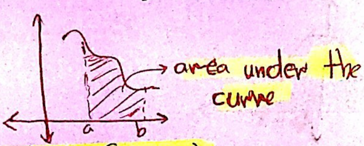
*Ch 6 : Continuous probability distribution :- التوزيع الاحتمالي المتصل

- Continuous probability distribution: A probability distribution in which the random variable X can take on any value.

- Because there are infinite values that X could assumes the probability of X taking on any one specific value is zero. « $P(X=a)=0$ ».

- If X is a continuous random variable, then :-

$P(a \leq X \leq b) = \int_a^b f(x)\,dx$ where $f(x)$ is the probability density function.



- $P(X \leq a) = P(X < a)$ since $P(X=a)=0$.

- The total area under the curve $f(x)$, where it is the probability density function is 1.

- The normal distribution is of major importance because of its wide applicability, and its extensive use in statistical inference.

-Sec 6.1: Uniform probability distribution

التوزيع المنتظم

• Uniform probability distribution:

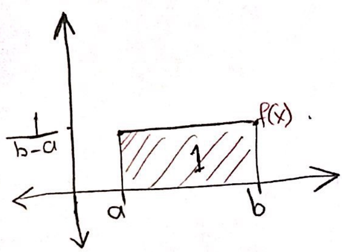
A continuous probability distribution for which the probability that the random variable will assume a value in any interval is the same interval of equal length.

أحد التوزيعات المتصلة، عندما تكون قيمة الكثافة الاحتمالية ثابتة على جميع الفترات التي لها نفس الطول.

• Uniform probability density function

* $$f(x) = \begin{cases} \frac{1}{b-a} & ; a \leq x \leq b \\ 0 & ; \text{elsewhere} \end{cases}$$

*



* $P(c \leq X \leq d)$ = area under the curve

* Total area $= 1 = \int_a^b f(x)\,dx$

• Area as a measure of probability: « Uniform ; $a \leq X \leq b$ »

① $P(X=c)=0$ « the probability of any single point is 0 »

② $P(c \leq X \leq d) = \int_c^d f(x)\,dx$;

المساواة للتوزيع cont

③ $E(X) = \mu = \frac{a+b}{2}$.

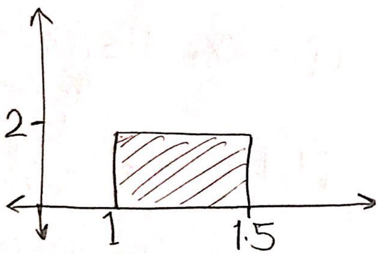
④ $Var(x) = \sigma^2 = \frac{(b-a)^2}{12}$

⑤ $\sigma = \frac{b-a}{\sqrt{12}}$

- Ex: The random variable X is Known to be uniformly distributed between 1 and 1.5.

a) Show the graph of the probability density function.

$$f(x) = \begin{cases} \frac{1}{1.5-1} = 2 & ; \ 1 \leq X \leq 1.5 \\ 0 & ; \ \text{elsewhere.} \end{cases}$$



b) Compute $P(1 \leq X \leq 1.25)$.

$$P(1 \leq X \leq 1.25) = \int_{1}^{1.25} 2\,dX = 2(1.25-1) = 0.5$$

c) Compute $P(X = 1.25)$.

$$P(X = 1.25) = 0$$

d) Compute $P(1.2 < X \leq 3)$.

$$P(1.2 < x \leq 3) = \int_{1.2}^{1.5} 2\,dx = 2(1.5 - 1.2) = 0.6$$

e) Comput the expected value.

$$E(x) = \frac{a+b}{2} = \frac{1+1.5}{2} = 1.25$$

f) Find the variance.

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(1.5-1)^2}{12} = 0.0208$$

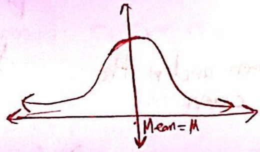
- Sec 6.2 : Normal Probability distribution : التوزيع الطبيعي

The most important probability distribution for describing a continuous random variable.

* The normal distribution has been used in a wide variety of practical applications in which the random variables are heights, weights, test scores, ---

* Normal curve:-

① It has a bell-shaped normal curve.



② Normal probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

; μ: mean.
σ: standard deviation.
π = 3.14159 ...
e = 2.718 ....

سنعتمد على standard normal من خلال الجداول لأن استخدام الصيغة صعباً.

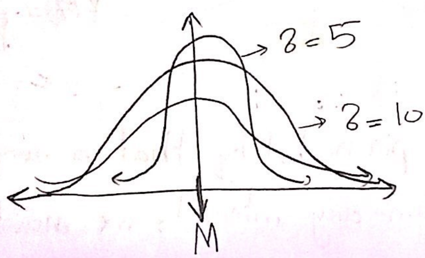
③ The normal curve has 2 parameters μ and σ.
They determine the location and shape of the normal distribution.

④ The highest point on the normal curve is at the mean, which is als the median and the mode of the distribution. (Mean = Median = Mode).

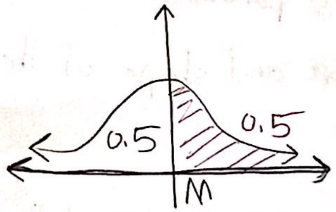
→ the mean can be: -ve, 0 or +ve

⑤ The normal distribution is symmetric.
→ the normal distribution isn't skewed «its skewness is 0».

⑥ The standard deviation determines how flat and wide the normal curve is.



⑦ The total area under the curve is 1.
→ Because the distribution is symmetric, the area under the curve to the left of the mean is 0.5 and the area to the right of the mean is 0.5

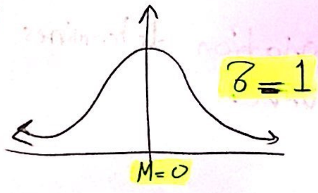


قواعد empirical rule متحققة في normal distribution

- **Standard normal probability distribution :-**

A random variable has a normal distribution with a mean of 0 and a standard deviation of 1.

- The letter Z is commonly used to designate this particular normal random variable.



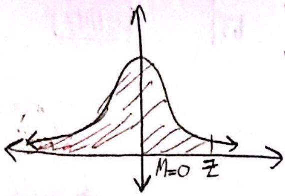
* standard normal density function:-

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

- To find the probability that a normal random variable is within any interval, we must compute the area under the curve over that interval.

- For the standard normal distribution, areas under the curve have been computed and are available in tables that can be used to compute probabilities.

* Standard normal distribution:



1. the total area under the curve is 1.
2. the area to the left of 0 is 0.5
3. the area to the right of 0 is 0.5
4. Probability = area under the curve.

- Z-table: a table that is used to find the probability for the normal distribution.

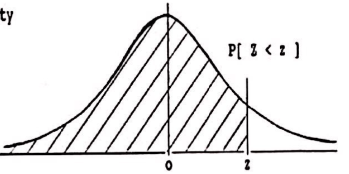
Entries in the table give the area under the curve to the left of the positive Z-value

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\,Z < z\,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$

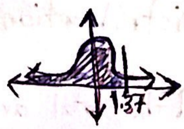


z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

- Ex: Find

① $P(Z < 1.37)$



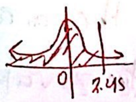
	0.07
1.3	$\boxed{0.9147}$

$\longrightarrow P(Z<1.37) = 0.9147$

② $P(Z \leq 1.37)$

$\longrightarrow P(Z \leq 1.37) = P(Z < 1.37) = 0.9147$ since $P(Z=a)=0$ "cont."

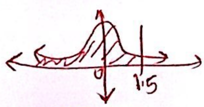
③ $P(Z < 2.45)$



	0.05
2.4	$\boxed{0.9929}$

$\longrightarrow P(Z<2.45) = 0.9929$

④ $P(Z < 1.5)$

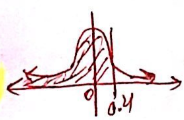


	0.00
1.5	$\boxed{0.9332}$

$\longrightarrow P(Z < 1.5\underline{0}) = 0.9332$

⑤ $P(Z < .4)$

$\rightarrow P(Z<0.4) = 0.6554$

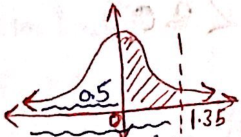


	0.00
0.4	$\boxed{0.6554}$

⑥ $P(Z<0)$

$\rightarrow P(Z<0) = 0.5$

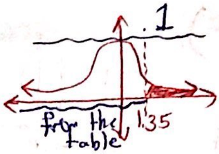
⑦ $P(0 < Z < 1.35)$



from the table
$P(Z<1.35) = 0.9115$
and we know $P(Z \leq 0) = 0.5$

$P(0<Z<1.35)$
$= 0.9115 - 0.5$
$= 0.4115$

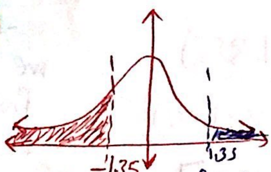
⑧ $P(Z > 1.35)$



$P(Z<1.35) = 0.9115$
and we know the total area is 1

$P(Z>1.35)$
$= 1 - 0.9115$
$= 0.0885$

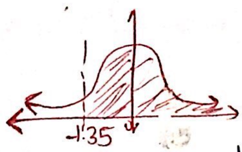
⑨ $P(Z < -1.35)$



we can find $P(Z>1.35)$ as part ⑧
and from the symmetry:-
$P(Z>1.35) = P(Z<-1.35)$

$P(Z<-1.35)$
$= P(Z>1.35)$
$= 0.0885$

⑩ $P(Z > -1.35)$
$= P(Z<1.35)$
$= 0.9115$



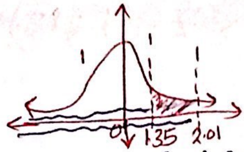
we can find $P(Z<1.35)$ from the table and from the symmetry $P(Z>-1.35) = P(Z<1.35)$

11 $P(1.35 < Z < 2.01)$

$= P(Z<2.01) - P(Z<1.35)$

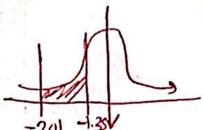
$= 0.9778 - 0.9115$

$= 0.0663$



we can find $P(Z<2.01)$ and $P(Z<1.35)$ from the table, then we subtract $P(Z<1.35)$ from $P(Z<2.01)$

12 $P(-2.01 < Z < -1.35)$

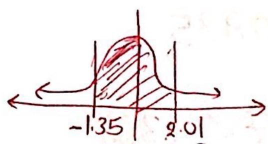


$= P(1.35 < Z < 2.01)$

$= 0.0663$

from the symmetry:
$P(-2.01 < Z < -1.35) = P(2.01 < Z < 1.35)$

13 $P(-1.35 < Z < 2.01)$



$= P(Z<2.01) - P(Z<-1.35)$

$= 0.9778 - [1 - P(Z<1.35)]$

$= 0.9778 - [1 - 0.9115]$

$= 0.9778 - 0.0885$

$= 0.8893$

we can $P(Z<2.01)$ from the table and $P(Z<-1.35)$ then
$P(-1.35<Z<2.01) = P(Z<2.01) - P(Z<-1.35)$

* In the preceding illustrations, we showed how to compute probabilities given specified z-values. In some situations, we are given a probability and are interested in working backward to find the corresponding Z-value.

إذا كانت معروفة المساحة = الاحتمال فكم تكون Z.

- Ex: Find a:

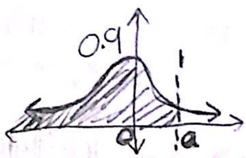
① $P(Z < a) = 0.8997$

	0.08
1.2	0.8997

بما أن الاحتمال أكبر من 0.5
∴ a عدد موجب فنبحث
في الجدول عن 0.8997

$a = 1.28$

② $P(Z \leq a) = 0.9$



→ $a = 1.28$

بما أن الاحتمال أكبر من 0.5
إذن a عدد موجب فنبحث
في الجدول عن 0.9 فإذا لم
نجدها نأخذ القيمة الأقرب عليها فإذا كانت
تقع بين قيمتين يبعدان
عن نفس البعد نأخذ
معدلهما.

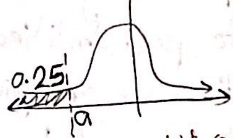
	0.08	0.09
1.2	0.8997	0.9015

هذا العدد أقرب على 0.9

3) $P(Z<a)=0.25$

	0.07	0.08
0.6	0.7486	0.7515

هذا العدد أقرب على 0.75



0.25 أقل من 0.5 ،
∴ a عدد سالب ،
ولكي نستطيع البحث
باستخدام الجدول نجد
$1-0.25=0.75$

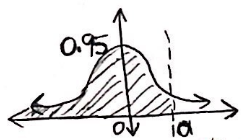
$P(Z>a)=P(Z<-a)$

$$\boxed{a=-0.67}$$

4) $P(Z<a)=0.95$

	0.04	0.05
1.6	$\boxed{0.9495}$	$\boxed{0.9505}$

0.95 نفس البعد على العددين

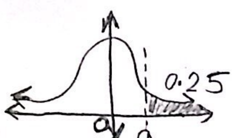


0.95 أكبر من 0.5 ،
∴ a عدد موجب ،
نبحث مباشرة في
الجدول على 0.95

$$\rightarrow \boxed{a=\frac{1.64+1.65}{2}=1.645}$$

5) $P(Z>a)=0.25$

	0.07	0.08
0.6	0.7486	0.7517



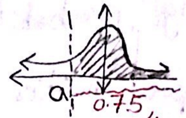
$P(Z<a)=1-0.25$
$=0.75$

∵ أكبر من 0.5 نبحث
في الجدول عن 0.75

$$\therefore \boxed{a=0.67}$$

6 $P(Z > a) = 0.75$

	0.07	0.08
0.6	0.7486	0.7517



أكبر من 0.5

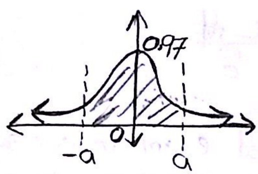
∴ a عدد سالب

نبحث في الجدول على 0.75

∴ $\boxed{a = -0.67}$

7 $P(-a < Z < a) = 0.97$ « or $P(|Z| < a) = 0.97$ »

	0.07
2.1	0.985



كي نتمكن من ايجاد a من الجدول

هذه هي المساحة من 0 $0.485 = \frac{0.97}{2}$

لكن نضيف اليها 0.5 نحصل على المساحة

كاملة تحت a ← $0.485 + 0.5$
$= 0.985$

نبحث عن هذا العدد في الجدول

∴ $a = 2.17$ and $-a = -2.17$

* Computing probabilities for any Normal distribution:-

→ the probabilities for all Normal distributions are computed by using the standard normal distribution.

→ Converting to the standard Normal random variable:

Let X be a normal random variable with mean μ and standard deviation σ, then the formula used to convert it to the standard normal variable Z

$$Z = \frac{X - \mu}{\sigma}$$

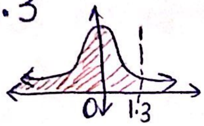
-Ex: The final exam in a statistics class were normally distributed with a mean of 65 and a standard deviation of 10.

$\mu = 65$, $\sigma = 10$, normal

a) Find the probability that a randomly selected student scored is less than 78 on the exam?

→ $P(X < 78)$

$$Z = \frac{X - \mu}{\sigma} = \frac{78 - 65}{10} = 1.3$$



$\therefore P(Z < 1.3) = 0.9032$

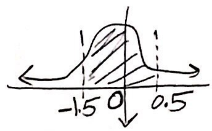
b) What is the percentage of the scores that are between 50 and 70?

$\longrightarrow P(50<X<70)$

$X=50: \quad Z=\dfrac{50-65}{10}=-1.5$

$X=70: \quad Z=\dfrac{70-65}{10}=0.5$

$\therefore P(-1.5<Z<0.5)$



$=P(Z<0.5)-P(Z<-1.5)$

From the table

$=P(Z<0.5)-(1-P(Z<1.5))$

from the table.

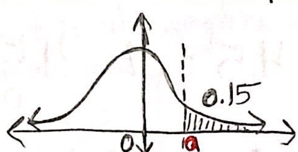
$=0.6915-(1-0.9332)$

$=0.6915-0.0668 = \boxed{0.6247}$

the percentage is 62.47%

c) Calculate the minimum score of the top 15% students on this exam?

$P(Z>a)=0.15$



لكي نتمكن من استخدام الجدول

نجد $P(Z<a)$

$=1-0.15=0.85 \rightarrow$ نبحث عن هذا الرقم

a: + since 15% less than 0.5

	0.03	0.04
1.0	0.8485	0.8508

$\rightarrow$ $a = 1.04$ «Z-score»

نحولها بدلالة X.

$$Z = \frac{X - \mu}{\sigma} \rightarrow 1.04 = \frac{X - 65}{10}$$

$$(1.04)(10) = X - 65$$

$$\therefore \boxed{X = 75.4}$$

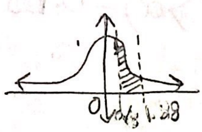
- Ex: The weights of Palestenian newborn babies are Normally distributed with a mean of 3Kg and a standard deviation of 0.8 Kg. $\mu = 3, \sigma = 0.8$, normal.

a) What is the probability of babies who weights between 3.5 and 4.5Kg?

$$X = 3.5 : Z = \frac{3.5 - 3}{0.8} = 0.625 \cong 0.63$$

$$X = 4.5 : Z = \frac{4.5 - 3}{0.8} = 1.875 \cong 1.88$$

$$\therefore P(0.63 < Z < 1.88)$$



$= P(Z<1.88) - P(Z<0.63)$

from the table   from the table

	0.03	0.08
0.6	$\boxed{0.7357}$	
1.8		$\boxed{0.9699}$

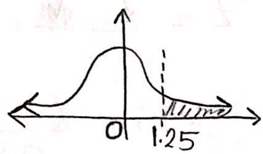
$= 0.9699 - 0.7357$

$= 0.2342$

b) If there were 100 babies this week, how many of them do you expect to weigh at least 4?

$P(X \geq 4)$

$X=4: \quad Z = \frac{4-3}{0.8} = 1.25$



$\rightarrow P(Z \geq 1.25)$

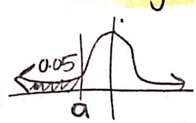
$= 1 - P(Z<1.25)$

$= 1 - 0.8944$

$= 0.1056$

$\therefore$ Expected number is $100(0.1056) = 10.56$

c) A baby is considerd unhealthy if its weights is in the bottom 5%. What is the weights of an unhealthy baby?



$\rightarrow P(Z<a) = 0.05$

a: −ve since 0.05 is less than 0.5

$1 - P(Z > a) = 0.05$

$\therefore P(Z > a) = 0.95$ } نبحث عن هذا الرقم في الجدول لنحصل a عدد سالب

	0.04	0.05
1.6	0.9495	0.9505

نفس البعد عن 0.95 نأخذ معدلهما

$$\therefore a = -\frac{1.64 + 1.65}{2} = -1.645$$

$$\rightarrow Z = \frac{X - \mu}{\sigma}$$

$$-1.645 = \frac{X - 3}{0.8} \rightarrow X = -1.645(0.8) + 3$$

$X = 1.684$ or less

- Sec 6.3 : skip

- Sec 6.4 : Exponential Probability Distribution.

التوزيع الأسي.

Cont. random variable

The exponential probability distribution may be used for random variables such as the time between arrivals at a car wash, the time required to load a truck, the distance between major defects in a highway, ---

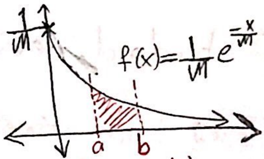
continuous يشبه ال poisson بس هاد

« it is useful in computing probabilities for the time it takes to complete a task »

* The exponential probability density function:-

$$f(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \quad ; \quad x \geq 0 , \mu > 0$$

where μ = expected value or mean.

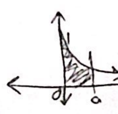


$P(a < X < b)$

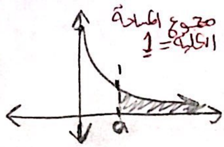
* Computing Probabilities for the exponential distribution:

① $P(X = a) = 0$ since it is continuous.

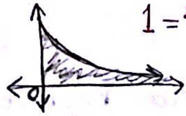
② $P(X \leq a) = \int_0^a \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = -e^{-\frac{x}{\mu}} \Big|_0^a = -e^{-\frac{a}{\mu}} - -1$
$= 1 - e^{-\frac{a}{\mu}}$



③ $P(X \geq a) = \int_a^\infty \frac{1}{\mu} e^{-\frac{x}{\mu}} dx \overset{\text{or}}{=} 1 - P(X < a)$
$= 1 - (1 - e^{-\frac{a}{\mu}})$
$= e^{-\frac{a}{\mu}}$



④ $P(X \geq 0) = 1$



⑤ $P(a \leq X \leq b) = \int_a^b \frac{1}{\mu} e^{\frac{-x}{\mu}} dx$

$\overset{or}{=} P(X \leq b) - P(X \leq a).$

$= (1 - e^{\frac{-b}{\mu}}) - (1 - e^{\frac{-a}{\mu}})$

$= e^{\frac{-a}{\mu}} - e^{\frac{-b}{\mu}}$

⑥ $E(X) = \mu$

⑦ $Var(X) = \delta^2 = \mu^2$

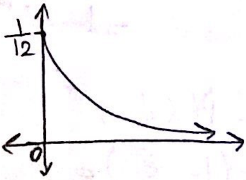
⑧ standard deviation $= \delta = \mu$.

- Ex: The time between arrivals of vehicles at a particular intersection follows an exponential probability distribution with a mean of 12 seconds.

a) Sketch this exponential probabity distribution.

$\mu = 12$

$\therefore f(x) = \frac{1}{12} e^{\frac{-x}{12}} \quad ; \quad x \geq 0.$



b) What is the probability that the arrival time between vehicles is 6 seconds or less?

using calculator:-

$$\rightarrow P(x \leqslant 6) = 1 - e^{\frac{-6}{12}}$$

$1 - \boxed{\text{shift}}\ \boxed{\ln}\ (-0.5) =$

$$= 1 - e^{-0.5}$$

$$= 0.3935$$

c) What is the probability that the arrival time between vehicles 12 or less?

$$\rightarrow P(x \leqslant 12) = 1 - e^{\frac{-12}{12}}$$

$1 - \boxed{\text{shift}}\ \boxed{\ln}\ (-1) =$

$$= 1 - e^{-1}$$

$$= 0.6321$$

d) What is the probability of 30 or more seconds between vehicles arrivals?

$$\rightarrow P(x \geqslant 30) = e^{\frac{-30}{12}}$$

$\boxed{\text{shift}}\ \boxed{\ln}\ \boxed{-2.5} =$

$$= e^{-2.5} = 0.0821$$

e) What is the probability that the arrival time between vehicales is between 6 and 12?

$$\rightarrow P(6 < X < 12) = P(X < 12) - P(X \leqslant 6)$$

$$= 0.6321 - 0.3935$$

$$= 0.2386$$

لو طلب percentage نضربو في 100%
لو اعطانا عدد كلي 60 وطلب العدد لهذا الاحتمال نضربه في العدد.

f) What is the variance?

$\rightarrow \sigma^2 = \mu^2 = 12^2 = 144$ second^2.