ASIL SHAAR (PROBABILITY THEORY(STAT3321))

CHAPTER 1

Chapter 1

Probability and Distributions

1.1 Introduction

* Experiment: process that gives Certain possible outcoms العملية التي تعطي نتيجة اكيدة.

Experiment → deterministic
Experiment → probabilistic
التجربة

* Sample space: Set of all possible out comes كل العناصر الممكنة
* Sample point: one possible outcome
* event: subset from the sample space. مجموعة محددة من تمام العينة

Rules for assigning probabilities

$S = [E_1, \dots, E_k]$

① $p(E_i) \geq 0$

② $\sum_{i=1}^{k} p(E_i) = 1$

$\Rightarrow$

Approaches for assigning probabilities

1. Classical method — يعتمد على فكرة انه حدث معين له فرصة الظهور نفسها
2. Relative freq. method — احتمال وقوع الاحداث المختلفة
3. subjective method. — تحديد قيمة الاحتمال هو رأي شخصي

Ex: إعتدال قطعة النقد

Exp: Toss a "fair" Coin (يعني لا يكون احتمال ظهور الصورة أكبر او أصغر من الكتابة)

$S = \{H, T\}$

- $P(H) = 0.5 \quad P(T) = 0.5$] classical method
- $P(H) = \frac{3}{9} = 0.33 \quad P(T) = \frac{6}{9} = 0.67$] Relat. freq. method.
- $P(H) = 0.2 \quad P(T) = 0.8$] subjectiv method.

Out come	freq	
H	5	3
T	4	6
Sum	9	

1.2 Set Theory

- Axioms مسلمات
- Rules of logic قواعد المنطق (T, F)
- undefined term (set, element

Def

سكربت C

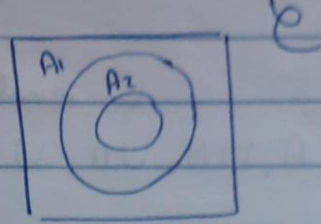
let A_1, A_2 be events from a sample space $\mathcal{E}$

lower case	upper case	script letter
a	A	$\mathcal{A}$
b	B	$\mathcal{B}$
c	C	$\mathcal{C}$

⇒

① $A_2 \subseteq A_1$ means

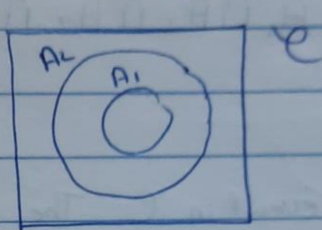
$x \in A_2 \Rightarrow x \in A_1$



Venn Diggram

② $A_1 \subseteq A_2$ means

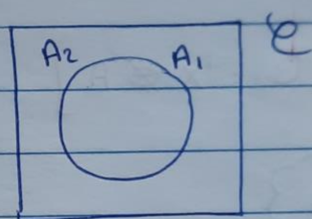
$x \in A_1 \rightarrow x \in A_2$



③ $A_1 = A_2$ if

$A_1 \subseteq A_2$ and $A_2 \subseteq A_1$

if $x \in A_1 \Leftrightarrow x \in A_2$



<u>Def</u>

let $\mathcal{E}$ be a sample space we define the null set N

as : $N = \{ \ \} = \varnothing$

Note: N has no elements

<u>Def</u> let $\mathcal{E}$ be a sample space let A and B be events

① we define A union B as

$A \cup B = [x \in \mathcal{E} : x \in A$ <u>or</u> $x \in B]$

② we define A intersection B as:

$A \cap B = [x \in \mathcal{E} : x \in A$ <u>and</u> $x \in B]$

$\Rightarrow$

Def

let $A_1, A_2, \ldots$ be events in $\mathcal{C}$

① $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots$

② $\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots$

Def

let A be an event in $\mathcal{C}$ The complement of A is defind as:

$A^* = [x \in \mathcal{C} : x \notin A]$

Proposition

let A, B, C be events from $\mathcal{C}$

① $\phi^* = \mathcal{C}$

② $\mathcal{C}^* = \phi$

③ $A \cup \mathcal{C} = \mathcal{C}$

④ $A \cap \mathcal{C} = A$

⑤ $A \cup A = A$

⑥ $A \cap A = A$

⑦ $A \cup \phi = A$

⑧ $A \cap \phi = \phi$

⑨ $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C) = (A \cup C) \cup B$

⑩ $A \cap B \cap C = (A \cap B) \cap C = (A \cap C) \cap B = (B \cap C) \cap A$

⑪ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

⑫ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$\Rightarrow$

(13) $(A^*)^* = A$

(14) $A \cup A^* = \mathcal{C}$

(15) $A \cap A^* = \phi$

(16) $(A \cup B)^* = A^* \cap B^*$

(17) $(A \cap B)^* = A^* \cup B^*$

proof : Not required.

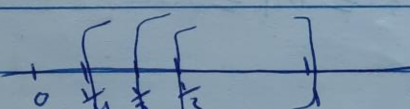
Ex

Define :

$A_k = \left\{ x \in \mathbb{R} : \frac{1}{K+1} \leq x \leq 1 \right\}$ K, 1, 2, 3, ...

$A_k = \left[\frac{1}{K+1}, 1 \right]$

$A_1 = \left[\frac{1}{2}, 1 \right]$



$A_2 = \left[\frac{1}{3}, 1 \right]$

$A_3 = \left[\frac{1}{4}, 1 \right]$

$\vdots$

$\bigcap_{i=1}^{\infty} A_i = \left[\frac{1}{2}, 1 \right]$

$\bigcup_{i=1}^{\infty} A_i = (0, 1]$

Ex

$A_k = \{x \in R : 0 < x < \frac{1}{k}\}$

$k = 1, 2, 3, \ldots$

$A_k = (0, \frac{1}{k})$

$A_1 = (0, 1)$

$A_2 = (0, \frac{1}{2})$

$A_3 = (0, \frac{1}{3})$

$\vdots$

$\bigcup_{i=1}^{\infty} A_i = (0, 1)$

$\bigcap_{i=1}^{\infty} A_i = \phi$

Def

$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$

$\lim_{k \to \infty} A_k = \bigcup_{k=1}^{\infty} A_k$

Def $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots$

$\lim_{k \to \infty} A_k = \bigcap_{k=1}^{\infty} A_k$

Ex:

$\mathcal{C} \subseteq R$

$Q : A \longrightarrow N$

$A \subseteq R$

$Q(A)$: number of positive integers in A

$A = (0, 3)$

$Q(A) = 2$

$B = \{\ldots, -3, -2, -1, 0\}$

$Q(B) = 0$

$\Rightarrow$

$C = \{ \ldots\ldots, -2, -1, 0, 1, 2 \}$

$Q(C) = 2$

black board $\mathbb{Z}$

$D = \{ \ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots \} = \mathbb{Z}$

$Q(D)$ undefined

Ex: $\mathcal{C} = \mathbb{R}^2$

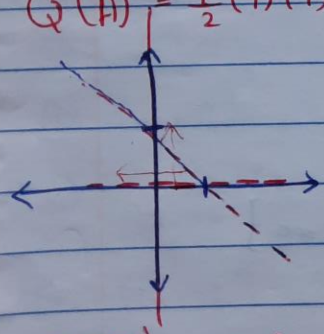
$Q : A \longrightarrow [0, \infty)$

$A \subseteq \mathbb{R}^2$

$Q(A) = \text{area of } A = \iint_A dx\, dy$

$A = \{ (x, y) \in \mathbb{R}^2 : x + y < 1, \ x \geq 0, \ y \geq 0 \}$

$Q(A) = \frac{1}{2}(1)(1) = \frac{1}{2}$



$Q(A^*)$ undefined

$B = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

$Q(B) = 0$

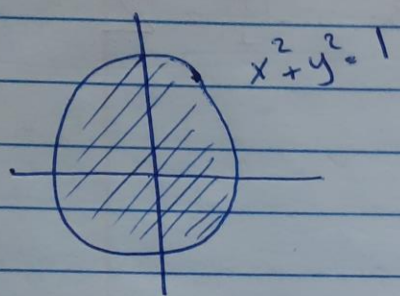
$\Rightarrow$

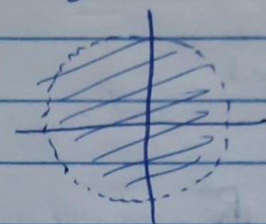
$C = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \}$

$Q(C) = \pi$

$\pi(r^2)$

انا بدون المساواة مش رح يتغير اشي بس خط الدائرة بيصير مقطع





major number ← set هو انه نحاول نعطي

Ex :

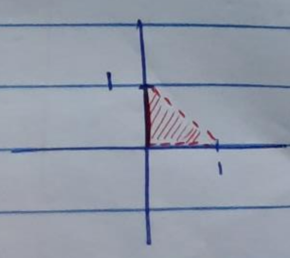
$\mathcal{C} = \mathbb{R}^3$

$Q : \mathcal{C} \to [0, \infty)$

$Q(A) = \text{Volum}(A) = \iiint_A dx\, dy\, dz$

$A \subseteq \mathcal{C}$

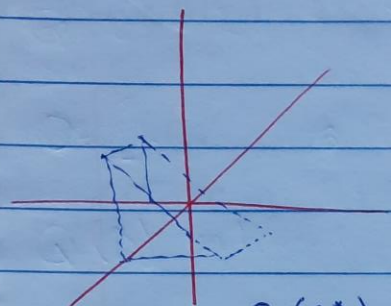
$A = \{ (x,y,z) : x + y < 1 ,\ x \geq 0 ,\ y \geq 0 ,\ z = 0 \}$



$Q(A) = 0$

$\Rightarrow$

$B = \{(x, y, z) : x + y \le 1 \;.\; x + y < 1,\; x \geq 0,\; y \geq 0,\; 1 < z < 3\}$



$Q(B) = \int$

$=$

$=$

$Q(A^*)$ undefined

$Q(B^*)$ undefin

$$\int_1^3\int_0^1\int_0^{1-y} dx\,dy\,dz = \int_1^3\int_0^1 \left(x\Big|_0^{1-y}\right) dy\,dz$$

$$= \int_1^3\int_0^1 (1-y)\,dy\,dz$$

$$= \int_1^3 \left(y - \frac{y^2}{2}\Big|_0^1\right) dz = \int_1^3 \frac{1}{2}\,dz$$

$$= \frac{1}{2} z \Big|_1^3 = \frac{3}{2} - \frac{1}{2} = \frac{2}{2}$$

$$= \boxed{1}$$

Ex:

$\mathcal{C} = \mathbb{R}$

$Q : \mathcal{C} \to \mathbb{R}$

$Q(A) = \sum_{x \in A} f(x)$, $f(x) = \begin{cases} (2) & \\ 0, & \text{else} \end{cases}$

$A \subseteq \mathbb{R}$

$A = (0, 1)$

$Q(A) = \sum_{x \in A} f(x) = 0$

$B = (0, 1]$

$Q(B) = \sum_{x \in B} f(x) = f(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$C = [0, 3]$

$Q(C) = \sum_{x \in C} f(x) = f(1) + f(2) + f(3)$

$= \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \boxed{\frac{7}{8}}$

$\Rightarrow$

$$Q(\mathcal{C}) = Q(\mathbb{R}) = \sum_{x \in \mathbb{R}} f(x)$$

$$= \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \cdots$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = \boxed{1}$$

probabilit major $\longrightarrow$ لازم يكون major واحد

1.3 The probability set function

<u>Def</u> let $\mathcal{C}$ be a sample space.

let P have the following propeties:

① $P(C) \geq 0 \quad \forall C \subseteq \mathcal{C}$

② $P\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} P(C_i) \quad \forall C_i \subseteq \mathcal{C}$

$C_i \cap C_j = \phi \quad i, j \,,\; i \neq j$

③ $P(\mathcal{C}) = 1$

Then we say the p is called a <u>probability set function</u>

<u>Th 1</u>

$P(C^*) = 1 - P(C) \quad \forall C \subseteq \mathcal{C}$

<u>Th 2</u>

$P(\phi) = 0$

<u>Th 3</u>

$C_1 \subseteq C_2$, C_1, C_2 events in $\mathcal{C}$

Then : $P(C_1) \leq P(C_2)$

$\Rightarrow$

Th 4

$0 \le p(C) \le 1 \quad \forall C \subset \mathscr{C}$

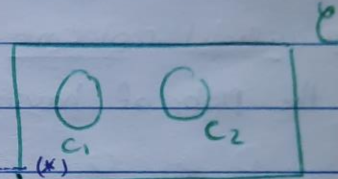
Th 5

C_1, C_2 events in $\mathscr{C}$ Then:

$$p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2)$$

Note: Try to prove Th 1, ..., Th 5

proof for Th 5.



Case 1: $C_1 \cap C_2 = \phi$

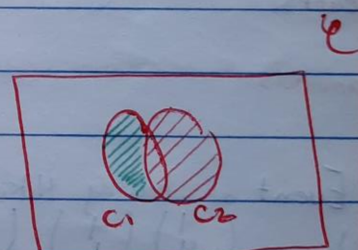
$p(C_1 \cup C_2) = p(C_1) + p(C_2)$ (def 2) ... (*)

$p(C_1 \cap C_2) = p(\phi) = 0$ (Th 2) .... (**)

$p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2)$ from (*) and (**)

$$\left.\begin{aligned} C_1 \cup C_2 &= C_2 \cup (C_1 \cap C_2^*) \\ &= (C_2 \cup C_1) \cap (C_2 \cup C_2^*) \\ &\ (C_2 \cup C_1) \cap (\mathscr{C}) \\ &= (C_1 \cup C_2) \end{aligned}\right]$$

هاد عل جنب



Case 2:

$C_1 \cap C_2 \ne \phi$

$p(C_1 \cup C_2) = p(C_2 \cup (C_1 \cap C_2^*))$

$\boxed{p(C_1 \cup C_2) \overset{\text{def2}}{=} p(C_2) + p(C_1 \cap C_2^*)}$ ..... Eq 3

$C_1 = (C_1 \cap C_2^*) \cup (C_1 \cap C_2)$

$p(C_1) = p(C_1 \cap C_2^*) \cup (C_1 \cap C_2)$

$\overset{\text{def2}}{=} \boxed{p(C_1 \cap C_2^*) + p(C_1 \cap C_2)}$ ----- Eq 4

$\Rightarrow$

Eq ③ - Eq ④

$$\Rightarrow p(C_1 \cup C_2) = p(C_1) + p(C_2) + \cancel{p(C_1 \cap C_2^*)} - \cancel{p(C_1 \cap C_2^*)}$$
$$- p(C_1 \cap C_2)$$

$$\boxed{p(C_1 \cup C_2) = p(C_1) + p(C_2) - p(C_1 \cap C_2)}$$

Ex:

Draw 5 Cards from an ordinary deck of 52 Cards, at random without replacment → (Hyper geometric Distribution) بتستخدم

① Find the prob. of having 4 Kings and 1 queen

♣ Clubs A 2 3 4 5 6 7 8 9 10 J Q K

♠ Spades

♥ Hearts

♦ Diamonds

① E: Event of having 4 Kings and 1 queen.

$$p(E) = \frac{\binom{4}{4}\binom{4}{1}\binom{44}{0}}{\binom{52}{5}}$$ haipor geimatrice

$$= \frac{1}{649740} = 1.54 \times 10^{-6}$$

② Find the prop. of having 2 Kings, 2 queens and 1 jack

F: event of having 2 Kings, 2 queens, 1 jack

$$p(F) = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{40}{0}}{\binom{52}{5}} = \frac{3}{54145} = 5.54 \times 10^{-5}$$

prove :-

$\boxed{\text{Th 1}}$ $C \subseteq \mathcal{C}$, $P(C^*) = 1 - p(c)$

proof : we have $\mathcal{C} = C \cup C^*$

where $C \cap C^* = \phi$. Thus

from def ② and ③

$$P(\mathcal{C}) = P(C \cup C^*)$$

$$P(\mathcal{C}) = P(C) + P(C^*)$$

$$1 = P(C) + P(C^*)$$

$$P(C^*) = 1 - P(C) \quad \#$$

$\boxed{\text{Th 2}}$ $P(\phi) = 0$

$C = \phi$ so that $C^* = \mathcal{C}$

by Th1 $P(\phi) = 1 - P(C^*) = 1 - \underbrace{P(\mathcal{C})}_{\text{def ①}} = 1 - 1 = \boxed{0}$ #

$\boxed{\text{Th 3}}$ $C_1 \subseteq C_2$ then $p(C_1) \leq p(C_2)$

proof: العلاقة

$$C_2 = C_1 \cup (C_2 \cap C_1^*)$$

bey def ⓑ : $C_1 \cap (C_2 \cap C_1^*) = \phi$

$$p(C_2) = p(C_1) + \underbrace{p(C_2 \cap C_1^*)}_{\geq 0}$$

but $p(C_2 \cap C_1^*) \geq 0$ by def part (a)

$\therefore p(C_2) \geq p(C_1)$

$\Rightarrow$

$\boxed{\text{Thum 4}}$ $0 \le p(C) \le 1 \quad \forall C \subseteq \mathcal{C}$

Proof: $\phi \subseteq C \subseteq \mathcal{C}$ we have by $\boxed{\text{Th 3}}$ that $p(\phi) \le p(C) \le p(\mathcal{C})$ so

$$\underbrace{0}_{\text{by Th (2)}} \le p(C) \le \underbrace{1}_{\text{by def part (c)}}$$

1.4 Conditinal prob. and Independence

Def: let C_1, C_2 be events in $\mathcal{C}$.

* The Conditional prob. of C_1 given C_2 is defined as:

$$p(C_1/C_2) = \frac{p(C_1 \cap C_2)}{p(C_2)}, \quad p(C_2) > 0$$

* The Conditional prob. of C_2 given C_1 is defined as:

$$p(C_2/C_1) = \frac{p(C_1 \cap C_2)}{p(C_1)}, \quad p(C_1) > 0$$

Note: for any event E given C in $\mathcal{C}$

1. $p(E/C) \ge 0$ for any $E \subseteq \mathcal{C}$
2. $p\left(\bigcup_{i=1}^{\infty} E_i/C\right) = \sum_{i=1}^{\infty} p(E_i/C)$ for any disjoint E_i
3. $p(\mathcal{C}/C) = 1$

we say $p(\,\cdot\,/C)$ is a prob. set function.
(الي بتوخذ اي اشي)

Try to prove the above 3 statments.

$\Rightarrow$

prop. multiplication law

$p(C_1 \cap C_2) = p(C_1 / C_2) \cdot p(C_2) \quad , \quad P(C_2) > 0$

$P(C_1 \cap C_2) = p(C_2 / C_1) \cdot p(C_1) \quad , \quad p(C_1) > 0$

prop. mutiplication law

$p(C_1 \cap C_2 \cap C_3) = p(C_1) \cdot p(C_2 / C_1) \cdot p(C_3 / C_1 \cap C_2)$

$p(C_1) \neq 0 \quad , \quad p(C_1 \cap C_2) \neq 0$

Proof:

$$*\, p(C_3 / C_1 \cap C_2) = \frac{P(C_1 \cap C_2 \cap C_3)}{P(C_1 \cap C_2)} \quad , \quad P(C_1 \cap C_2) \neq 0$$

$$p(C_2 / C_1) = \frac{P(C_1 \cap C_2)}{P(1)} \quad , \quad P(C_1) \neq 0$$

$p(C_1 \cap C_2 \cap C_3) = p(C_3 / C_1 \cap C_2) \cdot p(C_1 \cap C_2) \quad , \quad p(C_1 \cap C_2) \neq 0$

$= p(C_3 / C_1 \cap C_2) \cdot p(C_2 / C_1) \cdot p(C_1) \quad , \quad p(C_1 \cap C_2) \neq 0$

$p(C_1) \neq 0$

Def:

$C_1, C_2, \ldots, C_K$ events in $\mathcal{C}$

① $\bigcup_{i=1}^{K} C_i = \mathcal{C}$

② $C_i \cap C_j = \Phi \quad , \quad i \neq j$

③ $P(C_i) \neq 0$ for all i

we say $C_1, \ldots, C_K$ form a partition for $\mathcal{C}$

$\Rightarrow$

prop. law of Total prob.

$$P(A) = \sum_{j=1}^{k} P(A/C_j) \cdot P(C_j)$$

$C_1, C_2, C_3, \ldots, C_k$ partition for e

proof:

$$A = (A \cap C_1) \cup (A \cap C_2) \cup \ldots \cup (A \cap C_k)$$

Since $C_1, \ldots, C_k$ partitions for e

$$(A \cap C_i) \cap (A \cap C_i) = \Phi \quad \text{for } i \neq j$$

$$P(A) = P\left(\bigcup_{j=1}^{k} (A \cap C_j)\right) = \sum_{j=1}^{k} P(A \cap C_j)$$

$$P(A \cap C_j) = P(A/C_j) \cdot P(C_j)$$

$$\boxed{P(A) = \sum_{j=1}^{k} P(A/C_j) \cdot P(C_j)}$$

Theorm Bay's Theorem

$C_1, C_2, \ldots, C_k$ partition for e

$$P(C_j/A) = \frac{P(A/C_j) \cdot P(C_j)}{\sum_{i=1}^{k} P(A/C_i) \cdot P(C_i)}, \quad P(A) \neq 0$$

proof:

$$P(C_j/A) = \frac{P(C_j \cap A)}{P(A)} \quad P(A) \neq 0$$

$$= \frac{P(A/C_j) \cdot P(C_j)}{\sum_{i=1}^{k} P(A/C_i) \cdot P(C_i)}$$

$$\Rightarrow$$

$$P(A/B) = \frac{P(B/A)\, P(A)}{P(B)}$$

Def: C_1, C_2 are independent events if

$p(C_1/C_2) = p(C_1)$, $p(C_2) \neq 0$

$p(C_2/C_1) = p(C_2)$, $p(C_1) \neq 0$

proposition (multiplication law for independent events)

C_1, C_2 are independent events then:

$p(C_1 \cap C_2) = p(C_1) \cdot p(C_2)$, $p(C_1) \neq 0$
$p(C_2) \neq 0$

proof :-

$$p(C_1/C_2) = \frac{p(C_1 \cap C_2)}{p(C_2)} \quad , p(C_2) \neq 0$$

$p(C_1 \cap C_2) = p(C_1/C_2) \cdot p(C_2)$, $p(C_2) \neq 0$

similarly , $p(C_1 \cap C_2) = p(C_2/C_1) \cdot p(C_1)$, $p(C_1) \neq 0$

Now use C_1, C_2 independent , then

$$\begin{aligned} p(C_1 \cap C_2) &= p(C_1) \cdot p(C_2) && , p(C_2) \neq 0 \\ &= p(C_2) \cdot p(C_1) && , p(C_1) \neq 0 \\ &= p(C_1) \cdot p(C_2) && , p(C_1) \neq 0 , p(C_2) \neq 0 \end{aligned}$$

Def: C_1, C_2, C_3 events in $\mathcal{C}$

1) C_1, C_2, C_3 are pair wise independent

if: $p(C_1 \cap C_2) = p(C_1) \cdot p(C_2)$

$p(C_1 \cap C_3) = p(C_1) \cdot p(C_3)$

$p(C_2 \cap C_3) = p(C_2) \cdot p(C_3)$

where $p(C_1), p(C_2), p(C_3)$ are all nonzero

② C_1, C_2, C_3 are mutually independent if

$$p(C_1 \cap C_2 \cap C_3) = p(C_1) \cdot p(C_2) \cdot p(C_3)$$

where $p(C_1)$, $p(C_2)$, $p(C_3)$ are all nonzero

Def: $C_1, \ldots, C_n$ events in $\mathscr{C}$

where $p(C_i) \neq 0 \quad \forall i = 1, \ldots, n$

① $C_1, \ldots, C_n$ pair wise independent

$$p(C_i \cap C_j) = p(C_i)\, p(C_j), \quad i \neq j$$

② $C_1, \ldots, C_n$ mutually independent if

$$p(C_{d_1} \cap \ldots \cap C_{d_k}) = p(C_{d_1}) \cdot p(C_{d_2}) \ldots p(C_{d_k})$$

where $\{d_1, \ldots, d_k\} \subseteq \{1, \ldots, n\}$

Note mutual independence $\Rightarrow$ pairwise independence

Note $C_1, \ldots, C_n$ indep: means mutually independent.

Example 1

5 cards are taken at random from an ordinary deck of 52 cards without replacement. The prob. of an all-spade hand relative to the hypothesis that there is at least 4 spades in the hand.

E_1: event of all-spade hand

E_2: event of all spade

$E_1 \subseteq E_2$

$E_1 \cap E_2 = E_1$

$$p(E_1 / E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)} = \frac{p(E_1)}{p(E_2)}$$

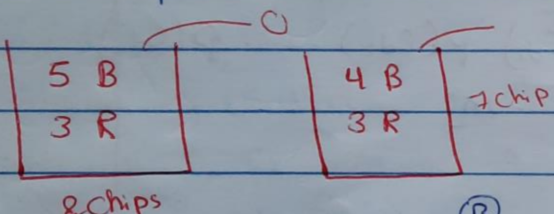
→

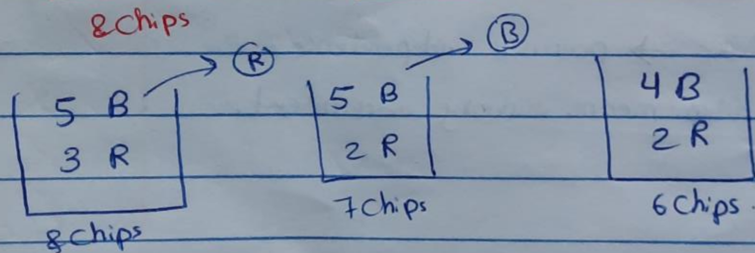
$$\Rightarrow = \frac{\dfrac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}}{\left[\dfrac{\binom{13}{4}\binom{39}{1}}{\binom{52}{5}} + \dfrac{\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}\right]} = \frac{3}{68} = 0.0441$$

Ex 2

Two chips are drawn successively at random without replacment from a bowl with 8 chips (5 blue and 3 red)

Find the prob. that the first is red that the second is blue





E_1: event 1^{st} ball is red (R)

E_2: event 2^{nd} ball is blue (B)

$$p(E_1 \cap E_2) = p(E_2 / E_1) \cdot p(E_1)$$

$$= \left(\frac{5}{7}\right) \cdot \left(\frac{3}{8}\right) = \frac{15}{56} = 0.2679$$

$$p(E_1 / E_2) = ?$$

Ex 3

Bowl I	Bowl II
3 R	8 R
7 B	2 B

A die is Cast   5, 6 → Bowl I

1, 2, 3, 4 → Bowl II

A red ball is taken

(1) find the prob. that we chose bowl I given we get a red ball

(2) " " " " " " " II " " " "

R: event having a red ball

I: " choosing bowl I

II: " " " II

(1) $$p(I/R) = \frac{p(I \cap R)}{p(R)} = \frac{p(R/I) \cdot p(I)}{p(R/I) \cdot p(I) + p(R/II) \cdot p(II)}$$

$$= \frac{\left(\frac{3}{10}\right)\left(\frac{2}{6}\right)}{\left(\frac{3}{10}\right)\left(\frac{2}{6}\right) + \left(\frac{8}{10}\right)\left(\frac{4}{6}\right)} = \frac{3}{19}$$

(2) $$p(II/R) = \frac{p(R/II) \cdot p(II)}{p(R/II) \cdot p(II) + p(R/I) \cdot p(I)}$$

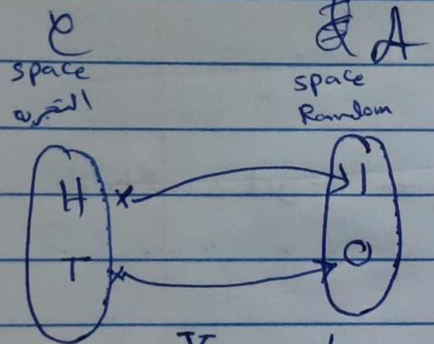
$$\frac{\left(\frac{8}{10}\right)\left(\frac{4}{6}\right)}{\left(\frac{8}{10}\right)\left(\frac{4}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{10}\right)} = \frac{\frac{32}{60}}{\frac{38}{60}} = \frac{32 \div 2}{38 \div 2} = \boxed{\frac{16}{19}}$$

1.5 Random Variables of Discret Type

1.6 Random Variabls of Continuous Type

Ex : Toss a Coin

$\mathcal{C} = \{H, T\}$ $p(H) = 0.5$ $p(T) = 0.5$



$\mathcal{C} \xrightarrow{X} \mathcal{A}$

$X : \mathcal{C} \to \mathcal{A}$

$X(c) = a = x$

$X(H) = 1$ $X = 0$

X : number of heads

بطريقة اخرى

$$X(c) = \begin{cases} 1 & , c = H \\ 0 & , c = T \end{cases}$$

Def let $\mathcal{C}$ be a sample space of an experiment A function X, which assigns to element $c \in \mathcal{C}$ one and only one real number $X(c) = x$, is called a random variable.

The space of X is the set of real numbers.

$$\mathcal{A} = \{x \in \mathbb{R} : x = X(c),\ c \in \mathcal{C}\}$$

Note: ممكن يكون محدود أو لا

$\mathcal{A}$ discrict set (Countable set) , X discrete r.v

$\mathcal{A}$ Continouse set (unCountable set) , X Continous r.v

~~Def~~

الافتراض الاول انه coin ← fair

Ex: Toss a Coin twice

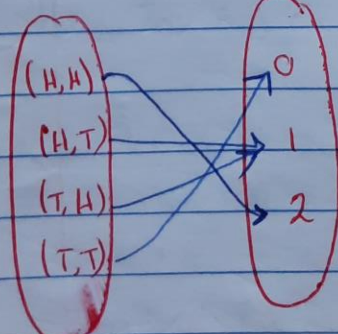
$\mathcal{C} = \{(H,H), (H,T), (T,H), (T,T)\}$

X : number of heads

$\mathcal{A} = \{0, 1, 2\}$

$\mathcal{A}$ discreate set, X discreate r.v

$\mathcal{C} \xrightarrow{X} \mathcal{A}$ → onto بطلع بهاي الطريقة.



Assume:

1. fair Coin
2. Tosses are independent

$P(H,H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(H,T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(T,H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(T,T) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

c	(H,H)	(H,T)	(T,H)	(T,T)	Total
$P(c)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

$P_r(X=0) = P_{\mathcal{C}}(T,T) = \frac{1}{4}$

$P_r(X=2) = P(H,H) = \frac{1}{4}$

$P_r(X=1) = P(H,T) + P(T,H) = \frac{1}{2}$

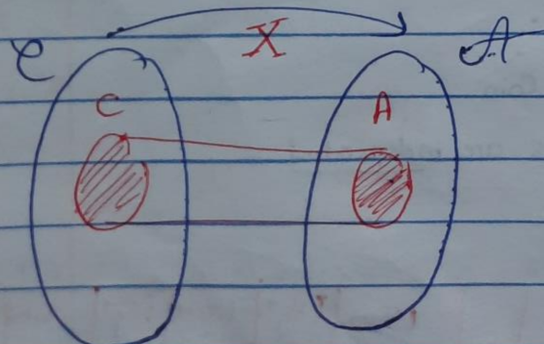
x	0	1	2	Total
$P_r(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

<u>Def</u> let P be a prob. set function of the space $\mathcal{C}$. Let X be a random variable from $\mathcal{C}$ to $\mathcal{A}$.
The <u>induced probability</u> P_X is defined on $\mathcal{A}$ as follows:

$P_X(A) = P_r(X \in A)$

[illegible] $= P(C)$

$A \subseteq \mathcal{A}$, $C \subseteq \mathcal{C}$, $X(C) = A$



$\Rightarrow$

<u>proposition:</u> P_X is a prob. set function.

<u>proof</u> Try to write it.

<u>Def</u>

let X be a discrete random variable.

let $f(x)$ be a function such that:

$f(x) > 0 \quad , \quad x \in A$ ، استر

$f(x) = 0 \quad , \quad x \in \mathbb{R} \setminus A$

$(f(x) \geq 0 \, , \; x \in \mathbb{R})$

and $\sum_{x \in \mathbb{R}} f(x) = \sum_{x \in A} f(x) = 1$

where

$p(A) = \sum_{x \in A} f(x) \quad , \; A \subseteq \mathbb{R}$

then we say $f(x)$ is a prob. density function (p.d.f

<u>Def</u> let X be a continuous random variable from $\mathcal{C}$ to A, where A is a continous set. let $f(x)$ be a function such that:

* $f(x) > 0$ for all $x \in A$

$f(x) = 0$ for all $x \in \mathbb{R} / A$

$(f(x) \geq 0$ for all $x \in \mathbb{R})$

* $\int_{-\infty}^{\infty} f(x) \, dx = \int_{\mathbb{R}} f(x) \, dx = \int_{A} f(x) \, dx = 1$

where $P(A) = \int_A f(x)\,dx$

Then we say that $f(x)$ is a prob. density function (p.d.f)

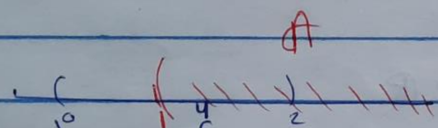
Ex: X is a random variable with the following p.d.f

$$f(x) = \begin{cases} \frac{2}{x^3}, & 1 < x < \infty \\ 0, & \text{else} \end{cases}$$

(1) what is A?

$A = [x \in \mathbb{R} : x \in (1, \infty)] = (1, \infty)$

note: X is a cont r.v



(2) $P_r(X = 4) = \int_4^4 f(x)\,dx = 0$

(3) $P_r(X = 0, 1, 2, 3, 4, \ldots) = \int_0^0 f(x)\,dx + \int_1^1 f(x)\,dx + \ldots = 0$

(4) $P_r(0 \leq X < 2) = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

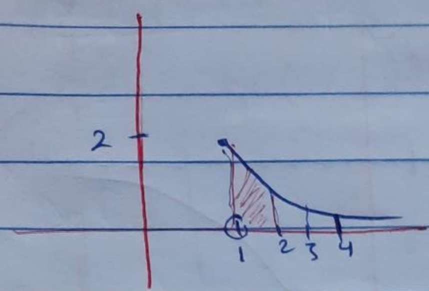
$= P_r(0 < X \leq 2)$

$= P_r(0 < X < 2)$

$= P_r(0 \leq X \leq 2)$

$0 + \int_1^2 \frac{2}{x^3}\,dx = 2\frac{x^{-2}}{-2}\Big|_1^2$

$= \frac{1}{x^2}\Big|_2^1 = 1 - \frac{1}{4} = \frac{3}{4}$



<u>Def</u>

let X be a r.v from $\mathcal{C}$ to $\mathbb{R}$

let $f(x)$ be the p.d.f of X

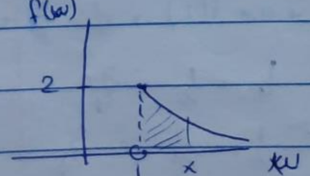
we define the Cumulative Distribution Function (CDF) or Distribution function as Follows:

$$F(x) = P_r(X \leq x)$$
$$= P_r(X \in (-\infty, x])$$

<u>Note</u>: $F(x) = \begin{cases} \sum_{w \leq x} f(w) & , X \text{ discrete r.v} \\ \int_{-\infty}^{x} f(w)\,dw & , X \text{ Continuous r.v} \end{cases}$

Ex: [illegible]

Ex (Continued) $f(w) = \begin{cases} \frac{2}{w^3} & , 1 < w < \infty \\ 0 & , \text{else} \end{cases}$



(3) Find the CDF of r.v X

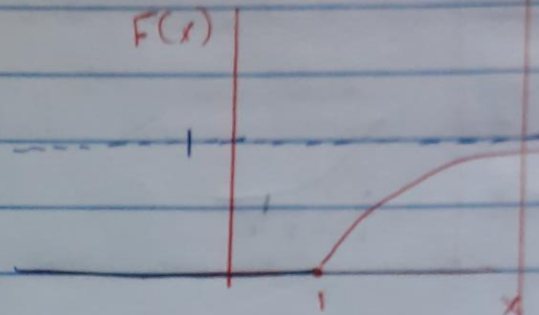
$F(x) = P_r(X \leq x) = \int_{-\infty}^{x} f(w)\,dw$

$\Rightarrow$

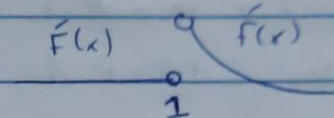
$$= \begin{cases} 0 & , \; x \le 1 \\ \int_1^x \frac{2}{w^3}\,dw & , \; x > 1 \end{cases}$$

$$= \left. \frac{-1}{w^2} \right|_1^x$$

$$= \left. \frac{1}{w^2} \right|_x^1 = \frac{1}{x^2}$$



$$F'(x) = \frac{dF(x)}{dx} = \begin{cases} 0 & , \; x < 1 \\ \text{DNE} & , \; x = 1 \\ +\frac{2}{x^3} & , \; x > 1 \end{cases}$$



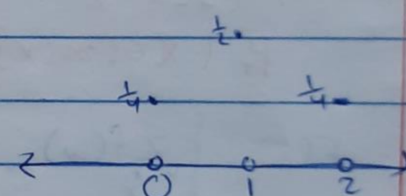
~~DNE: Doesn't exist~~

Note: $F'(x) = f(x)$ almost everywhere

$\mathcal{X} = \{x \in \mathbb{R} : x > 1\} = (1, \infty)$

Ex:

$$f(x) = \begin{cases} \frac{1}{4} & , \; \tilde{x} = 0 \\ \frac{1}{2} & , \; \tilde{x} = 1 \\ \frac{1}{4} & , \; \tilde{x} = 2 \\ 0 & , \; \text{else} \end{cases}$$



① $P_r(X = 4) = \sum_{x \in \{4\}} f(x) = f(4) = 0$

② $Pr(0 \le X \le 2) = \sum_{x \in [0,2]} f(x) = f(0) + 0 + f(1) + 0 + f(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = \boxed{1}$

③ $pr(0 \le x < 2) = \sum_{x \in [0,2)} f(x) = f(0) + f(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

④ $P_r(0 < x \le 2) = \sum_{x \in (0,2]} f(x) = f(1) + f(2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

⑤ $pr(0 < x < 2) = \sum_{x \in (0,2)} f(x) = f(1) = \frac{1}{2}$

$\Rightarrow$

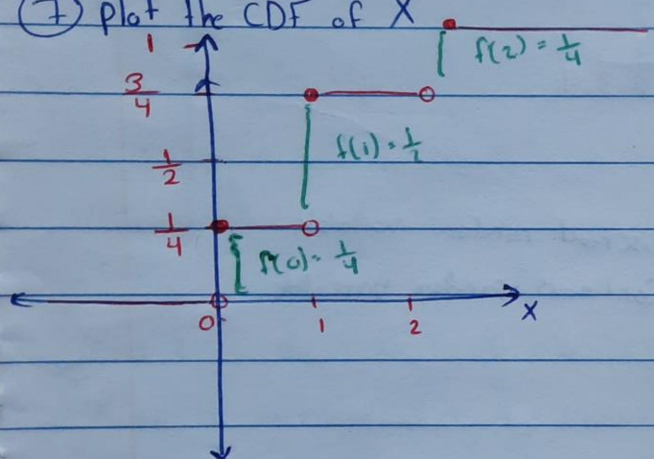
⑥ Find CDF of X

$F_X(x) = P_r(X \le x) = \sum_{w \le x} f(x) =$

اذا في اكثر من random variable

$$\begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{3}{4}, & 1 \le x < 2 \\ 1, & 2 \le x \end{cases}$$

⑦ Plot the CDF of X



Step function

nondecreasing

1.7 properties of the Distribution Function

X : real-valued random variable.

Recall $F(x) = P_r(X \le x)$

$= P_r(X \in (-\infty, x])$

property 1 : $0 \le F(x) \le 1$

property 2 : F is non-decreasing

That is,

$a < b \Rightarrow F(a) \le F(b)$

$\Rightarrow$

property 3: $F(\infty) = 1$, $F(-\infty) = 0$

where $F(\infty) = \lim_{x \to \infty} F(x)$

$F(-\infty) = \lim_{x \to -\infty} F(x)$

property 4: F is right continuous

$\lim_{x \to a^+} F(x) = F(a)$, $a \in R$

$F(a+) = F(a)$, $a \in R$

Note: Try to prove properties 1, 2, 3, 4

proposition:

$$F(b) - F(-b) = \begin{cases} f(b) , & X \text{ discreat random variable} \\ 0 , & X \text{ Continous random variable} \end{cases}$$

Note: Try to prove it

proposition:

X Continous random Variable

$F'(x) = f(x)$

where x is a point of Continuity of $f(x)$

Note: Try to prove it

$\Rightarrow$

المطلوب عادة عن pdf سؤال مختلف

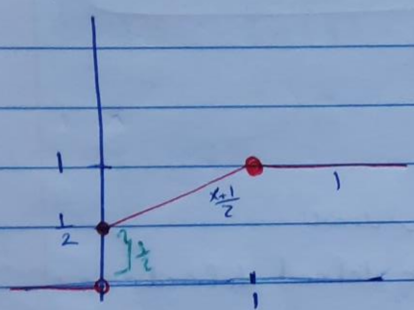
Ex: لا حول من CDF ← pdf بنشتق بالعكس بكامل

X is a random variable with CDF

$$F(x) = \begin{cases} 0 & , \; x < 0 \\ \frac{x+1}{2} & , \; 0 \le x < 1 \\ 1 & , \; x \ge 1 \end{cases}$$

ممكن تكون دمج

① plot $F(x)$



Domain $(-\infty, \infty)$

Function $(0, 1]$

nondecreasing

right continous

لا هو continous ولا step function

② what is the type of r.v X?

X is a mixture of both discrete and continous

This type is called Censored r.v

$$P_r(X=a) = \begin{cases} f(a) = F(a) - F(\cancel{a}^-) & , \; X \text{ discret} \\ 0 & , \; X \text{ continous} \end{cases}$$

③ $P_r(X=0) = F(0) - F(0^-)$

$= \frac{1}{2} - 0 = \frac{1}{2}$

$\Rightarrow$

④ $P_r(X=1) = 0$

⑤ $P_r(-3 \leq X \leq \frac{1}{2})$

proposition.

$$\left.\begin{array}{l} F(b) = P_r(X \leq b) \\ F(a) = P_r(X \leq a) \\ a < b \end{array}\right\} \Rightarrow F(b) - F(a) = P_r(a < X \leq b)$$

$= F(\frac{1}{2}) - F(-3)$

$= \dfrac{\frac{1}{2}+1}{2} - 0 = \boxed{\dfrac{3}{4}}$

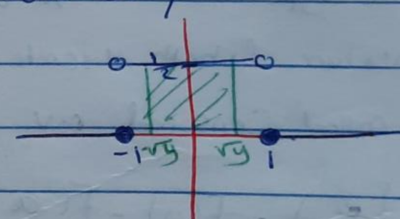
Transformation of r.v ثابتة على فترة ومعرفة

Ex: Let X be a uniform random variable defined defined on the interval $[-1, 1]$

Let $Y = X^2$ Find CDF, p.d.f of Y

Solution:

$$f(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \quad \frac{1}{b-a} \\ 0, & \text{else where} \end{cases}$$



X	x	F	f
Y	y	G	g
$G = F_y$ (CDF)		$g = f_y$ (Pdf)	
$F = F_x$		$f = f_x$	

أحولها لـ X لأنه بتساوي Y

$G(y) = P_r(Y \leq y) \xrightarrow{\text{Contious}} = P_r(X^2 \leq y)$

$P_r(-\sqrt{y} \leq X \leq \sqrt{y})$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx \Rightarrow$$

$$= \begin{cases} 0 & , y < 0 \\ \displaystyle\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2}\, dx = \frac{1}{2} x \Big|_{-\sqrt{y}}^{\sqrt{y}} = \sqrt{y} & , 0 \le y < 1 \\ \displaystyle\int_{-\sqrt{y}}^{-1} 0\, dx + \int_{-1}^{1} \frac{1}{2}\, dx + \int_{1}^{\sqrt{y}} 0\, dx = 1 & , 1 \le y \end{cases}$$

CDF

$$F_Y(y) = G(y) = \begin{cases} 0 & , y < 0 \\ \sqrt{y} & , 0 \le y < 1 \\ 1 & , 1 \le y \end{cases}$$

لما اشتقهم بعطيني DNE

Pdf

$$f_Y(y) = g(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & , 0 < y < 1 \\ 0 & , \text{elsewhere} \end{cases}$$

Note: $G'(y)$ DNE where $y = 0, 1$

1.8 Expectation of a random variable

Recall

Sample mean : $\bar{x} = \frac{\sum x_i}{n}$

weighted mean: $\bar{x} = \frac{\sum x_i f_i}{\sum f_i}$

Expected value $\mu = \sum_x X . f(x)$

Def :

let X be a random variable with pdf $f(x)$. The expectation /mathematical expedation /expected value is defined as :

$$E(X) = \begin{cases} \sum_x x f(x) & , X \text{ discrete r.v} \\ \int_{\mathbb{R}} x f(x)\, dx & , X \text{ Continous r.v} \end{cases}$$

Provided that

$$\begin{cases} \sum_x |x| f(x) \text{ Converges} & , X \text{ discrete r.v} \\ \int |x| f(x)\, dx \text{ Converges} & , X \text{ Continous r.v} \end{cases}$$

ما بنفحصهم إلا إذا السؤال طلب.

proposition 1

$$E(u(x)) = \begin{cases} \sum_{x=-\infty}^{\infty} u(x) f(x) & , X \text{ discrete r.v} \\ \int_{-\infty}^{\infty} u(x) f(x)\, dx & , X \text{ continuous r.v} \end{cases}$$

Proof : Not required.

proposition 2   X r.v with pdf f(x)

① $E(\underbrace{K}_{\text{constant}}) = K$, for any $K \in \mathbb{R}$

② $E(K\, v(x)) = K\, E(v(x))$, for any $K \in \mathbb{R}$
for any function v

③ $E\left(\sum_{i=1}^{m} k_i v_i(x)\right) = \sum_{i=1}^{m} k_i E(v_i(x))$, for any $k_1, \dots, k_m \in \mathbb{R}$
for any functions $v_1, \dots, v_m$

proof : Try to write it

Ex : X is a r.v with pdf

$$f(x) = \begin{cases} 4x^3 & , 0 < x < K \\ 0 & , \text{else where} \end{cases}$$

$A = (0, K)$, $K > 0$
X Cont. r.v

① Find K

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{K} 4x^3\, dx + \int_{K}^{\infty} 0\, dx = 1$$

$$\Rightarrow x^4 \Big|_0^K = 1 \quad \Rightarrow K^4 = 1 \quad \Rightarrow K^4 - 1 = 0$$

$\Rightarrow (k^2-1)(k^2+1)=0$

$\Rightarrow k^2-1=0$

$\Rightarrow (k-1)(k+1)=0$

$\Rightarrow k=-1$ or $k=+1$

$\Rightarrow \boxed{k=1}$ $(k>0)$

② $E(x)$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{k} x(4x^3)\, dx = \int_{0}^{k} 4x^4\, dx$$

$$= \frac{4x^5}{5}\Big|_{0}^{k} = \frac{4k^5}{5} \quad \text{But } k=1$$

$$E(x) = \frac{4}{5}$$

Ex:

let X be a r.v with p.d.f

x	1	2	3	4	Total
$f(x)$	$\frac{4}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	

$A = \{1, 2, 3, 4\}$, X discrete r.v

① $E(x) = \sum_{x \in A} x \cdot f(x) = (1)\left(\frac{4}{10}\right) + (2)\left(\frac{1}{10}\right) + (3)\left(\frac{3}{10}\right)$

$+ (4)\left(\frac{2}{10}\right) = \frac{23}{10} = \boxed{2.3}$

$\Rightarrow$

② $E(x^2) = \sum_{x \in R} x^2 f(x)$

$= (1)^2\left(\frac{4}{10}\right) + (2)^2\left(\frac{1}{10}\right) + (3)^2\left(\frac{3}{10}\right) + (4)^2\left(\frac{2}{10}\right)$

$\frac{67}{10} = \boxed{6.7}$

③ $E(7X + 2X^2)$ بس الجمع بتوزع

$= E(7X) + E(2X^2)$

$= 7E(X) + 2E(X^2) = (7)\left(\frac{23}{10}\right) + (2)\left(\frac{67}{10}\right)$

$= \boxed{29.5}$

1.9 Some special Expectation

Def

let X be a r.v with p.d.f $f(x)$

① The mean value of X is defined as: $\mu = E(X)$

② The variance of X is defined as: $\sigma^2 = var(X) = E[(X - \mu)^2]$

③ The standard deviation of X is defined as:

$\sigma = \sqrt{\sigma^2} = \sqrt{var(X)}$

Note: $E(X)$ 1^{st} moment

$E(X^2)$ 2^{nd} moment

⋮

$E(e^{tX})$ exists for $-h < t < h$

$\Rightarrow M(t) = E(e^{tX})$ m.g.f

proposition:

$$Var(X) = E(X^2) - (E(X))^2$$

$$\sigma^2 = E(X^2) - \mu^2$$

proof:

$$Var(X) = E((X-\mu)^2)$$

$$= E(X^2 - 2X\mu + \mu^2)$$

$$= E(X^2) + E(-2X\mu) + E(\mu^2)$$

$$Var(X) = E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu\mu + \mu^2$$

$$= E(X^2) - \mu^2$$

Def $\sigma^2 = E[(X-\mu)^2]$

prop $\sigma^2 = E(X^2) - \mu^2$

Def let X be a random variable with p.d.f $f(x)$. suppose that there is a positive number h such that for $-h<t<h$ the expectation

$$E(e^{tX}) = \begin{cases} \sum_{X \in R} e^{tX} f(X) & , X \text{ discrete r.v} \\ \int e^{tx} f(x)\,dx & , X \text{ continous r.v} \end{cases}$$

exists

Then, this expectation is called The moment-generating function m.g.f of X and is denoted by

$$M(t) = E(e^{tX})$$

Example let X be a r.v with p.d.f $f(x)$

X	1	2	3	4	Sum
$f(x)$	0.1	0.2	0.3	0.4	1

$A = \{1, 2, 3, 4\}$

Find the m.g.f

A discrete $\Rightarrow X$ discrete

Solution $E(e^{tx}) = \sum_{x \in \mathbb{R}} e^{tx} . f(x) =$

$e^{t.1} f(1) + e^{t.2} f(2) + e^{t.3} f(3) + e^{t.4} f(4)$

$= 0.1 e^{t} + 0.2 e^{2t} + 0.3 e^{3t} + 0.4 e^{4t}, \; t \in \mathbb{R}$

Ex: Let X be a r.v with p.d.f $= f(x) = \begin{cases} e^{-x}, & x > 0 \\ \text{Zero}, & \text{else where} \end{cases}$

Find the m.g.f of X $\quad A = (0, \infty)$

Continous r.v

Solution:

$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{0}^{\infty} e^{tx} e^{-x}\, dx = \int_{0}^{\infty} e^{(t-1)x}\, dx =$

$\left. \frac{e^{(t-1)x}}{(t-1)} \right|_{0}^{\infty}$ $\quad t < 1 \quad = \lim_{x \to \infty} \frac{e^{(t-1)x}}{(t-1)} - \frac{e^{0}}{t-1}$

$0 - \frac{1}{t-1} = \boxed{\frac{1}{1-t}}$

$X \sim f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{else} \end{cases}$ $\quad E(e^{tx}) = \frac{1}{1-t}, \quad t < 1$

m.g.f : $M(t) = \frac{1}{1-t}$, $t < 1$

Note: if $E(e^{tx})$ exist for $t < -1$

$\Rightarrow$ m.g.f doesn't exist

proposition let X be a r.v with p.d.f $f(x)$, and m.g.f $M(t)$. Then:

① The derivatives of $M(t)$ of all orders exist at $t=0$

② $M^{(k)}(0) = \frac{d^k M(t)}{dt^k}\Big|_{t=0} = E(X^k)$, moment لما تشتقه وحط 0

$k=1, 2, \ldots$

proof

$M(t) = E(e^{tx})$, $-h < t < h$

$\frac{dM(t)}{dt} = M'(t) = \frac{d}{dt} E(e^{tx})$, X Cont.

$$= \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\right) f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x e^{tx} f(x)\, dx$$ عوض المشتقة عن 0

$$M'(0) = \frac{dM(t)}{dt}\Big|_{t=0} = \int_{-\infty}^{\infty} x f(x)\, dx = E(x)$$

$$\frac{d^2 M(t)}{dt^2} = M''(t) = (M'(t))' = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dt} e^{tx}\right) x f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} x^2 f(x)\, dx$$

$\Rightarrow$

$$M''(0) = \frac{d^2 M(t)}{dt^2}\Big|_{t=0} = \int_{-\infty}^{\infty} x^2 f(x)\, dx = E(x^2)$$

and so we can show:

$$M^{(k)}(0) = \frac{d^k M(t)}{dt^k}\Big|_{t=0} = E(X^k) \text{ , } \cancel{k=1,2,}\dots$$

Similary, we can show the same results if X is discreate

Note:

$M(t)$ exists

1. $\mu = E(x) = M'(0)$
2. $\sigma^2 = M''(0) - (M'(0))^2$
3. $M(0) = 1$

$M(t) = E(e^{tx})$

$M(0) = E(e^{0x}) = E(e^0) = E(1) = 1$

Ex:

X has p.d.f

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{else} \end{cases}$$

Find $E(x^3)$ — اذا فش m.g.f احنا فعلياً ورطنا

$$E(x^3) = \int_{-\infty}^{\infty} x^3 f(x)\, dx = \int_{0}^{\infty} x^3 e^{-x}\, dx = \dots$$

But, using m.g.f

$$M(t) = \frac{1}{1-t}, \quad t < 1$$

$\Rightarrow$

$M(t) = (1-t)^{-1}$, $t<1$

$M'(t) = +(1-t)^{-2}$, $t<1$

$M''(t) = +2(1-t)^{-3}$, $t<1$

$M^{(3)}(t) = +6(1-t)^{-4}$, $t<1$

$\mu = M'(0) = 1$

$\sigma^2 = (M''(0)) - (M'(0))^2 = 2 - (1)^2 = 1$ $\qquad \sigma = 1$

$E(X^3) = M^{(3)}(0) = 6$

* Check Example 3 and Example 4 page 63.
* $E(e^{itX}) = \varphi(t)$ characteristic function. ← مش معنا

1.10 : Chebyshev's Inequality

Theorem 6

let $u(X)$ be a non-negative function of the random variable X

If $E(u(x))$ exists, then, for every positive constant c,

$$pr(u(x) \geq c) \leq \frac{E(u(x))}{c}$$

Theorem 7 Chebyshev's Inequality

let X be a random variable with finite variance σ^2.

Then for every $k>0$, $Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$

or $pr(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

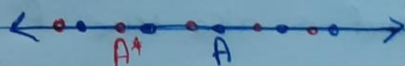
proof of Th 6

let X be a continuous r.v with p.d.f. $f(x)$

$$E[u(x)] = \int_{-\infty}^{\infty} u(x) f(x)\, dx \quad \text{(exists)}$$

$\mathbb{R} = A \cup A^*$ where $A = \{x \in \mathbb{R} : u(x) \geq c\}$

and $u(x) \geq 0$ and $c > 0$



$$E[u(x)] = \int_{\mathbb{R}} u(x) f(x)\, dx = \int_{A} u(x) f(x)\, dx + \int_{A^*} u(x) f(x)\, dx$$

$u(x) \geq 0$ for all $x \in \mathbb{R}$

$u(x) \geq 0$ for all $x \in A^*$

$f(x) \geq 0$ for all $x \in \mathbb{R}$

$f(x) \geq 0$ for all $x \in A^*$

$\Rightarrow u(x) \cdot f(x) \geq 0$ for all $x \in A^*$

$\Rightarrow \int_{A^*} u(x) f(x)\, dx \geq 0$

$$\Rightarrow \boxed{E[u(x)] \geq \int_{A} u(x) f(x)\, dx}$$

$A = \{x \in \mathbb{R} : u(x) \geq c\}$

$u(x) \geq c$ for all $x \in A$

$u(x) f(x) \geq c f(x)$ for all $x \in A$ $\quad (f(x) \geq 0$ for all $x \in A)$

$$\int_{A} u(x) f(x)\, dx \geq \int_{A} c\, f(x)\, dx = c \int_{A} \underbrace{f(x)\, dx}_{\text{probability}} = c\, p(A)$$

$$\boxed{\int_{A} u(x) f(x)\, dx \geq c\, pr(u(x) \geq c)}$$

$\Rightarrow$

$$\Rightarrow E[u(x)] \geq \int_A u(x) f(x)\, dx \geq c\, p(u(x) \geq c)$$

$$\Rightarrow E[u(x)] \geq c\, Pr(u(x) \geq c)$$

$$\Rightarrow \frac{E[u(x)]}{c} \geq pr(u(x) \geq c) \qquad (c > 0)$$

$$\Rightarrow \boxed{pr(u(x) \geq c) \leq \frac{E[u(x)]}{c}}$$

#

Similary , we can show that the proof is also true if X is a discreat r.v

Theorem 7 chebyshev's Inequality

let X be a r.v with mean μ and variance σ^2. Then for every $k > 0$

$$pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

بدي اثبت النظرية 7

or

$$pr(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

باستخدام النظرية 6

proof of Th 7

Th 6 $\rightsquigarrow pr(u(x) \geq c) \leq \frac{E(u(x))}{c}$, $c > 0$

let $u(x) = (x - \mu)^2$ and let $c = k^2\sigma^2$

using Th 6 $\rightarrow pr((x-\mu)^2 \geq k^2\sigma^2) \leq \frac{E((x-\mu)^2)}{k^2\sigma^2}$

$x^2 \geq y^2 \Rightarrow |x| \geq |y|$

$$\Rightarrow pr(|x - \mu| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

$$\Rightarrow \boxed{pr(|x - \mu| \geq k\sigma) \leq \frac{1}{k^2}} \qquad k > 0$$

$\Rightarrow$

$\Rightarrow 1 - pr\left(|X-\mu| < k\sigma\right) \leq \frac{1}{k^2}$

$\Rightarrow 1 - \frac{1}{k^2} \leq pr\left(|X-\mu| < k\sigma\right)$

$\Rightarrow \boxed{pr\left(|X-\mu| < k\sigma\right) \geq 1 - \frac{1}{k^2}}$, $k > 0$

<u>Example</u>

X r.v with p.d.f. $f(x) = \begin{cases} \frac{1}{2\sqrt{3}} & , -\sqrt{3} < X < \sqrt{3} \\ 0 & , \text{else} \end{cases}$

① Find μ

$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} x \frac{1}{2\sqrt{3}}\, dx = 0$ (check)

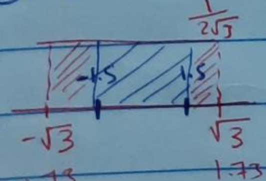
② Find σ

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}}\, dx = \frac{1}{2\sqrt{3}} \cdot 2\sqrt{3} = \boxed{1}$
(check)

$\sigma^2 = E(x^2) - (E(x))^2 = 1 - (0)^2 = 1$

$\sigma = 1$

③ $pr\left(|x| \geq \frac{3}{2}\right) = 1 - pr\left(|x| < \frac{3}{2}\right)$



$= 1 - pr\left(\frac{-3}{2} < X < \frac{3}{2}\right)$

$= 1 - \int_{-3/2}^{3/2} f(x)\, dx$

$= 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}}\, dx = 1 - \frac{3}{2\sqrt{3}}$

$= 1 - \frac{\sqrt{3}}{2}$

$\cong 0.1340$

(4) $= pr\left(|x| \geq \frac{3}{2}\right) = 1 - \frac{\sqrt{3}}{2} \cong 0.1340$

Assume we only know $\mu = 0$, $\sigma = 1$

Using Chebyshev's Inequality

$pr(|X - \mu| \geq K\sigma) \leq \frac{1}{K^2}$

$pr(|X - 0| \geq K) \leq \frac{1}{K^2}$

$pr(|X| \geq K) \leq \frac{1}{K^2}$, $K > 0$

$pr\left(|X| \geq \frac{3}{2}\right) \leq \frac{1}{\left(\frac{3}{2}\right)^2} = \frac{4}{9}$

$pr\left(|X| \geq \frac{3}{2}\right) \leq 0.4444$

check example 2 page 70