(A) = 0.6$, $P(B) = 0.5$, $P(A \cap B) = 0.4$

then 1) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.5} = 0.8$

2) $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.4}{0.6} = 0.6667$

3) $P(A|B') = \frac{P(A \cap B')}{P(B')}$

$= \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{0.2}{0.5} = 0.4$

Endepent Events

Two events A and B are independent if

$P(A|B) = P(A)$, $P(B|A) = P(B)$

Otherwise the two events are dependent.

Ex: 1) If $P(A) = 0.5$, $P(B) = 0.6$, $P(A \cap B) = 0.2$

then $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.5} = 0.4 \neq P(A)$

$\Rightarrow$ A, B are dependent

2) $P(A) = 0.4$, $P(B) = 0.5$, $P(A \cap B) = 0.2$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = 0.5 = P(A)$

$\Rightarrow$ independent

Ex: If $P(A) = 0.8$, $P(B) = 0.7$, $P(A \cup B) = 0.9$
Are A, B mutually exclusive? why
Are A, B independent? why

$$P(A \cap B) = -P(A \cup B) + P(A) + P(B)$$
$$= 0.6$$

$P(A \cap B) \neq 0 \Rightarrow$ Not mutually exclusive

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.6}{0.7} \neq P(A) \quad \text{Dependent}$$

Multiplication Law

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)\, P(B)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \cdot P(A)$$

If A, B are independent, then

$$P(A|B) = P(A) \Rightarrow P(A \cap B) = P(A) \cdot P(B)$$
$$P(B|A) = P(B) \Rightarrow P(A \cap B) = P(B) \cdot P(A)$$

A, B are independent $\Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

Ex: ① $P(A) = 0.7$, $P(B) = 0.6$, $P(A \cap B) = 0.5$
$P(A \cap B) = 0.5 \neq P(A) \cdot P(B) = 0.42$ Dep.
② $P(A) = 0.8$, $P(B) = 0.5$, $P(A \cap B) = 0.4$
$P(A \cap B) = 0.4 = P(A) \cdot P(B)$ Independent.

Ex: A supermarket obtains the following data on the age and the marital status of 200 customers

	Age	Marital status: Single S	Not Single S'
U	Under 30	70	60
U'	30 or over	50	20

1) Develop a joint probability table
2) Use marginal to comm on the percentage of customers regarding the marital status.
3) What is the probability of selecting a customer with age 30 or over?
$P(U')$
4) What is the probability of selecting a single customer?
$P(S)$
5) What is the probability of selecting a customer whose single and under the age of 30?
$P(S \cap U)$
6) What is the probability of selecting a customer whose is not single and age ≥ 30?
$P(S' \cap U')$, $P(S' \cup U')$ or
7) If a customer is under 30, what is the probability that he is single
$P(S | U)$
9) If a customer is not single, what is the probability that he is ≥ 30? $P(U'/S')$
10) Are martal status and age independent? Explain